ON THE POSITIVE INTEGER POINTS OF CERTAIN TWO PARAMETER FAMILIES OF HYPERBOLAS

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Abstract. In this paper, in Sections 2-4, we consider the problem of finding all the positive integer points of a certain two parameter family of quadratic Diophantine equations. Geometrically, this family of quadratic Diophantine equations is a two parameter family of hyperbolas. We proceed by linking the positive integer points of this family of hyperbolas to the positive integer points of a two parameter family of Pell equations. Along the way, we prove a theorem that gives the fundamental solution of the two parameter family of Pell equations in question. In Section 5, we summarize our findings regarding the positive integer points of another two parameter family of quadratic Diophantine equations.

1. Introduction

In this paper, we follow convention and take a Pell (Diophantine) equation to be an equation

$$x^2 - dy^2 = e, \tag{1.1}$$

in which d is a positive integer that is not a perfect square, and $e \neq 0$ is an integer. We take an *integer* solution (or integer point) (x, y) of (1.1) to be a solution in which both x and y are integers. A positive solution is one where $x > 0$ and $y > 0$. As is customary in this topic, we refer to the solution (x, y), or to the solution $x + y\sqrt{d}$, interchangeably.

Pell equations have a long history. For readers of this journal who may be interested in a Fibonacci connection, we recommend the references [1]– [7]. For a general introduction to Pell equations, we recommend Nagell [8]. Nagell presents results not normally contained in a first course, and proceeds independently of the theory of continued fractions.

In [8], Section 59, Nagell explores the problem of obtaining all the integer points of certain conics. The present paper was inspired by his treatment of the hyperbola $5x^2 - 14xy + 7y^2 = -1$. Nagell obtains all the integer points of this hyperbola by linking these points to the integer points of a certain Pell equation, all of whose solutions are easily obtained. After studying Nagell's elegant method, we wondered if there exists an entire family of hyperbolas whose integer points can be similarly determined. The purpose of this paper is to present two such families of hyperbolas. Specifically, we begin by finding all the positive integer points of any hyperbola of the form

$$mx^2 - \left(mc^2 - 1\right) y^2 = 1, \tag{1.2}$$

where $m \geq 2$ and $c \geq 1$ are integers. Following Nagell, we link the integer points of (1.2) to the integer points of the Pell equation

$$u^2 - m\left(mc^2 - 1\right) v^2 = 1. \tag{1.3}$$

It is easy to see that m and $mc^2 - 1$ cannot be perfect squares simultaneously. Therefore, with the constraints on m and c, the integer points of (1.2) are not merely the intersection points of two pairs of lines. Also, $m\left(mc^2 - 1\right)$ can never be a perfect square. On the contrary, suppose that, for some positive integer f, $f^2 = m\left(mc^2 - 1\right) = m^2c^2 - m$. Then the maximum possible value of f is $mc-1$. But $(mc-1)^2 - \left(m^2c^2 - m\right) = 1 + m(1-2c) < 0$, which shows that such an f does not exist.

For $m = 1$ and $c \geq 2$, (1.2) becomes

$$x^2 - \left(c^2 - 1\right) y^2 = 1, \tag{1.4}$$

with *fundamental* (smallest) solution $(c, 1)$. All the positive integer points of (1.4) are therefore $\left(c + \sqrt{c^2-1}\right)^n$, $n \geq 1$. This follows from a central theorem in the theory of Pell equations. See, for instance, Theorem 104 in Nagell [8], which we require for the proof of our main theorem in Section 4.

In Section 2, we show that the fundamental solution of (1.3) is $\left(2mc^2 - 1, 2c\right)$. In Section 3, we link the integer points of (1.2) and the integer points of (1.3) with the use of two linear transformations from the Cartesian plane to the Cartesian plane. In Section 4, we prove our main theorem. In this theorem, we express the positive integer points of (1.2) in terms of a second order linear recursive sequence. Finally, in Section 5, we consider another family of hyperbolas whose positive integer points we determine with the same methods that we use in Sections 2-4.

2. A Theorem on the Fundamental Solution of (1.3)

In Sections 2-4 of this paper, we take the constraints on the integers m and c to be those specified for (1.2), namely $m \geq 2$ and $c \geq 1$. We devote this section to the proof of a theorem that gives the fundamental solution of (1.3). This theorem is of importance in its own right, because all the positive integer solutions of (1.3) arise as positive integer powers of this fundamental solution.

