ON THE EVALUATION OF SUMS OF EXPONENTIATED MULTIPLES OF GENERALIZED CATALAN NUMBER LINEAR COMBINATIONS USING A HYPERGEOMETRIC APPROACH

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ABSTRACT. Infinite series comprising exponentiated multiples of p-term linear combinations of Catalan numbers arise naturally from a related power series expansion for $\sin(2p\alpha)$ (in odd powers of $\sin(\alpha)$) which itself has an interesting history. In this article some explicit results generated previously by the author (for $p = 1, 2, 3$) are discussed in the context of this general problem of series summation, and new evaluations made for the cases $p = 4, 5$ by way of further examples. A powerful hypergeometric approach is adopted which offers, from the analytical formulation developed, a means to achieve these particular evaluations and in principle many others for even greater values of p.

1. INTRODUCTION

1.1. **Background.** Consider the following expansion, in odd powers of $\sin(\alpha)$, of the trigonometric function $\sin(2p\alpha)$ for integer $p \geq 1$:

$$\sin(2p\alpha)/2 = \sum_{n=1}^{p} \alpha_n^{(p)} \sin^{2n-1}(\alpha) + \sum_{n=1}^{\infty} \frac{h_p(c_{n-1}, \ldots, c_{n+p-2})}{2^{2(n+p)-3}} \sin^{2(n+p)-1}(\alpha), \tag{1.1}$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the $(n+1)$th term $(n \geq 0)$ of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ with (ordinary) generating function

$$G(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_{n \geq 0} c_n x^n. \tag{1.2}$$

This standardized form of expansion was chosen by Xinrong [6] who showed in 2004 (using umbral calculus) that—beyond an initial p stand-alone terms with individual numerical coefficients, the functional coefficient $h_p(c_{n-1}, \ldots, c_{n+p-2})$ of each remaining term in the expansion (1.1) comprises a specific and identifiable linear combination of the p Catalan elements $c_{n-1}, \ldots, c_{n+p-2}$. Low order cases had already been formulated (in a slightly different, non-standard form) by Larcombe [2] in which such linearity was discerned and further postulated as a definitive feature of this type of expansion for $\sin(2p\alpha)$. By way of example, note that the $p = 1$ version of (1.1) reads

$$\sin(2\alpha)/2 = \sin(\alpha) - \sum_{n=1}^{\infty} \left[\frac{c_{n-1}}{2^{2n-1}}\right] \sin^{2n+1}(\alpha) \tag{1.3}$$

(for which $\alpha_1^{(1)} = 1$, $h_1(c_{n-1}) = -c_{n-1}$), while $p = 2$ (with $\alpha_1^{(2)} = 2$, $\alpha_2^{(2)} = -5$, $h_2(c_{n-1}, c_n) = 2(8c_{n-1} - c_n)$) gives

$$\sin(4\alpha)/2 = 2\sin(\alpha) - 5\sin^3(\alpha) + \sum_{n=1}^{\infty} \left[\frac{8c_{n-1} - c_n}{4^n}\right] \sin^{2n+3}(\alpha), \tag{1.4}$$

and (with $\alpha_1^{(3)} = 3$, $\alpha_2^{(3)} = -35/2$, $\alpha_3^{(3)} = 189/8$, $h_3(c_{n-1}, c_n, c_{n+1}) = -(256c_{n-1} - 64c_n + 3c_{n+1}))$ for $p = 3$ we have

$$\sin(6\alpha)/2 = 3\sin(\alpha) - (35/2)\sin^3(\alpha) + (189/8)\sin^5(\alpha) - \sum_{n=1}^{\infty}\left[\frac{256c_{n-1} - 64c_n + 3c_{n+1}}{2^{2n+3}}\right]\sin^{2n+5}(\alpha); \tag{1.5}$$

the next two functions h_4 and h_5 are

$$\begin{aligned} h_4(c_{n-1}, \dots, c_{n+2}) &= 4(1024c_{n-1} - 384c_n + 40c_{n+1} - c_{n+2}), \\ h_5(c_{n-1}, \dots, c_{n+3}) &= -(65536c_{n-1} - 32768c_n + 5376c_{n+1} - 320c_{n+2} + 5c_{n+3}), \end{aligned} \tag{1.6}$$

and so the process continues with $h_6, h_7, h_8, \dots$, becoming ever more lengthy and complex.

Before we present our analysis, we outline the remit of the paper and give some results established already within the context of a generalized problem.

