A BALANCING PROBLEM ON A BINARY RECURRENCE AND ITS ASSOCIATE

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Abstract. A balancing problem associated with two integer sequences is introduced. The problem is studied using the sequences obtained from a binary recurrence and its associate sequence. We provide an algorithm to solve, under some circumstances, the Diophantine equation $G_0 + G_1 + \cdots + G_x = H_0 + H_1 + \cdots + H_y$ in the non-negative integer unknowns x and y, where the sequence $\{H_n\}_{n=0}^{\infty}$ is the associate of $\{G_n\}_{n=0}^{\infty}$.

1. Introduction

According to Behera and Panda [1], a natural number n is a balancing number if $1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$ for some natural number r. By slightly modifying the defining equation of balancing numbers, Panda and Ray [5] called a natural number n a cobalancing number with cobalancer r if $1+2+\cdots+n = (n+1)+\cdots+(n+r)$. The concepts of balancing and cobalancing numbers have been extended in several directions. Panda [6] felt that the natural numbers used in the definition of balancing and cobalancing numbers could possibly be replaced by an arbitrary integer sequence $a_m, m = 1, 2, \ldots$. He called a term a_n of this sequence a sequence balancing number if $a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + \cdots + a_{n+r}$ for some r, while he called a_n a sequence cobalancing number if $a_1 + a_2 + \cdots + a_n = a_{n+1} + \cdots + a_{n+r}$ for some r. He studied the existence of sequence balancing and cobalancing numbers in the sequence of odd and even natural numbers. However, for $n > 1$, he failed to find any sequence balancing or cobalancing number in the sequence $a_k = k^n$.

In 2011, Szakács [8] thought of replacing the additions used in the definitions of balancing and cobalancing numbers by multiplications and defined multiplying balancing numbers n as natural numbers satisfying the Diophantine equation $1\cdot 2\cdot \cdots \cdot(n-1) = (n+1)(n+2)\cdots(n+r)$ for some r and proved that 7 is the only multiplying balancing number. T. Kovács, K. Liptai and P. Olajos [4] extended the concept of balancing numbers to arithmetic progressions and studied sequence balancing numbers with terms from an arithmetic progression. They called $an+b$ an (a,b)-balancing number if $(a+b)+\cdots+(a(n-1)+b) = (a(n+1)+b)+\cdots+(a(n+r)+b)$ holds for some r where $a > 0$ and $b > 0$ are coprime integers. They provided certain conditions for the existence of such numbers. Komatsu and Szalay [3] replaced natural numbers occurring in the definition of balancing numbers with binomial coefficients and studied certain special cases of sequence balancing numbers.

Simple algebraic manipulations in the defining equations of balancing and cobalancing numbers allow us to redefine balancing and cobalancing numbers as follows.

A natural number n is a balancing number if

$$1 + 2 + \cdots + m = O_1 + O_2 + \cdots + O_n \tag{1}$$

for some natural number m, and n is a cobalancing number if

$$1 + 2 + \cdots + \ell = E_1 + E_2 + \cdots + E_n \tag{2}$$

for some natural number ℓ, where O_n and E_n denote the nth odd number and the nth even number, respectively. These definitions motivate us to consider Diophantine equations of the form

$$a_1 + a_2 + \cdots + a_m = b_1 + b_2 + \cdots + b_n, \tag{3}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are integer sequences. This paper considers binary recurrences of certain type and their associate sequences.