Theorem 2.1. *For $m \geq 2$ and $c \geq 1$, the fundamental solution of (1.3) is*

$$\left(2mc^2 - 1, 2c\right).$$

Proof. By substitution, $\left(2mc^2 - 1, 2c\right)$ is a solution of (1.3). Furthermore, with the constraints on m and c, this solution is positive. If this is not the fundamental solution, then there exist positive integers $0 < k < 2c$ and u_0 such that $(u_0, 2c-k)$ is a solution of (1.3). Substituting into (1.3), we see that

$$\begin{aligned} u_0^2 &= m\left(mc^2 - 1\right)(2c-k)^2 + 1 \\ &= m^2c^2(2c-k)^2 - m(2c-k)^2 + 1 \\ &< m^2c^2(2c-k)^2, \end{aligned} \tag{2.1}$$

which implies that the maximum possible value of u_0 is $mc(2c-k) - 1$. However, from the first line of the array (2.1), we have

$$(mc(2c-k) - 1)^2 - m\left(mc^2 - 1\right)(2c-k)^2 - 1 = -km(2c-k) < 0.$$

This shows that k and u_0 do not exist, and so $\left(2mc^2 - 1, 2c\right)$ is indeed the fundamental solution of (1.3). $\square$

We remark that (1.4) represents a one parameter family of Pell equations of the form (1.1), with $e = 1$, whose fundamental solution is known. In Theorem 2.1, we present a two parameter family of such Pell equations.

3. Two Linear Transformations

We define two linear transformations from the Cartesian plane to the Cartesian plane as follows:

$$(u,v) = S(x,y) = \left(cmx - \left(mc^2-1\right)y, -x+cy\right),$$
$$(x,y) = T(u,v) = \left(cu + \left(mc^2-1\right)v, u+mcv\right).$$

The linear transformation S can be represented by a 2×2 matrix, as can the linear transformation T. Furthermore, as the reader can easily verify, each of these matrices has determinant 1, and these matrices are inverses of one another.

Suppose that (x_0, y_0) is any integer point of (1.2). Then, upon substituting the point $(u_0, v_0) = S(x_0, y_0)$ into the left side of (1.3), we obtain $mx_0^2 - \left(mc^2-1\right)y_0^2$. This shows that (u_0, v_0) is an integer point of (1.3). Likewise, any integer point of (1.3) is mapped by T to an integer point of (1.2). These observations, together with the facts recorded in the previous paragraph, demonstrate that each of the linear transformations S and T is a one-to-one correspondence between the integer points of (1.2), and the integer points of (1.3).

We now focus on the *positive* integer points of (1.2) and (1.3). Consider the graph of the hyperbola (1.2) in the first quadrant of the Cartesian plane, with origin $O(0,0)$. The point $A(c,1)$ is a positive integer point of this graph. Let $B(x_0, y_0)$ be another positive integer point of this graph, where $x_0 > c$ and $y_0 > 1$. Then

$$y_0^2 = \frac{mx_0^2-1}{mc^2-1} > \frac{x_0^2}{c^2},$$

so that

$$-x_0 + cy_0 > 0. \tag{3.1}$$

Next,

$$mx_0^2 - \left(mc^2-1\right)y_0^2 > 0 \Rightarrow mx_0 - \left(mc^2-1\right)y_0\frac{y_0}{x_0} > 0,$$

and so from (3.1) we have

$$cmx_0 - \left(mc^2-1\right)y_0 > 0. \tag{3.2}$$

Let $\mathcal{P}'$ designate the set of positive integer points of (1.2) that are to the right of $A(c,1)$. Let $\mathcal{P}''$ designate the set of positive integer points of (1.3). Then the inequalities (3.1) and (3.2) show that S maps points in $\mathcal{P}'$ to points in $\mathcal{P}''$. Clearly, T maps points in $\mathcal{P}''$ to points in $\mathcal{P}'$. Given the nature of the linear transformations S and T, we record the observations that we have just made in the lemma that follows.