1.2. **Remit and Previous Results.** As shown in [2], beginning with a first principles formulation of $\sin(2\alpha)$ as the power series (1.3), those expansions for $\sin(4\alpha), \sin(6\alpha), \sin(8\alpha), \dots$, build on one another sequentially and functions $h_2, h_3, h_4, \dots$, are duly determined explicitly (the manner in which series are developed becomes intractable beyond the first few values of p, however, due to the level of algebraic manipulation involved). Since that 2000 article it has been shown that every coefficient within the linear function h_p has, for arbitrary p, a known closed form following analysis elsewhere—an overview of such work is given in a more recent article [5], where convergence of the generic expansion (1.1) is dealt with as its main theme and the natural principal interval of convergence $|\alpha| < \frac{\pi}{2}$ extended to $|\alpha| \leq \frac{\pi}{2}$ analytically. The history of power series of type (1.1) is an interesting one—dating back to initial work in China from well over two hundred years ago—and the reader who seeks further information on it is referred to relevant citations also in [5]. We emphasize that for $p \geq 1$ the appearance of p-term linear combinations of the celebrated Catalan numbers in such expansions is a remarkable phenomenon, and yet further evidence of their mathematical ubiquity which in this instance affords the opportunity to evaluate a suite of derivative infinite series containing them.

Following on from publications [3, 4] by the author, we wish here to extend these works and consider the evaluation of series of general form

$$I_p(\beta) = (-1)^p \sum_{n=1}^{\infty} \beta^n h_p(c_{n-1}, \dots, c_{n+p-2}) \tag{1.7}$$

for arbitrary p (and non-zero β), taking a hypergeometric route. We are interested particularly in those values $\beta = \frac{1}{4}, \frac{3}{16}, \frac{1}{8}$ and $\frac{1}{16}$ associated with evaluations of (1.1) at $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ and $\frac{\pi}{6}$, respectively. The latter have been undertaken for the $p = 1$ instance in [3] and for the $p = 2, 3$ instances in [4], values of α having been chosen since each one (i) lies within the interval of convergence of (1.1) and (ii) has the property that $\sin(\alpha)$ can be written in exact (that is, error-free) form; these are important in so far as (i) guarantees that those particular series produced are actually summable, and (ii) means that corresponding β values realized are rationals (displaying a linearity in their progression). Actual results already obtained are, in terms of the series (1.7),

$$\begin{aligned} I_1(1/4) &= 1/2, \\ I_1(3/16) &= 1/4, \\ I_1(1/8) &= (1-1/\sqrt{2})/2, \\ I_1(1/16) &= (1-\sqrt{3}/2)/2, \end{aligned} \tag{1.8}$$

from [3], and

$$\begin{aligned} I_2(1/4) &= 6, \\ I_2(3/16) &= 10/3, \\ I_2(1/8) &= 2, \\ I_2(1/16) &= 2(2\sqrt{3}-3), \end{aligned}$$

$$\begin{aligned} I_3(1/4) &= 73, \\ I_3(3/16) &= 45, \\ I_3(1/8) &= 5+16\sqrt{2}, \\ I_3(1/16) &= 13, \end{aligned} \tag{1.9}$$

from [4], containing a mix of integers, rationals and irrationals between them.

In the next section we recall some existing results in detail, and clarify what is required in order that $I_p(\beta)$ be recast in a form suitable for direct evaluation. It becomes evident that it is instructive to effect the transformation of a certain ${}_3F_2(4x/(1+x)^2)$ hypergeometric series to a (finite) polynomial in x of degree p, which is described fully in Section 3. Special case results follow thereafter, in Section 4, where evaluations are made accordingly for the two particular series $I_4(\beta) = \sum_{n\geq 1} \beta^n h_4(c_{n-1}, \ldots, c_{n+2})$ and $I_5(\beta) = -\sum_{n\geq 1} \beta^n h_5(c_{n-1}, \ldots, c_{n+3})$ as representative examples of the general analysis developed; these results are new, and give a flavor of how such series may be handled. It is believed that no series of type $I_p(\beta)$ has been summed before, and as such the topic presented here—in combination with the forerunner works [2, 3, 4] and others by the author—together constitute an interesting modern day chapter in the timeline of the Catalan sequence.

2. The Hypergeometric Approach

2.1. **Hypergeometric Form of $I_p(\beta)$.** It is no surprise that, for arbitrary p, the series $I_p(\beta)$ has an accessible hypergeometric representation that reflects the deep influence of the Catalan numbers within (1.1) and whose structure is informed by previous studies. What initially drives the formulation here is the recursion

$$c_{n+1} = 2\frac{(2n+1)}{(n+2)}c_n \tag{2.1}$$

(valid for $n \geq 0$, given $c_0 = 1$) established originally by Euler. This permits, in principle, the reduction of (Catalan) variable dependency in h_p so that it becomes a functional multiple of the single Catalan number c_{n-1} for $p > 1$. For a few small values of p, hypergeometric conversion of $I_p(\beta)$ is relatively straightforward by hand and gives a useful indication of the general form sought. It is seen, for example, that in addition to $h_1(c_{n-1}) = -c_{n-1}$ we may write