Let $\{G_n\}_{n=0}^{\infty}$ be a sequence defined by the binary recurrence $G_n = AG_{n-1} + BG_{n-2}$, $n \geq 2$ where A, B, G_0 and G_1 are given integers and let $D = A^2 + 4B$. It is well-known that if $D \neq 0$, then the explicit form (also popularly known as the Binet form) for G_n is given by

$$G_n = \frac{(G_1 - \beta G_0)\alpha^n - (G_1 - \alpha G_0)\beta^n}{\alpha - \beta}, \quad n \geq 0, \tag{4}$$

where $\alpha = (A + \sqrt{D})/2$ and $\beta = (A - \sqrt{D})/2$ are the two distinct roots of the characteristic equation $x^2 - Ax - B = 0$ of the sequence $\{G\}_{n=0}^{\infty}$. If $D = 0$, then $\alpha = \beta = A/2$ and

$$G_n = n\alpha^{n-1}G_1 - (n-1)\alpha^n G_0, \quad n \geq 0. \tag{5}$$

Let $\{H_n\}_{n=0}^{\infty}$ be the sequence defined by the recurrence relation $H_n = AH_{n-1} + BH_{n-2}$ where $H_0 = 2G_1 - AG_0$, $H_1 = AG_1 + 2BG_0$. $\{H_n\}_{n=0}^{\infty}$ is called the associate sequence of $\{G\}_{n=0}^{\infty}$. According to $D \neq 0$ or $D = 0$,

$$H_n = (G_1 - \beta G_0)\alpha^n + (G_1 - \alpha G_0)\beta^n \quad \text{or} \quad H_n = (2G_1 - AG_0)\alpha^n, \tag{6}$$

respectively. The objective of this paper is to study the Diophantine equation

$$G_0 + G_1 + \cdots + G_x = H_0 + H_1 + \cdots + H_y, \tag{7}$$

which we call the balancing problem associated with the sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$. We describe an algorithm which, under some conditions, solves (7). At the end of the paper we illustrate the method and obtain the following result.

Theorem 1. *The only solutions to equation* (7) *is* $(x, y) = (2, 0)$ *if* $\{G\}_{n=0}^{\infty}$ *is the Fibonacci sequence or the Jacobsthal sequence. Further, (7) has no solution if* $\{G\}_{n=0}^{\infty}$ *is the Pell sequence. Moreover, in the case of the sequence* $G_n = G_{n-1} + 9900G_{n-2}$, $G_0 = 0$, $G_1 = 1$ *equation* (7) *possesses only the solution* $(x, y) = (2, 0)$ *again.*

2. Preliminaries

As usual let $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ be binary recurrences as defined in the last section. The following lemma, which deals with the sum formulas for the sequence $\{G_n\}_{n=0}^{\infty}$, has an important role in the solution of the problem (7).

Lemma 2. *For every natural number* n,

$$\sum_{i=0}^{n} G_i = \begin{cases} \frac{G_{n+1} + BG_n - G_1 + (A-1)G_0}{A+B-1} & \text{if } D \neq 0 \text{ and } A + B \neq 1, \\ \frac{G_{n+1} - G_0 + (-G_1 + (A-1)G_0)(n+1)}{A-2} & \text{if } D \neq 0 \text{ and } A + B = 1, \\ \frac{G_{n+1} + BG_n - G_1 + (A-1)G_0}{A+B-1} & \text{if } D = 0 \text{ and } A + B \neq 1, \\ \frac{(G_n + G_0)(n+1)}{A} & \text{if } D = 0 \text{ and } A + B = 1. \end{cases} \tag{8}$$

Note that the last case occurs if and only if $(A, B) = (2, -1)$. Observe that if $A + B \neq 1$, then the sum of the terms does not depend on the value of D.

Proof. The proof relies on the explicit forms (4) and (5), and the summation formula of the consecutive terms of a geometric progression.

The results of Lemma 2 are well-known. The first case was shown by Horadam [2]. Moreover, Lemma 2 is equivalent to the statements of Theorem 1 in [7] due to Russell. Since the article [7] gives only hints to the proofs, here we go into details, but only in the third case when $D = 0$ and $A+B \neq 1$. Then the appropriate conditions imply $\alpha = \beta \neq 1$. Note that $\alpha = A/2$, $\alpha^2 = -B$ and $(\alpha-1)^2 = 1-A-B$ hold now. Clearly,