Lemma 3.1. *Let the sets $\mathcal{P}'$ and $\mathcal{P}''$ be as defined above. Then $S(\mathcal{P}') = \mathcal{P}''$, and $T(\mathcal{P}'') = \mathcal{P}'$. Also, each of the linear transformations S and T is a one-to-one correspondence between $\mathcal{P}'$ and $\mathcal{P}''$. Furthermore, the positive integer points of (1.2) consist of the point $(c,1)$, together with the points $T(\mathcal{P}'')$.*

In the next section, we conveniently express the positive integer points of (1.2) in terms of a second order linear recursive sequence.

4. Main Theorem on the Positive Integers Points of (1.2)

In this section, we present our main theorem, which gives all the positive integer points of (1.2). To this end, for integers $m \geq 2$ and $c \geq 1$, we define the integer sequence $\{R_n\}$, for all n, by

$$R_n = R_n(m,c) = \left(4mc^2-2\right)R_{n-1} - R_{n-2}, R_0 = 0, R_1 = 1. \tag{4.1}$$

It is a simple matter to show that $\{R_n\}$ is strictly increasing for $n \geq 0$. All the positive integer points of (1.2) are given in our main theorem, which follows.

Theorem 4.1. *For integers $m \geq 2$ and $c \geq 1$, all the positive integer points of (1.2) are given by*

$$(c(R_{n+1} - R_n), R_{n+1} + R_n), n \geq 0. \tag{4.2}$$

Proof. At the end of Section 3, we saw that the positive integer points of (1.2) consist of the point $(c, 1)$, together with $T(\mathcal{P}'')$, where $\mathcal{P}''$ is the set of positive integer points of (1.3). For $n = 0$, (4.2) becomes $(c, 1)$, which is the smallest positive integer point of (1.2).

By Theorem 2.1, the fundamental solution of (1.3) is $\left(2mc^2 - 1, 2c\right)$. Therefore, by that well-known theorem in the theory of Pell equations to which we alluded to at the end of Section 1, $\mathcal{P}''$ consists of the points

$$u_n + v_n\sqrt{m(mc^2 - 1)} = \left(2mc^2 - 1 + 2c\sqrt{m(mc^2 - 1)}\right)^n, n \geq 1. \tag{4.3}$$

To prove Theorem 4.1, we are required to prove that

$$T(u_n, v_n) = (c(R_{n+1} - R_n), R_{n+1} + R_n), n \geq 1. \tag{4.4}$$

We have

$$\begin{aligned} T(u_1, v_1) &= T\left(2mc^2 - 1, 2c\right) \\ &= \left(4mc^3 - 3c, 4mc^2 - 1\right) \\ &= (c(R_2 - R_1), R_2 + R_1), \end{aligned} \tag{4.5}$$

and so (4.4) is true for $n = 1$.

Next, suppose that for some integer $k \geq 1$,

$$T(u_k, v_k) = (c(R_{k+1} - R_k), R_{k+1} + R_k). \tag{4.6}$$

Now, setting $d = d(m, c) = m\left(mc^2 - 1\right)$, we have

$$\begin{aligned} u_{k+1} + v_{k+1}\sqrt{d} &= \left(2mc^2 - 1 + 2c\sqrt{d}\right)^{k+1} \\ &= \left(2mc^2 - 1 + 2c\sqrt{d}\right)^k \left(2mc^2 - 1 + 2c\sqrt{d}\right) \\ &= \left(u_k + v_k\sqrt{d}\right)\left(2mc^2 - 1 + 2c\sqrt{d}\right). \end{aligned} \tag{4.7}$$

Upon expanding the last expression in the array (4.7), replacing d by $m\left(mc^2 - 1\right)$, and reverting to Cartesian coordinates, we see that

$$\begin{aligned} (u_{k+1}, v_{k+1}) &= \left(\left(2mc^2 - 1\right)u_k + 2mc\left(mc^2 - 1\right)v_k, 2cu_k + \left(2mc^2 - 1\right)v_k\right) \\ &= W(u_k, v_k). \end{aligned}$$

Here, like S and T, W is a linear transformation that can be represented by a 2×2 matrix. Accordingly, by matrix multiplication, we have

$$\begin{aligned} T(u_{k+1}, v_{k+1}) &= TW(u_k, v_k) \\ &= TWT^{-1}(c(R_{k+1} - R_k), R_{k+1} + R_k) \qquad \text{by (4.6)} \\ &= TWS(c(R_{k+1} - R_k), R_{k+1} + R_k) \qquad \text{since } T^{-1} = S. \end{aligned}$$