$$\begin{aligned}
h_2(c_{n-1}) &= h_2(c_{n-1}, c_n(c_{n-1})) \\
&= 2[8c_{n-1} - c_n(c_{n-1})] \\
&= 2\left[8c_{n-1} - 2\frac{(2n-1)}{(n+1)}c_{n-1}\right] \\
&= 4\frac{(2n+5)}{(n+1)}c_{n-1}, \qquad (2.2)
\end{aligned}$$

and, further,

$$\begin{aligned}
h_3(c_{n-1}) &= h_3(c_{n-1}, c_n(c_{n-1}), c_{n+1}(c_{n-1})) \\
&= -[256c_{n-1} - 64c_n(c_{n-1}) + 3c_{n+1}(c_{n-1})] \\
&= -\left[256c_{n-1} - 64\cdot 2\frac{(2n-1)}{(n+1)}c_{n-1} + \ 3\cdot 2\frac{(2n+1)}{(n+2)}\cdot 2\frac{(2n-1)}{(n+1)}c_{n-1}\right] \\
&= -12\frac{(2n+7)(2n+9)}{(n+1)(n+2)}c_{n-1}, \qquad (2.3)
\end{aligned}$$

with

$$\begin{aligned}
h_4(c_{n-1}) &= h_4(c_{n-1}, c_n(c_{n-1}), c_{n+1}(c_{n-1}), c_{n+2}(c_{n-1})) \\
&= 4[1024c_{n-1} - 384c_n(c_{n-1}) + 40c_{n+1}(c_{n-1}) - c_{n+2}(c_{n-1})] \\
&\ \vdots \\
&= 32\frac{(2n+9)(2n+11)(2n+13)}{(n+1)(n+2)(n+3)}c_{n-1}, \qquad (2.4)
\end{aligned}$$

and so on, from which a clear and encouraging pattern emerges. Thus we see, in the first instance of (1.7), that $I_1(\beta) = -\sum_{n\geq 1}\beta^n h_1(c_{n-1}) = \sum_{n\geq 1}\beta^n c_{n-1}$ which hypergeometrically has form

$$I_1(\beta) = \beta\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, 1\\ 2\end{matrix}\ \right|\ 4\beta\right), \qquad (2.5)$$

while looking at the next two cases we find, utilizing (2.2) and (2.3),

$$\begin{aligned}
I_2(\beta) &= \sum_{n\geq 1}\beta^n h_2(c_{n-1}, c_n) \\
&= 2\sum_{n\geq 1}\beta^n[8c_{n-1} - c_n] \\
&= 14\beta\, {}_3F_2\left(\left.\begin{matrix}\frac{9}{2}, \frac{1}{2}, 1\\ 3, \frac{7}{2}\end{matrix}\ \right|\ 4\beta\right), \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned} I_3(\beta) &= -\sum_{n\geq 1} \beta^n h_3(c_{n-1}, c_n, c_{n+1}) \\ &= \sum_{n\geq 1} \beta^n [256c_{n-1} - 64c_n + 3c_{n+1}] \\ &= 198\beta\, {}_3F_2 \left(\begin{matrix} \frac{13}{2}, \frac{1}{2}, 1 \\ 4, \frac{9}{2} \end{matrix} \;\middle|\; 4\beta \right). \end{aligned} \tag{2.7}$$

Equations (2.2)–(2.4) are, in fact, merely special cases of the result

$$h_p(c_{n-1}) = (-1)^p p(n+p) \frac{[2(n+2p-1)]!n!}{(n+2p-1)![2(n+p)]!} c_{n-1}, \qquad p, n \geq 1 \tag{2.8}$$

(see, for example, [5, Eq. (1.10), p. 237]; it also gives $h_1(c_{n-1}) = -c_{n-1}$), through which, as will be shown, a general hypergeometric form of $I_p(\beta)$ can be obtained that recovers the $p = 1, 2, 3$ cases of (2.5)–(2.7).

Theorem 2.1. *For integer $p \geq 1$,*

$$I_p(\beta) = (pc_{2p}/2)\beta\, {}_3F_2 \left(\begin{matrix} 2p + \frac{1}{2}, \frac{1}{2}, 1 \\ p+1, p+\frac{3}{2} \end{matrix} \;\middle|\; 4\beta \right).$$

Proof. Consider, using the form (2.8) of the function $h_p = h_p(c_{n-1})$,

$$I_p(\beta) = (-1)^p \sum_{n=1}^{\infty} \beta^n h_p(c_{n-1}) = (-1)^p \sum_{n=1}^{\infty} \beta^n h_p(n), \tag{P.1}$$

denoting $h_p(c_{n-1})$ by $h_p(n)$ for convenience. Rewriting,

$$I_p(\beta) = (-1)^p \beta \sum_{n=0}^{\infty} \beta^n h_p(n+1), \tag{P.2}$$

and we establish a hypergeometric form for the series $Z(\beta; p) = \sum_{n=0}^{\infty} \beta^n h_p(n+1)$ by appeal solely to (2.8). First, we note that the initial term in the series (and so a hypergeometric multiplier) is

$$\begin{aligned} h_p(1) &= (-1)^p p(p+1) \frac{(4p)!1!}{(2p)!(2p+2)!} c_0 \\ &= \frac{(-1)^p p}{2} \frac{1}{2p+1} \binom{4p}{2p} \\ &= \frac{(-1)^p p}{2} c_{2p}. \end{aligned} \tag{P.3}$$

