CLOSED FORMS FOR FINITE SUMS IN WHICH THE DENOMINATOR OF THE SUMMAND IS A PRODUCT OF TRIGONOMETRIC FUNCTIONS

R. S. MELHAM

Abstract. In this paper, we present closed forms for certain finite sums. In each case, the denominator of the summand is a product of sine or cosine functions. Furthermore, in each case, the arguments of the trigonometric functions in the denominator of the summand increase in arithmetic progression.

1. Introduction

In a sequence of recent papers, we present closed forms for finite sums where, in each case, the denominator of the summand is a product of terms from generalized Fibonacci sequences. Such finite sums can therefore be specialized to the Fibonacci sequence. For instance, as a by-product of this work, we have

$$\sum_{i=1}^{n-1} \frac{1}{F_i F_{i+1} F_{i+2} F_{i+3}} = \frac{7}{4} - \frac{1}{2}\left(\frac{F_{n-1}}{F_n} + \frac{3F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}}\right), \tag{1.1}$$

which is given in [1]. Reference [1] contains references to some of the papers mentioned above.

A search of the literature revealed nothing analogous to (1.1) for the sine or cosine functions. This prompted us to begin the research that led to the present paper. In each of the sums that we consider in this paper, the denominator of the summand is a product of sine or cosine functions. Furthermore, the arguments of the functions in the denominator of each summand increase in arithmetic progression. Each finite sum that we present is categorized according to the number of distinct terms in the denominator of its summand. Accordingly, we use the notation $S_k^j(n,d)$ to denote a finite sum with k distinct terms in the denominator of its summand. For instance, in Section 3, each of $S_2^0 = S_2^0(n,d)$ and $S_2^1 = S_2^1(n,d)$ has two distinct terms in the denominator of its summand.

At the outset, we set the constraints on the parameters n and d that occur in the previous paragraph, and henceforth we do not restate these constraints. In any sum $S_k^j = S_k^j(n,d)$, $n \geq 2$ and $d \geq 1$, are assumed to be integers.

We limit the scope of this paper so that each finite sum that we consider has at most five distinct factors in the denominator of its summand, guaranteeing that our results do not become too unwieldy. In order to keep the presentation to a reasonable length, we give the closed forms for only a selection of the sums that we define.

In Section 2, we define two finite sums, Φ and Ψ, in terms of which we express all our results. We present our main results in Sections 3, 4, 5, and 6, and demonstrate the method of proof that can be used to prove each of these results in Section 5.

2. The Finite Sums Φ and Ψ

There are two finite sums that we use to express the closed forms for all the sums that occur in this paper. For integers $0 \le l_1 < l_2$, these finite sums are

$$\Phi(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{1}{\sin(i+2)\sin(i+n)},$$
$$\Psi(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{1}{\cos(i+2)\cos(i+n)}.$$

In Lemma 2.1, we give identities for Φ and Ψ that are required for the proofs of all the theorems in this paper. We prove only the identity for Φ, since the proof of the analogous identity for Ψ proceeds along similar lines.

In order to prove the identity for Φ, we require certain identities from elementary trigonometry. Two of these identities are

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}, \tag{2.1}$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{2.2}$$

We require also identities (2.3) and (2.4), each of which can be proved by the application of (2.1) to each product on the left, followed by the use of (2.2).

$$\sin(\alpha-\beta)\sin(n+\alpha+1) + \sin 1 \sin(n+\beta) = \sin(\alpha-\beta+1)\sin(n+\alpha), \tag{2.3}$$
$$\sin(n-1)\sin(i+n) - \sin(n-2)\sin(i+n+1) = \sin 1 \sin(i+2). \tag{2.4}$$

Finally, for integers $0 \le l_1 < l_2$, we require the sum

$$\sum_{i=l_1}^{l_2-1} \frac{1}{\sin(i+n)\sin(i+n+1)} = \frac{\sin(l_2-l_1)}{\sin 1 \sin(n+l_1)\sin(n+l_2)}, \tag{2.5}$$

which can be proved by straightforward induction on l_2. Here, one makes use of (2.3) in the inductive step. We leave the details to the interested reader.

Lemma 2.1, which follows, gives the identities for Φ and Ψ to which we refer in the second paragraph of this section.