$$\begin{aligned}\sum_{i=0}^{n} G_i &= \sum_{i=0}^{n}\left(i\alpha^{i-1}G_1 - (i-1)\alpha^i G_0\right) = G_1\sum_{i=0}^{n}\left(\alpha^i\right)' - \alpha^2 G_0 \sum_{i=0}^{n}\left(\alpha^{i-1}\right)' \\ &= G_1\left(\sum_{i=0}^{n}\alpha^i\right)' - \alpha^2 G_0\left(\sum_{i=0}^{n}\alpha^{i-1}\right)' \\ &= G_1\frac{n\alpha^{n+1}-(n+1)\alpha^n+1}{(\alpha-1)^2} - G_0\frac{(n-1)\alpha^{n+2}-n\alpha^{n+1}+2\alpha-1}{(\alpha-1)^2} \\ &= \frac{\alpha^2 G_n - G_{n+1} + G_1 - (2\alpha-1)G_0}{(\alpha-1)^2},\end{aligned}$$

and then it immediately leads to the statement. Thus the proof of the third case is complete. □

From this point onward we assume $A > 0$, $B \neq 0$, $A+B \neq 1$, $D > 0$, and also $|G_0|+|G_1| \neq 0$. Thus, α and β are real numbers, $\alpha > 1$, $\alpha > |\beta|$, $\beta \neq 0$, $\beta \neq 1$. Clearly, $\alpha+\beta = A$, $\alpha\beta = -B$ and $\alpha-\beta = \sqrt{D}$ also hold. The first case of Lemma 2 allows us to replace equation (7) by

$$G_{x+1} + BG_x = H_{y+1} + BH_y + \eta, \tag{9}$$

where $\eta = (A-1)G_1 + (-A^2-2B+1)G_0$.

We also assume that $G_n \geq 0$ and $H_n \geq 0$ for all $n \in \mathbb{N}$. Consequently, by (4) and first part of (6), $G_1 - \beta G_0 = (\sqrt{D}G_n + H_n)/(2\alpha^n) \geq 0$ follows. We claim that $G_1 - \beta G_0 > 0$. Observe that if $G_1 - \beta G_0 = 0$, then $G_n = -(G_1-\alpha G_0)\beta^n/\sqrt{D}$ and $H_n = (G_1-\alpha G_0)\beta^n$ have opposite signs (since $G_1 - \alpha G_0 \neq 0$), which is a contradiction. Solving the system $H_0 = 2G_1 - AG_0 \geq 0$ and $H_1 = AG_1 + 2BG_0 \geq 0$, we find that the initial values G_0 and G_1 must satisfy $0 \leq G_0 \leq 2G_1/A$, $0 < G_1$ (independently from the sign of B). Note that $H_0 = 0 = H_1$ would imply $D = 0$.

The solution of (9) essentially relies on the following lemma.

Lemma 3. *For any non-negative integer n_0 and for all $n \geq n_0$*

$$\alpha^{n+\varepsilon_1(n_0)} \leq G_n \leq \alpha^{n+\varepsilon_2(n_0)} \qquad \text{and} \qquad \alpha^{n+\delta_1(n_0)} \leq H_n \leq \alpha^{n+\delta_2(n_0)}, \tag{10}$$

where

$$\varepsilon_j(n_0) = \log_\alpha \frac{(G_1-\beta G_0) + (-1)^j|G_1-\alpha G_0|\left(\frac{|\beta|}{\alpha}\right)^{n_0}}{\sqrt{D}} \tag{11}$$

and

$$\delta_j(n_0) = \varepsilon_j(n_0) + \log_\alpha \sqrt{D}, \tag{12}$$

$j = 1, 2$.