Then we form the summand term ratio $\frac{\beta^{n+1} h_p(n+2)}{\beta^n h_p(n+1)} = \beta h_p(n+2)/h_p(n+1)$, which simplifies algebraically to

$$\beta \frac{h_p(n+2)}{h_p(n+1)} = 2\beta \frac{(2n+4p+1)(2n+1)}{(n+p+1)(2n+2p+3)} = 4\beta \frac{(n+2p+\frac{1}{2})(n+\frac{1}{2})}{(n+p+1)(n+p+\frac{3}{2})}, \tag{P.4}$$

so that by (P.3), (P.4),

$$Z(\beta; p) = \frac{(-1)^p p}{2} c_{2p}\, {}_3F_2 \left(\begin{matrix} 2p + \frac{1}{2}, \frac{1}{2}, 1 \\ p+1, p+\frac{3}{2} \end{matrix} \;\middle|\; 4\beta \right), \tag{P.5}$$

and $I_p(\beta) = (-1)^p \beta Z(\beta; p)$ is immediate. $\square$

2.2. **Formalizing the Problem.** In [4] a hypergeometric approach to the evaluations of $T(\beta) = \frac{1}{2} I_2(\beta)$ and $U(\beta) = I_3(\beta)$ was based on application of the individual (respective) results

$$ {}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2} \right) = \frac{1}{7}(1+x)(7-x) \tag{2.9} $$

and

$$ {}_3F_2\left(\begin{matrix} \frac{13}{2}, \frac{1}{2}, 1 \\ 4, \frac{9}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2} \right) = \frac{1}{198}(1+x)(198 - 55x + 3x^2). \tag{2.10} $$

Although not used in [3] (due to the very simple nature of $S(\beta) = I_1(\beta)$ examined therein), the corresponding result is

$$ {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ 2 \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2} \right) = 1 + x, \tag{2.11} $$

as was noted in Remark 3.2 of [4, p. 258]. Having arrived at Theorem 2.1, therefore, we wish to find the equivalent closed form polynomial in x for the hypergeometric series

$$ F_p(x) = {}_3F_2\left(\begin{matrix} 2p + \frac{1}{2}, \frac{1}{2}, 1 \\ p+1, p+\frac{3}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2} \right) \tag{2.12} $$

that recovers (2.9)–(2.11) and allows evaluations of $I_p(\beta)$ to be made for new values of $p > 3$ in routine fashion on simply setting $x(\beta) = G(\beta) - 1$ (as the solution $x(\beta)$ to the quadratic equation formed by equating β with $x/(1+x)^2$; see the Appendix A proof of Lemma 2.1 in [4, p. 259]). This, as we shall see, is no trivial task analytically.

3. Transformation Analysis

While it is possible, for any value of $p \geq 1$, to compute $F_p(x)$ via (2.12) algebraically by computer, there is little in its form to suggest that (rather than being an infinite power series) it is in fact a finite polynomial for arbitrary p, and an alternating sign one. The following theorem provides a transformation that reveals both of these properties, through which low order cases may be validated and hitherto unseen evaluations for $I_4(\beta)$ and $I_5(\beta)$ given for the first time.

Theorem 3.1. *For integer $p \geq 1$,*

$$ F_p(x) = 1 + \frac{2(4p+1)}{(p+1)(2p+3)} x \, {}_3F_2\left(\begin{matrix} -(p-\frac{1}{2}), -(p-1), 2 \\ p+2, p+\frac{5}{2} \end{matrix} \,\middle|\, -x \right). $$

As alluded to above, the proof process is a multi-step one which is characteristic of the kind of task in hand and will be familiar to anyone working in this particular field. It is, though, useful to arrive at an alternative hypergeometric description of $F_p(x)$ that offers more insight into its structure than (2.12), and re-assuring to see previous results recovered, too, as a check.