Lemma 2.1. *With Φ and Ψ as defined at the beginning of this section,*

$$\sin(n-1)\Phi(n+1, l_1, l_2) - \sin(n-2)\Phi(n, l_1, l_2) = \frac{\sin(l_2-l_1)}{\sin(n+l_1)\sin(n+l_2)}, \tag{2.6}$$
$$\sin(n-1)\Psi(n+1, l_1, l_2) - \sin(n-2)\Psi(n, l_1, l_2) = \frac{\sin(l_2-l_1)}{\cos(n+l_1)\cos(n+l_2)}. \tag{2.7}$$

Proof. Set $D(n,l_1,l_2) = \sin(n-1)\Phi(n+1,l_1,l_2) - \sin(n-2)\Phi(n,l_1,l_2)$. Then

$$\begin{aligned}
D(n,l_1,l_2) &= \sum_{i=l_1}^{l_2-1} \frac{1}{\sin(i+2)} \left(\frac{\sin(n-1)\sin(i+n) - \sin(n-2)\sin(i+n+1)}{\sin(i+n)\sin(i+n+1)} \right) \\
&= \sum_{i=l_1}^{l_2-1} \frac{\sin 1}{\sin(i+n)\sin(i+n+1)}, \text{ by (2.4)} \\
&= \frac{\sin(l_2-l_1)}{\sin(n+l_1)\sin(n+l_2)}, \text{ by (2.5).}
\end{aligned}$$

This completes the proof of (2.6). The proof of (2.7) is similar. □

3. The Summand has Two Distinct Factors in the Denominator

The reader should recall the constraints upon the parameters n and d that we state in the introduction, and that apply throughout this paper. In this section, we give closed forms for the finite sums

$$S_2^0(n,d) = \sum_{i=1}^{n-1} \frac{1}{\sin i \sin(i+d)},$$

$$S_2^1(n,d) = \sum_{i=1}^{n-1} \frac{1}{\cos i \cos(i+d)}.$$

We present the closed forms for S_2^0 and S_2^1 in the following theorem.

Theorem 3.1. *With S_2^0 and S_2^1 as defined above,*

$$\sin d \left(S_2^0(n,d) - S_2^0(2,d) \right) = \sin(n-2)\Phi(n,0,d), \tag{3.1}$$

$$\sin d \left(S_2^1(n,d) - S_2^1(2,d) \right) = \sin(n-2)\Psi(n,0,d). \tag{3.2}$$

For $d = 1$, formulas (3.1) and (3.2) yield, respectively,

$$\sum_{i=1}^{n-1} \frac{1}{\sin i \sin(i+1)} = \csc 1(\cot 1 - \cot n),$$

$$\sum_{i=1}^{n-1} \frac{1}{\cos i \cos(i+1)} = \csc 1 \tan n - \sec 1.$$

We remark that S_2^0 and S_2^1 are the only finite sums, with two distinct terms in the denominator of the summand, whose closed forms we have been able to find.

4. The Summand has Three Distinct Factors in the Denominator

In this section, we present only the closed forms for S_3^0 and S_3^1, defined below. Define

$$S_3^0(n,d) = \sum_{i=1}^{n-1} \frac{\cos(i+d)}{\sin i \sin(i+d)\sin(i+2d)},$$

$$S_3^1(n,d) = \sum_{i=1}^{n-1} \frac{\sin(i+d)}{\cos i \cos(i+d)\cos(i+2d)}.$$

There are further similar finite sums, with three distinct terms in the denominator of the summand, whose closed forms we have been able to find. In the summand of S_3^0, replace the numerator by $\cos i$ and also by $\cos(i+2d)$. Likewise, in the summand of S_3^1, replace the numerator by $\sin i$ and also by $\sin(i+2d)$. Then we have found the closed forms for the corresponding four finite sums.

We present the closed forms for S_3^0 and S_3^1 in the theorem that follows.