Proof. Recall (4), and note that if $G_1 - \alpha G_0 = 0$, then

$$G_n = \frac{G_1-\beta G_0}{\sqrt{D}}\alpha^n = \alpha^{n+\varepsilon} \qquad \text{and} \qquad H_n = (G_1-\beta G_0)\alpha^n = \alpha^{n+\delta}$$

hold for the constants $\varepsilon = \log_\alpha((G_1 - \beta G_0)/\sqrt{D})$ and $\delta = \log_\alpha(G_1 - \beta G_0)$ (i.e. $\varepsilon_1(n_0) = \varepsilon = \varepsilon_2(n_0)$ and $\delta_1(n_0) = \delta = \delta_2(n_0)$ are valid independently from the value n_0). Assume now that $G_1 - \alpha G_0 \neq 0$. Let $c_1 = \sqrt{D}$ for the sequence $\{G\}_{n=0}^\infty$ and $c_1 = 1$ for the sequence $\{H\}_{n=0}^\infty$. Then, by (4) we have

$$\left.\begin{array}{c} G_n \\ H_n \end{array}\right\} \leq \frac{(G_1 - \beta G_0)\alpha^n + |G_1 - \alpha G_0||\beta|^n}{c_1} = \alpha^n \frac{(G_1 - \beta G_0) + |G_1 - \alpha G_0| \left(\frac{|\beta|}{\alpha}\right)^n}{c_1}.$$

If $n \geq n_0$, then the last part is not more than

$$\alpha^n \frac{(G_1 - \beta G_0) + |G_1 - \alpha G_0| \left(\frac{|\beta|}{\alpha}\right)^{n_0}}{c_1} = \left\{ \begin{array}{l} \alpha^{n+\varepsilon_2(n_0)} \\ \alpha^{n+\delta_2(n_0)} \end{array} \right. .$$

In order to get lower bounds, we must be sure that $(G_1 - \beta G_0)\alpha^n - |G_1 - \alpha G_0||\beta|^n$ is never negative. Clearly,

$$(G_1 - \beta G_0)\alpha^n - |G_1 - \alpha G_0||\beta|^n \geq \min\{\sqrt{D}G_n, H_n\} \geq 0.$$

Then, obviously,

$$\begin{aligned} \left.\begin{array}{c} G_n \\ H_n \end{array}\right\} &\geq \frac{(G_1 - \beta G_0)\alpha^n - |G_1 - \alpha G_0||\beta|^n}{c_1} = \alpha^n \frac{(G_1 - \beta G_0) - |G_1 - \alpha G_0| \left(\frac{|\beta|}{\alpha}\right)^n}{c_1} \\ &\geq \alpha^n \frac{(G_1 - \beta G_0) - |G_1 - \alpha G_0| \left(\frac{|\beta|}{\alpha}\right)^{n_0}}{c_1} = \left\{ \begin{array}{l} \alpha^{n+\varepsilon_1(n_0)} \\ \alpha^{n+\delta_1(n_0)} \end{array} \right. . \end{aligned}$$

□

The principal idea of this paper is the following. Observe that if $n_0 \to \infty$, then the common limit of $\delta_1(n_0)$ and $\delta_2(n_0)$ is $\delta = \log_\alpha(G_1 - \beta G_0)$, while $\varepsilon = \lim_{n_0\to\infty} \varepsilon_1(n_0) = \lim_{n_0\to\infty} \varepsilon_2(n_0) = \varepsilon = \delta - \log_\alpha \sqrt{D}$. Therefore, if $\Delta = \log_\alpha \sqrt{D}$ is not an integer, then combining (9) and Lemma 3, one can conclude that the difference of x and y is not an integer which is clearly a contradiction.

We are going into details after making other observations. Consider first the case when $\Delta \in \mathbb{Z}$.

Lemma 4. *If $\Delta \in \mathbb{Z}$, then $\Delta = 0$, A is odd, $B = (1 - A^2)/4$ is negative. Moreover $\alpha = (A + 1)/2$, $\beta = (A - 1)/2$.*

Proof. In view of the value of Δ, we split the proof into parts. Clearly, we have to maintain the equation $\alpha - \beta = \alpha^\Delta$.