Proof. Writing

$$ (u)_k = u(u+1)(u+2)(u+3)\cdots(u+k-1) \tag{P.6} $$

to denote the rising factorial function defined for integer $k \geq 0$ (where $(u)_0 = 1$), then

$$\begin{aligned} F_p(x) &= \sum_{i=0}^{\infty} \frac{(2p+\frac{1}{2})_i(\frac{1}{2})_i(1)_i}{(p+1)_i(p+\frac{3}{2})_i} \cdot \frac{[4x/(1+x)^2]^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{(2p+\frac{1}{2})_i(\frac{1}{2})_i}{(p+1)_i(p+\frac{3}{2})_i}(4x)^i(1+x)^{-2i}. \end{aligned} \tag{P.7}$$

Since $(1+x)^{-\alpha} = \sum_{j=0}^{\infty} \binom{-\alpha}{j} x^j = \sum_{j=0}^{\infty} \binom{\alpha+j-1}{j}(-x)^j$ (by negation) $= \sum_{j=0}^{\infty} \frac{(\alpha)_j}{j!}(-x)^j$, (P.7) becomes

$$\begin{aligned} F_p(x) &= \sum_{i=0}^{\infty} \frac{(2p+\frac{1}{2})_i(\frac{1}{2})_i}{(p+1)_i(p+\frac{3}{2})_i}(4x)^i \sum_{j=0}^{\infty} \frac{(2i)_j}{j!}(-x)^j \\ &= \sum_{i,j=0}^{\infty} \frac{4^i(2p+\frac{1}{2})_i(\frac{1}{2})_i}{(p+1)_i(p+\frac{3}{2})_i} \frac{(-1)^j(2i)_j}{j!} x^{i+j}, \end{aligned} \tag{P.8}$$

whose term in x^n has, for $n \geq 0$, coefficient

$$\begin{aligned} [x^n]\{F_p(x)\} &= \sum_{i+j=n} \frac{4^i(2p+\frac{1}{2})_i(\frac{1}{2})_i}{(p+1)_i(p+\frac{3}{2})_i} \frac{(-1)^j(2i)_j}{j!} \\ &= 4^n \sum_{j=0}^{n} \frac{4^{-j}(2p+\frac{1}{2})_{n-j}(\frac{1}{2})_{n-j}}{(p+1)_{n-j}(p+\frac{3}{2})_{n-j}} \frac{(-1)^j(2n-2j)_j}{j!}. \end{aligned} \tag{P.9}$$

Now, the easily established relation $(\alpha)_n/(\alpha)_{n-j} = (-1)^j(-\alpha-n+1)_j$ allows us to make the representations

$$\begin{aligned} (2p+1/2)_{n-j} &= \frac{(2p+\frac{1}{2})_n}{(-1)^j(-2p+\frac{1}{2}-n)_j}, \\ \frac{1}{(p+1)_{n-j}} &= \frac{(-1)^j(-p-n)_j}{(p+1)_n}, \\ \frac{1}{(p+\frac{3}{2})_{n-j}} &= \frac{(-1)^j(-p-\frac{1}{2}-n)_j}{(p+\frac{3}{2})_n}. \end{aligned} \tag{P.10}$$

In addition it can be shown, with some work, that

$$(1/2)_{n-j}(2n-2j)_j = 4^j \frac{(\frac{1}{2})_n(1-n)_j}{(1-2n)_j}, \tag{P.11}$$

so, noting that $[x^0]\{F_p(x)\} = 1$ and $(1-n)_n = 0$, (P.10) and (P.11) combine to give, for $n \geq 1$, $[x^n]\{F_p(x)\}$ (P.9) as

$$\begin{aligned} [x^n]\{F_p(x)\} &= 4^n \frac{(2p+\frac{1}{2})_n(\frac{1}{2})_n}{(p+1)_n(p+\frac{3}{2})_n} \sum_{j=0}^{n} \frac{(1-n)_j(-p-n)_j(-p-\frac{1}{2}-n)_j}{(1-2n)_j(-2p+\frac{1}{2}-n)_j} \cdot \frac{1}{j!} \\ &= 4^n \frac{(2p+\frac{1}{2})_n(\frac{1}{2})_n}{(p+1)_n(p+\frac{3}{2})_n} \sum_{j=0}^{n-1} \frac{(1-n)_j(-p-n)_j(-p-\frac{1}{2}-n)_j}{(1-2n)_j(-2p+\frac{1}{2}-n)_j} \cdot \frac{1}{j!} \\ &= M(n;p)H(n;p), \end{aligned} \tag{P.12}$$

say, where

$$M(n;p) = 4^n \frac{(2p+\frac{1}{2})_n(\frac{1}{2})_n}{(p+1)_n(p+\frac{3}{2})_n} \tag{P.13}$$

and $H(n;p)$ is the finite series in j written as the ${}_3F_2(1)$ hypergeometric function

$$H(n;p) = {}_3F_2\left(\left.\begin{matrix} -(n-1), -(p+n), -(p+n+\frac{1}{2}) \\ -(2n-1), -(2p+n-\frac{1}{2}) \end{matrix}\,\right|\, 1\right), \tag{P.14}$$

remarking also that over the range $j = 0, \ldots, n-1$ the rising factorials $(-p-n)_j$ and $(1-2n)_j$ are non-zero. From (P.12), therefore, we may write

$$\begin{aligned} F_p(x) &= 1 + \sum_{n\geq 1} M(n;p)H(n;p)x^n \\ &= 1 + x\sum_{n\geq 0} M(n+1;p)H(n+1;p)x^n, \end{aligned} \tag{P.15}$$

and, looking at Theorem 3.1, it remains that the functions $M(n+1;p)$ and $H(n+1;p)$ be manipulated into the form desired so as to complete the proof.