Theorem 4.1. *With S_3^0 and S_3^1 as defined above,*

$$2\sin^2 d\left(S_3^0(n,d)-S_3^0(2,d)\right)=\sin(n-2)\left(\Phi(n,0,d)-\Phi(n,d,2d)\right), \tag{4.1}$$

$$-2\sin^2 d\left(S_3^1(n,d)-S_3^1(2,d)\right)=\sin(n-2)\left(\Psi(n,0,d)-\Psi(n,d,2d)\right). \tag{4.2}$$

For $d=1$, formulas (4.1) and (4.2) yield, respectively,

$$\sum_{i=1}^{n-1}\frac{\cos(i+1)}{\sin i\sin(i+1)\sin(i+2)}=\frac{(2\cos 2+1)\left(\cos 3-\cos(2n+1)\right)}{4\sin 1\sin 2\sin 3\sin n\sin(n+1)},$$

$$\sum_{i=1}^{n-1}\frac{\sin(i+1)}{\cos i\cos(i+1)\cos(i+2)}=\frac{\cos 3-\cos(2n+1)}{\sin 4\cos n\cos(n+1)}.$$

As we state at the end of the introduction, in the next section we demonstrate a method of proof that can be used to prove each result in this paper.

5. The Summand has Four Distinct Factors in the Denominator

We have found that, as the number of factors in the denominator increases, the number of finite sums for which we can write down closed forms also increases. Accordingly, we have discovered closed forms for the eight finite sums that we define below. Define

$$S_4^0(n,d)=\sum_{i=1}^{n-1}\frac{1}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)},$$

$$S_4^1(n,d)=\sum_{i=1}^{n-1}\frac{\cos(i+d)\cos(i+2d)}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)},$$

$$S_4^2(n,d)=\sum_{i=1}^{n-1}\frac{\sin(2i+3d)}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)},$$

$$S_4^3(n,d)=\sum_{i=1}^{n-1}\frac{\cos(2i+3d)}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)}.$$

Define also

$$S_4^4(n,d) = \sum_{i=1}^{n-1} \frac{1}{\cos i \cos(i+d)\cos(i+2d)\cos(i+3d)},$$

$$S_4^5(n,d) = \sum_{i=1}^{n-1} \frac{\sin(i+d)\sin(i+2d)}{\cos i \cos(i+d)\cos(i+2d)\cos(i+3d)},$$

$$S_4^6(n,d) = \sum_{i=1}^{n-1} \frac{\sin(2i+3d)}{\cos i \cos(i+d)\cos(i+2d)\cos(i+3d)},$$

$$S_4^7(n,d) = \sum_{i=1}^{n-1} \frac{\cos(2i+3d)}{\cos i \cos(i+d)\cos(i+2d)\cos(i+3d)}.$$

In order to keep the presentation to a reasonable length, we give the closed forms for only a selection of the sums defined above.

We present the closed forms for S_4^0 and S_4^4 together, since the results are similar. Furthermore, we demonstrate our method of proof by proving the first of these. For brevity, put $c = c(d) = \sin d \sin(2d)\sin(3d)$. We then have the following theorem.

Theorem 5.1. *With S_4^0 and S_4^4 as defined above,*

$$c\left(S_4^0(n,d) - S_4^0(2,d)\right) = \sin(n-2)\left(\Phi(n,0,d) - 2\cos(2d)\Phi(n,d,2d) + \Phi(n,2d,3d)\right), \tag{5.1}$$

$$c\left(S_4^4(n,d) - S_4^4(2,d)\right) = \sin(n-2)\left(\Psi(n,0,d) - 2\cos(2d)\Psi(n,d,2d) + \Psi(n,2d,3d)\right). \tag{5.2}$$

Proof. What follows is a proof of (5.1). Denote the quantities on the left and right sides of (5.1) by $L(n,d)$ and $R(n,d)$, respectively. It is immediate that

$$L(n+1,d) - L(n,d) = \frac{\sin d \sin(2d)\sin(3d)}{\sin n \sin(n+d)\sin(n+2d)\sin(n+3d)}. \tag{5.3}$$

With the use of (2.6), we see that

$$\begin{aligned} R(n+1,d) - R(n,d) = {} & \frac{\sin d}{\sin n \sin(n+d)} - \frac{2\sin d\cos(2d)}{\sin(n+d)\sin(n+2d)} \\ & + \frac{\sin d}{\sin(n+2d)\sin(n+3d)}. \end{aligned} \tag{5.4}$$