Expressed in matrix form, the product TWS is

$$TWS = \begin{pmatrix} 2mc^2 - 1 & 2mc^3 - 2c \\ 2mc & 2mc^2 - 1 \end{pmatrix},$$

from which it follows that

$$T\left(u_{k+1}, v_{k+1}\right) = \left(c\left(4mc^2 R_{k+1} - 3R_{k+1} - R_k\right), 4mc^2 R_{k+1} - R_{k+1} - R_k\right). \tag{4.8}$$

Finally, using the recurrence (4.1) to eliminate R_k in (4.8), we arrive at

$$T\left(u_{k+1}, v_{k+1}\right) = \left(c\left(R_{k+2} - R_{k+1}\right), R_{k+2} + R_{k+1}\right).$$

Theorem 4.1 is thus proved by induction. □

5. The Positive Integer Points of $mx^2 - \left(mc^2+1\right)y^2 = -1$

In this section, we address the problem of finding the positive integer points of the family of hyperbolas

$$mx^2 - \left(mc^2+1\right)y^2 = -1, \tag{5.1}$$

in which $m \geq 1$ and $c \geq 1$ are integers. To accomplish this task, we link the integer points of (5.1) to the integer points of

$$u^2 - m\left(mc^2+1\right)v^2 = 1. \tag{5.2}$$

Since the procedures that we employ are analogous to those set forth in Sections 2-4, we merely give an outline of the method.

Considerations similar to those in the paragraph following (1.3) show that (5.1) represents a family of hyperbolas, and in (5.2), $m\left(mc^2+1\right)$ is never a perfect square. Next, we state a theorem that gives the fundamental solution of (5.2). This theorem is the counterpart to Theorem 2.1, and is proved similarly.

Theorem 5.1. *For $m \geq 1$ and $c \geq 1$, the fundamental solution of (5.2) is*

$$\left(2mc^2+1, 2c\right).$$

From this point on, in order to highlight the similarities with our analysis in Sections 2-4, we use the same notation that we use in those sections.

Following our approach in Section 3, we link the integer points of (5.1) to the integer points of (5.2). We do this with the use of two linear transformations, S and T. These two linear transformations, from the Cartesian plane to the Cartesian plane, are defined as follows:

$$(u,v) = S(x,y) = \left(-cmx + \left(mc^2+1\right)y, x - cy\right),$$
$$(x,y) = T(u,v) = \left(cu + \left(mc^2+1\right)v, u + mcv\right).$$

Each of these linear transformations can be represented by a 2×2 matrix of determinant -1, and are inverses of one another. Furthermore, each of S and T is a one-to-one correspondence between the integer points of (5.1), and the integer points of (5.2).

Next, we focus on the *positive* integer points of (5.1) and (5.2). Let $\mathcal{P}'$ denote the set of positive integer points of (5.1) that are to the right of $(c,1)$, which is a positive integer point of (5.1). Let $\mathcal{P}''$ denote the set of positive integer points of (5.2). After establishing inequalities that are analogous to (3.1) and (3.2), we obtain the lemma that follows.

Lemma 5.2. *Let the sets $\mathcal{P}'$ and $\mathcal{P}''$ be as defined in the previous paragraph. Then $S\left(\mathcal{P}'\right) = \mathcal{P}''$, and $T\left(\mathcal{P}''\right) = \mathcal{P}'$. Also, each of the linear transformations S and T is a one-to-one correspondence between $\mathcal{P}'$ and $\mathcal{P}''$. Furthermore, the positive integer points of (5.1) consist of the point $(c,1)$, together with the points $T\left(\mathcal{P}''\right)$.*

For integers $m \geq 1$ and $c \geq 1$, define the integer sequence $\{R_n\}$, for all n, by

$$R_n = R_n(m, c) = \left(4mc^2 + 2\right) R_{n-1} - R_{n-2}, R_0 = 0, R_1 = 1. \tag{5.3}$$

We now state our main theorem concerning the positive integer points of (5.1). This theorem is analogous to Theorem 4.1, and its proof follows similar lines as the proof of Theorem 4.1.

Theorem 5.3. *For integers $m \geq 1$ and $c \geq 1$, all the positive integer points of (5.1) are given by*

$$\left(c\left(R_{n+1} + R_n\right), R_{n+1} - R_n\right), n \geq 0. \tag{5.4}$$

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