Consider first, then,

$$M(n+1;p) = 4^{n+1} \frac{(2p+\frac{1}{2})_{n+1}(\frac{1}{2})_{n+1}}{(p+1)_{n+1}(p+\frac{3}{2})_{n+1}} \tag{P.16}$$

which, on employing the simple relations $(2p+\frac{1}{2})_{n+1} = (2p+\frac{1}{2})(2p+\frac{3}{2})_n$, $(\frac{1}{2})_{n+1} = (\frac{1}{2})(\frac{3}{2})_n$, $(p+\frac{3}{2})_{n+1} = (p+\frac{3}{2})(p+\frac{5}{2})_n$ and $(p+1)_{n+1} = (p+1)(p+2)_n$, can be written

$$M(n+1;p) = 2^{2n+1} \frac{(4p+1)}{(p+1)(2p+3)} \frac{(2p+\frac{3}{2})_n(\frac{3}{2})_n}{(p+2)_n(p+\frac{5}{2})_n}. \tag{P.17}$$

Next, the required evaluation of

$$H(n+1;p) = {}_3F_2\left(\left.\begin{matrix} -n, -(p+n+1), -(p+n+\frac{3}{2}) \\ -(2n+1), -(2p+n+\frac{1}{2}) \end{matrix}\,\right|\, 1\right) \tag{P.18}$$

is achieved by applying the well-known identity

$${}_3F_2\left(\left.\begin{matrix} a, b, -t \\ c, a+b-c-t+1 \end{matrix}\,\right|\, 1\right) = \frac{(c-a)_t(c-b)_t}{(c)_t(c-a-b)_t} \tag{P.19}$$

of Pfaff-Saalschütz (with $a = -(p+n+1)$, $b = -(p+n+\frac{3}{2})$, $c = -(2p+n+\frac{1}{2})$ and $t = n$), yielding

$$H(n+1;p) = \frac{(\frac{1}{2}-p)_n(1-p)_n}{(-2p-n-\frac{1}{2})_n(n+2)_n}, \tag{P.20}$$

and our proof concludes rapidly—to finish we write, from (P.17) and (P.20),

$$\begin{aligned}M(n+1;p)H(n+1;p) &= 2^{2n+1}\frac{(4p+1)}{(p+1)(2p+3)}\frac{(\frac{1}{2}-p)_n(1-p)_n}{(p+2)_n(p+\frac{5}{2})_n}\\ &\qquad\times\frac{(2p+\frac{3}{2})_n}{(-2p-n-\frac{1}{2})_n}\frac{(\frac{3}{2})_n}{(n+2)_n}\\ &= 2^{2n+1}\frac{(4p+1)}{(p+1)(2p+3)}\frac{(\frac{1}{2}-p)_n(1-p)_n}{(p+2)_n(p+\frac{5}{2})_n}\frac{1}{(-1)^n}\frac{(2)_n}{4^n n!}\\ &= \frac{2(4p+1)}{(p+1)(2p+3)}\frac{(\frac{1}{2}-p)_n(1-p)_n(2)_n}{(p+2)_n(p+\frac{5}{2})_n}\frac{(-1)^n}{n!}\end{aligned}\tag{P.21}$$

after the simplifications shown, and substitution of (P.21) into (P.15) delivers Theorem 3.1 immediately; clearly, the ${}_3F_2(-x)$ series of Theorem 3.1 is a degree $p-1$ polynomial in x, with $F_p(x)$ degree p. □

Remark 3.1. A word of clarification is perhaps in order regarding application of the Pfaff-Saalschütz identity. Potential terms of the form 0/0 in the series $H(n+1;p)$ (P.18) (caused by the appearance of the upper parameters $-n$ or $-(p+n+1)$ combined with the lower parameter $-(2n+1)$) are avoided by a simple limiting argument that justifies the use of (P.19). Replacing $-(2n+1)$ with $-(2n+1)+\varepsilon$ in (P.18), and likewise $-(p+n+1)$ with $-(p+n+1)+\varepsilon$ ($|\varepsilon|\ll 1$), Pfaff-Saalschütz still applies. Both sides of the resulting equation (P.20) are rational functions of ε, which can then be set to zero as a limit.

4. Evaluations of $I_4(\beta)$ and $I_5(\beta)$

We finish by giving new evaluations of the series $I_4(\beta)$ and $I_5(\beta)$ as an illustration of our analysis, and add a couple of remarks.