Our aim is to prove that

$$L(n+1,d) - L(n,d) = R(n+1,d) - R(n,d). \tag{5.5}$$

To this end, we express the right side of (5.4) with the same denominator as the right side of (5.3). This shows that to prove (5.5) we are required to prove that

$$\sin(2d)\sin(3d) = \sin(n+3d)(\sin(n+2d) - 2\sin n\cos(2d)) + \sin n\sin(n+d). \tag{5.6}$$

Expanding $\sin(n+2d)$, we see that to prove (5.6) it is enough to prove that

$$\sin(2d)\sin(3d) = \sin(n+3d)\sin(2d-n) + \sin n\sin(n+d). \tag{5.7}$$

To prove (5.7), we apply (2.1) to each product on the right side of (5.7), and apply (2.2) to the result. This shows that (5.5) is true. Furthermore, $L(2,d) = R(2,d) = 0$, and this completes the proof of (5.1). □

Since the closed forms for S_4^2 and S_4^6 are relatively succinct, we present these closed forms in the following theorem.

Theorem 5.2. *With S_4^2 and S_4^6 as defined above,*

$$2\sin^2 d\cos d\left(S_4^2(n,d)-S_4^2(2,d)\right)=\sin(n-2)\left(\Phi(n,0,d)-\Phi(n,2d,3d)\right), \tag{5.8}$$

$$-\sin d\sin(2d)\left(S_4^6(n,d)-S_4^6(2,d)\right)=\sin(n-2)\left(\Psi(n,0,d)-\Psi(n,2d,3d)\right). \tag{5.9}$$

Let $d=1$. Then (5.8) and (5.9) yield, respectively,

$$\sum_{i=1}^{n-1}\frac{\sin(2i+3)}{\sin i\sin(i+1)\sin(i+2)\sin(i+3)}=\frac{\cos 4-\cos(2n+2)}{\sin^3 1(4\cos 2+2)\sin n\sin(n+2)},$$

$$\sum_{i=1}^{n-1}\frac{\sin(2i+3)}{\cos i\cos(i+1)\cos(i+2)\cos(i+3)}=\frac{\cos 4-\cos(2n+2)}{\sin 1\cos^2 1(4\cos 2-2)\cos n\cos(n+2)}.$$

We remark that there are variations of the sums defined in this section whose closed forms we have managed to find. For instance, consider the four sums S_4^2, S_4^3, S_4^6, and S_4^7. In each of these finite sums, replace the argument $2i+3d$ in the numerator of the summand by $2i+kd$, where k is an integer. Then we have managed to find the corresponding closed forms for certain small values of k. We have found that the case $k=3$ yields closed forms that are the least complicated.

6. The Summand has Five Distinct Factors in the Denominator

Define

$$S_5^0(n,d)=\sum_{i=1}^{n-1}\frac{\cos(i+2d)}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)\sin(i+4d)},$$

$$S_5^1(n,d)=\sum_{i=1}^{n-1}\frac{\cos(i+d)\cos(i+2d)\cos(i+3d)}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)\sin(i+4d)},$$

$$S_5^2(n,d)=\sum_{i=1}^{n-1}\frac{\sin(3(i+2d))}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)\sin(i+4d)},$$

$$S_5^3(n,d)=\sum_{i=1}^{n-1}\frac{\cos(3(i+2d))}{\sin i\sin(i+d)\sin(i+2d)\sin(i+3d)\sin(i+4d)}.$$

Also define

$$S_5^4(n,d)=\sum_{i=1}^{n-1}\frac{\sin(i+2d)}{\cos i\cos(i+d)\cos(i+2d)\cos(i+3d)\cos(i+4d)},$$

$$S_5^5(n,d)=\sum_{i=1}^{n-1}\frac{\sin(i+d)\sin(i+2d)\sin(i+3d)}{\cos i\cos(i+d)\cos(i+2d)\cos(i+3d)\cos(i+4d)},$$

$$S_5^6(n,d)=\sum_{i=1}^{n-1}\frac{\cos(3(i+2d))}{\cos i\cos(i+d)\cos(i+2d)\cos(i+3d)\cos(i+4d)},$$

$$S_5^7(n,d)=\sum_{i=1}^{n-1}\frac{\sin(3(i+2d))}{\cos i\cos(i+d)\cos(i+2d)\cos(i+3d)\cos(i+4d)}.$$

We have discovered closed forms for each of the eight sums defined above. There are variations of these sums that we have also investigated. For instance, in $S_5^0(n,d)$ replace the numerator of the summand by $\cos^3(i+2d)$, and in $S_5^4(n,d)$ replace the numerator of the summand by $\sin^3(i+2d)$. Then we have discovered closed forms for the corresponding finite sums. Furthermore, comments similar to those made in the paragraph at the end of Section 5 apply to each of the eight sums defined at the beginning of this section.