- Note that $\Delta < 0$ is not possible since $\alpha^{|\Delta|}\sqrt{D} = 1$ contradicts $\alpha > 1$ and $D \geq 1$.
- If $\Delta = 0$, then $\sqrt{D} = 1$ and subsequently $B = (1 - A^2)/4$ and $\alpha = (A + 1)/2$, $\beta = (A - 1)/2$. Trivially, A is odd and $B < 0$.
- Obviously, $\Delta = 1$ is impossible since $\alpha - \beta \neq \alpha$.
- Suppose $\Delta \geq 2$. If D is non-square, then $\sqrt{D} = \alpha^\Delta$ implies that the $\sqrt{D}$-free part R_Δ of $((A + \sqrt{D})/2)^\Delta = R_\Delta + \sqrt{D}S_\Delta$ is zero. But, clearly the rational number

$$R_\Delta = \frac{A^\Delta + \binom{\Delta}{2}A^{\Delta-2}D + \cdots}{2^\Delta} > 0.$$

Hence there exist a positive integer d such that $D = d^2$. Obviously, with $d = 1$ we arrive at a contradiction since $1 = ((A+1)/2)^\Delta$ leads to $A = 1$ and then $B = 0$. Thus, $d \geq 2$ and

$$d = \left(\frac{A+d}{2}\right)^\Delta,$$

i.e. d is a Δth-power. Assuming $d = t^\Delta$ $(t \geq 2)$, we get $t = (A + t^\Delta)/2$. Consequently, $A = 2t - t^\Delta > 0$ which contradicts $t \geq 2$ and $\Delta \geq 2$.

□

3. Solution of (9)

The details are only worked out for positive B, but a slight modification of the method together with separation of specific subcases is able to handle the case $B < 0$ as well. Observe that $B > 0$ entails $\sqrt{D} > 2$. Therefore (9) has no solution if $G_1 - \alpha G_0 = 0$ since $H_i = \sqrt{D} G_i > 2G_i$ for any $i \in \mathbb{N}$ (trivially, $G_0 = 0$ is not possible, because it would imply $G_1 = 0$). Further $\Delta \notin \mathbb{Z}$ also follows from $B > 0$. (See Lemma 4. In the investigation of the case $B < 0$ the relation $\Delta \in \mathbb{Z}$ is possible.) So assume $B > 0$. Then $A + B - 1 > 0$, and by (9), $G_{x+1} + BG_x$ and $H_{y+1} + BH_y + \eta$ are also positive. In order to get a good estimate for the difference of x and y, we apply Lemma 3 as follows. Let $\mu = \log_\alpha(1 + B/\alpha)$ and suppose that $x \geq x_0$, where x_0 is a suitable non-negative integer. Also assume $y \geq y_0$ for some integer $y_0 \geq 0$. Then

$$G_{x+1} + BG_x \leq \alpha^{x+1+\varepsilon_2(x_0)} + B\alpha^{x+\varepsilon_2(x_0)} = \alpha^{x+1+\varepsilon_2(x_0)+\mu}, \tag{13}$$

and

$$G_{x+1} + BG_x \geq \alpha^{x+1+\varepsilon_1(x_0)} + B\alpha^{x+\varepsilon_1(x_0)} = \alpha^{x+1+\varepsilon_1(x_0)+\mu}. \tag{14}$$

Similarly, we have

$$\alpha^{y+1+\delta_1(x_0)+\mu} \leq H_{y+1} + BH_y \leq \alpha^{y+1+\delta_2(x_0)+\mu}. \tag{15}$$

At this point we distinguish two cases.