Noting that Theorem 3.1 reproduces the right-hand side polynomials of (2.11), (2.9) and (2.10) (for $p=1,2,3$, respectively) as originally computed via (2.12), it is further found that

$$\begin{aligned}F_4(x) &= \frac{1}{715}(1+x)(715-273x+35x^2-x^3),\\ F_5(x) &= \frac{1}{41990}(1+x)(41990-19380x+3876x^2-285x^3+5x^4).\end{aligned}\tag{4.1}$$

We may evaluate $I_p(\beta)$ according to

$$I_p(\beta)=(pc_{2p}/2)\beta\,{}_3F_2\left(\begin{matrix}2p+\frac{1}{2},\frac{1}{2},1\\ p+1,p+\frac{3}{2}\end{matrix}\,\middle|\,\frac{4x(\beta)}{(1+x(\beta))^2}\right)=(pc_{2p}/2)\beta F_p(x(\beta))\tag{4.2}$$

by Theorem 2.1 and (2.12), in line with [4] where $x(\beta)=G(\beta)-1$ as already mentioned; in other words, $I_4(\beta)=4\sum_{n\geq 1}\beta^n[1024c_{n-1}-384c_n+40c_{n+1}-c_{n+2}]=(4c_8/2)\beta F_4(x(\beta))=2860\beta F_4(x(\beta))$ and $I_5(\beta)=\sum_{n\geq 1}\beta^n[65536c_{n-1}-32768c_n+5376c_{n+1}-320c_{n+2}+5c_{n+3}]=(5c_{10}/2)\beta F_5(x(\beta))=41990\beta F_5(x(\beta))$. Thus, noting that by (1.2),

$$\begin{aligned}x(1/4)&=G(1/4)-1=1,\\ x(3/16)&=G(3/16)-1=1/3,\\ x(1/8)&=G(1/8)-1=3-2\sqrt{2},\\ x(1/16)&=G(1/16)-1=7-4\sqrt{3},\end{aligned}\tag{4.3}$$

then in particular, when $p = 4$, (4.1) delivers

$$\begin{aligned}
I_4(1/4) &= 715F_4(x(1/4)) = 715 \cdot \frac{952}{715} = 952, \\
I_4(3/16) &= (2145/4)F_4(x(3/16)) = \frac{2145}{4} \cdot \frac{5216}{4455} = 16952/27, \\
I_4(1/8) &= (715/2)F_4(x(1/8)) = \frac{715}{2} \cdot \frac{784}{715} = 392, \\
I_4(1/16) &= (715/4)F_4(x(1/16)) = \frac{715}{4} \cdot \frac{32}{715}(245 - 128\sqrt{3}) \\
&= 8(245 - 128\sqrt{3}), \qquad (4.4)
\end{aligned}$$

all series having been checked numerically to a high degree of convergence accuracy. We find, in a similar fashion,

$$\begin{aligned}
I_5(1/4) &= 13103, \\
I_5(3/16) &= 727991/81, \\
I_5(1/8) &= 7127 - 1024\sqrt{2}, \\
I_5(1/16) &= 16384\sqrt{3} - 25657, \qquad (4.5)
\end{aligned}$$

the calculation details of which we leave as a straightforward reader exercise.

Remark 4.1. The transformation Theorem 3.1 gives a form for $F_p(x)$ that is more convenient to work with than (2.12), and more informative. As we have seen, all specific computations made for $p = 1, \ldots, 5$ reveal the presence of a factor $1 + x$ within $F_p(x)$, and those beyond $p = 5$ suggest that this is the case for general p. We confirm that such a conjecture is indeed true by means of a little known identity, tucked away in W. N. Bailey's revered 1935 text [1], which can be shown to lead independently to Theorem 3.1. Rather than allow it to clutter the main narrative here the details are set out in the Appendix, and it means that the factored form of $F_p(x)$ noted could, in principle, be reverse-engineered from Theorem 3.1 as a starting point, which is an observation worth making—for clarity, Theorem 3.1 is the preferred representation of $F_p(x)$ occurring as a consequence of our line of enquiry, and a fully factored form is produced in practice only by algebraic computation and simplification functionality (as evident in (2.9)–(2.11) and (4.1)).

Remark 4.2. Although we choose to give results for neither $I_6(\beta)$ nor $I_7(\beta)$ it is useful, for completeness, to see the associated evaluating polynomials as $F_6(x) = (1 + x)(312018 - 163438x + 43263x^2 - 5313x^3 + 253x^4 - 3x^5)/312018$ and $F_7(x) = (1+x)(9360540 - 5368545x + 1726725x^2 - 296010x^3 + 24570x^4 - 819x^5 + 7x^6)/9360540$, each with the basic structure expected from our analysis.

5. Summary

In this paper we have re-examined some of the results given in previous explicit series evaluations by the author, and developed new ones from consideration of a fully general problem. This has been achieved by appeal to some non-trivial hypergeometric theory, attesting to its power where applied appropriately; here it has offered both a route through the analysis and a clear understanding of the evaluations procedure.