On the negative side, in each of S_5^2, S_5^3, S_5^6, and S_5^7, we replaced the numerator of the summand by the cube of that numerator. However, in each case, we were not able to find the closed forms for the corresponding finite sums.

Below, we give only the closed forms for S_5^2 and S_5^4. Accordingly, set $e = e(d) = 2\cos(2d)-1$, and $f = f(d) = 2\cos(2d)+1$. We then have the following two theorems.

Theorem 6.1. *With S_5^2 as defined above,*

$$\begin{aligned}&\sin^2 d\sin(4d)\left(S_5^2(n,d) - S_5^2(2,d)\right)\\&= \sin(n-2)\left(e\Phi(n,0,d) - \Phi(n,d,2d) \;\; - \Phi(n,2d,3d) + e\Phi(n,3d,4d)\right).\end{aligned}$$

Theorem 6.2. *With S_5^4 as defined above,*

$$\begin{aligned}&2f\sin^2 d\sin^2(2d)\left(S_5^4(n,d) - S_5^4(2,d)\right)\\&= \sin(n-2)\left(-\Psi(n,0,d) + f\Psi(n,d,2d) \;\; - f\Psi(n,2d,3d) + \Psi(n,3d,4d)\right).\end{aligned}$$

The specializations of S_5^2 and S_5^4 for small values of d, including $d=1$, are rather unwieldy. We therefore write down only the closed form of S_5^4 for $d=1$. After much simplification, the sum in question becomes

$$\begin{aligned}&\sum_{i=1}^{n-1} \frac{2\sin 8\cos 3\cos 5\sin(i+2)}{\cos i\cos(i+1)\cos(i+2)\cos(i+3)\cos(i+4)} - 16\sin 1\sin 3\\&= (\cos 7 - \cos(2n+3))\,\frac{\cos 1 + \cos 3 + \cos 7 + \cos(2n+3)}{\cos n\cos(n+1)\cos(n+2)\cos(n+3)}.\end{aligned}$$

7. Concluding Comments

In order to present this paper succinctly, we have chosen to give all our results in abbreviated form. To write down our results in their most general form is easy. Let θ be any real number that is not a rational multiple of π. This condition on θ excludes the possibility of vanishing denominators. Then this entire paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ.

For example, redefine Φ as

$$\Phi(n,l_1,l_2,\theta) = \sum_{i=l_1}^{l_2-1} \frac{1}{\sin((2+i)\theta)\sin((n+i)\theta)},$$

and redefine S_3^0 as

$$S_3^0(n,d,\theta) = \sum_{i=1}^{n-1} \frac{\cos((i+d)\theta)}{\sin(i\theta)\sin((i+d)\theta)\sin((i+2d)\theta)}.$$

Then (4.1) generalizes to

$$2\sin^2(d\theta)\left(S_3^0(n,d,\theta) - S_3^0(2,d,\theta)\right) = \sin((n-2)\theta)\left(\Phi(n,0,d,\theta) - \Phi(n,d,2d,\theta)\right).$$

Results analogous to those presented here become more unwieldy as the number of factors in the denominator of the summand increases. It is for this reason that we have limited the scope of this paper. We trust that our presentation has given the reader an indication of the kinds of results that are possible.

Acknowledgement

The author would like to thank the referee for a careful reading of the original submission. The comments and suggestions offered by this referee have greatly enhanced the presentation.

References

[1] R. S. Melham, *On certain families of finite reciprocal sums that involve generalized Fibonacci numbers*, The Fibonacci Quarterly, **53.4** (2015), 323–334.

MSC2010: 11B99

Department of Mathematical Sciences, University of Technology Sydney, PO Box 123, Broadway, NSW 2007 Australia

E-mail address: ray.melham@uts.edu.au