Case 1. $\eta \geq 0$. Obviously, $\alpha^{y+1+\delta_1(x_0)+\mu} \leq H_{y+1} + BH_y + \eta$ and, if $u_0 \leq y + 1 + \delta_2(x_0) + \mu$ is a suitable positive number, then

$$H_{y+1} + BH_y + \eta \leq \alpha^{y+1+\delta_2(x_0)+\mu+\kappa(u_0)}, \tag{16}$$

where $\kappa(u_0) = \log_\alpha(1 + \eta/\alpha^{u_0})$. Combining (14) and (16), and then the lower estimate of (15) and (13), together with (9) one can obtain

$$y + \delta_1(y_0) - \varepsilon_2(x_0) \leq x \leq y + \delta_2(y_0) - \varepsilon_1(x_0) + \kappa(u_0).$$

In order to facilitate to reach our goal, assume that $x_0 = y_0$ and both are tending to infinity. Then $\delta_1(y_0) - \varepsilon_2(x_0) \to \Delta$, and $\delta_2(y_0) - \varepsilon_1(x_0) \to \Delta$. Letting $u_0 \to \infty$ and $y_0 \to \infty$, $\kappa(u_0) \to 0$ follows. Hence, if x_0 and u_0 are all large enough, then there exist small positive real numbers h_1 and h_2 such that

$$y + \Delta - h_1 \leq x \leq y + \Delta + h_2. \tag{17}$$

Supposing $\Delta \notin \mathbb{Z}$, we arrive at a contradiction. Indeed, since Δ is not an integer, $D \geq 2$ and $\alpha > 1$ implying $\Delta > 0$. Choosing suitable, sufficiently large x_0 and u_0, we have

$$K < \Delta - h_1 < \Delta + h_2 < K + 1,$$

where $K = \lfloor \Delta \rfloor < \Delta$. Thus there exists no integer between $y + \Delta - h_1$ and $y + \Delta + h_2$.

Case 2. $\eta < 0$. Then $G_{x+1} + BG_x + |\eta| = H_{x+1} + BH_x$ and we repeat the treatment of the previous case to conclude

$$y + \delta_1(y_0) - \varepsilon_2(x_0) - \kappa(v_0) \leq x \leq y + \delta_2(y_0) - \varepsilon_1(x_0), \tag{18}$$

where v_0 is a suitable positive number satisfying $v_0 \leq x+1+\varepsilon_2(x_0)+\mu$ and $\kappa(v_0)=\log_\alpha(1+|\eta|/\alpha^{v_0})$. The argument is the same: since $\Delta \notin \mathbb{Z}$, we can find sufficiently large $x_0 = y_0$ and v_0 such that (18) cannot hold.

Finally, to finish the proof, one needs to verify the small integral values $x \leq x_0 - 1$ or $y \leq y_0 - 1$.

4. Examples

In this section we completely solve equation (7) in the case of the four given sequences $\{G_n\}_{n=0}^{\infty}$ including the Fibonacci, the Pell, and the Jacobsthal numbers. We always chose the initial values $G_0 = 0$ and $G_1 = 1$. The value $x_0 = y_0$ is as sharp as possible. The following table contains the details of the calculations.

$G_n =$	$G_{n-1}+G_{n-2}$	$2G_{n-1}+G_{n-2}$	$G_{n-1}+2G_{n-2}$	$G_{n-1}+9900G_{n-2}$
$\alpha,\ \beta$	$\frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2}$	$1+\sqrt{2},\ 1-\sqrt{2}$	$2,\ -1$	$100,\ -98$
$H_0,\ H_1$	$H_0=2,\ H_1=1$	$H_0=2,\ H_1=2$	$H_0=2,\ H_1=1$	$H_0=2,\ H_1=1$
η	$\eta=0$	$\eta=1$	$\eta=0$	$\eta=0$
$x_0=y_0$	3	2	3	110
$\Delta-h_1$	1.439	1.112	1.222	1.00003
$\Delta+h_2$	1.905	1.302	1.948	1.29883

If $x \leq x_0 - 1$ or $y \leq y_0 - 1$, one must check by hand or by computer the remaining cases. Then we obtain the proof of Theorem 1.

Note that if $A = 1$, $G_0 = 0$ and $G_1 = 1$, then $G_2 = 1$ and hence, $H_0 = 2$. Thus, $G_0 + G_1 + G_2 = 2 = H_0$ is always true. Theorem 1 ascertains no solution in the aforesaid cases apart from the trivial one.

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