It is worth emphasizing, perhaps, that the study of any similar class of series seems to be absent from the literature (due almost certainly to its unusual nature). Note also that a method of evaluation for $I_p(\beta)$ based solely on the Catalan sequence generating function

$G(x)$—seen to be successful in the $p = 2, 3$ cases of [4] as so called Method III, and trivially so in the $p = 1$ instance of [3]—would appear to lend itself to an elegant generalization which accommodates arbitrary p; this, however, must be assigned to discussion in a future article.

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Appendix

In this Appendix we offer a formal route to Theorem 3.1 which stands independent of the Section 3 proof.

Proof. Consider a rather obscure result in Bailey [1, Example 4(iv), p. 97] (he attributes it to F. J. W. Whipple in fact), namely,

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} \,\middle|\, x\right) = (1-x)^{-a}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}a, \frac{1}{2}(1+a), 1+a-b-c \\ 1+a-b, 1+a-c \end{matrix} \,\middle|\, \frac{-4x}{(1-x)^2}\right), \tag{A.1} $$

which reads after rearrangement, on setting $a = 1$, $b = 1 - p$ and $c = \frac{1}{2} - p$,

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, 1, 2p+\frac{1}{2} \\ p+1, p+\frac{3}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2}\right) = F_p(x) = (1+x)\, {}_3F_2\left(\begin{matrix} 1, 1-p, \frac{1}{2}-p \\ p+1, p+\frac{3}{2} \end{matrix} \,\middle|\, -x\right), \tag{A.2} $$

where (and with reference to Remark 4.1) we see a factor of $1 + x$ intrinsic to it. We can reproduce Theorem 3.1 (in which $F_p(x)$ contains no such factor) by proceeding as follows:

$$ \begin{aligned} F_p(x) &= {}_3F_2\left(\begin{matrix} 1, 1-p, \frac{1}{2}-p \\ p+1, p+\frac{3}{2} \end{matrix} \,\middle|\, -x\right) + x\, {}_3F_2\left(\begin{matrix} 1, 1-p, \frac{1}{2}-p \\ p+1, p+\frac{3}{2} \end{matrix} \,\middle|\, -x\right) \\ &= \sum_{k\ge 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!} + x \sum_{k\ge 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!} \\ &= 1 + \sum_{k\ge 1} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!} + x \sum_{k\ge 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!} \\ &= 1 + \sum_{k\ge 0} \frac{(1)_{k+1}(1-p)_{k+1}(\frac{1}{2}-p)_{k+1}}{(p+1)_{k+1}(p+\frac{3}{2})_{k+1}} \cdot \frac{(-x)^{k+1}}{(k+1)!} \\ &\qquad + \; x \sum_{k\ge 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!}. \end{aligned} \tag{A.3} $$

The rising factorial terms within the first sum of (A.3) are now rewritten, giving

$$\begin{aligned}
F_p(x) &= 1 + \sum_{k\geq 0} \frac{(1+k)(1)_k \cdot (1-p+k)(1-p)_k \cdot (\frac{1}{2}-p+k)(\frac{1}{2}-p)_k}{(p+1+k)(p+1)_k \cdot (p+\frac{3}{2}+k)(p+\frac{3}{2})_k} \cdot \frac{(-x)^{k+1}}{(k+1)!} \\
&\qquad + \quad x \sum_{k\geq 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \cdot \frac{(-x)^k}{k!} \\
&= 1 + \sum_{k\geq 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \left[1 - \frac{(1-p+k)(\frac{1}{2}-p+k)}{(p+1+k)(p+\frac{3}{2}+k)}\right] \frac{(-1)^k x^{k+1}}{k!} \\
&= 1 + \sum_{k\geq 0} \frac{(1)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_k(p+\frac{3}{2})_k} \left[\frac{(4p+1)(1+k)}{(p+1+k)(p+\frac{3}{2}+k)}\right] \frac{(-1)^k x^{k+1}}{k!} \\
&= 1 + (4p+1)x \sum_{k\geq 0} \frac{(1)_{k+1}(1-p)_k(\frac{1}{2}-p)_k}{(p+1)_{k+1}(p+\frac{3}{2})_{k+1}} \cdot \frac{(-x)^k}{k!} \\
&= 1 + (4p+1)x \sum_{k\geq 0} \frac{(2)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+1)(p+2)_k \cdot (p+\frac{3}{2})(p+\frac{5}{2})_k} \cdot \frac{(-x)^k}{k!} \\
&= 1 + \frac{2(4p+1)}{(p+1)(2p+3)} x \sum_{k\geq 0} \frac{(2)_k(1-p)_k(\frac{1}{2}-p)_k}{(p+2)_k(p+\frac{5}{2})_k} \cdot \frac{(-x)^k}{k!} \\
&= 1 + \frac{2(4p+1)}{(p+1)(2p+3)} x\, {}_3F_2\left(\begin{matrix} 2, -(p-1), -(p-\frac{1}{2}) \\ p+2, p+\frac{5}{2} \end{matrix} \,\middle|\, -x\right),
\end{aligned} \tag{A.4}$$

as desired. $\square$

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