ON FRANKLIN AND COMPLETE MAGIC SQUARE MATRICES

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Abstract. We show that any complete (or most-perfect) magic square of order $8k$ ($k = 1, 2, \ldots$) can be transformed to a pandiagonal Franklin magic square by means of a special permutation matrix. However, not all pandiagonal Franklin magic squares can be obtained by this transformation for $k \geq 2$ and not all Franklin squares are pandiagonal. Since the number of complete magic squares of order $4k$ ($k \geq 2$) is known, our transformation gives a lower bound on the number of all order-$8k$ Franklin squares and pandiagonal ones.

General parameterizations, constructed here for Franklin and complete square matrices, lead to the fact that they are rank 3. Using this information, their spectra are studied. Then, odd matrix powers of pandiagonal Franklin squares are shown to be pandiagonal Franklin squares.

1. Introduction

Franklin squares originate in the work of Benjamin Franklin who published an order-8 and an order-16 Franklin square in 1769 as noted in the detailed history of Franklin's mathematical activities by Pasles [19]. Franklin squares are similar to magic squares in that all their rows and columns sum to the same index number m but their two main diagonals need not. Instead, four families of bent diagonals sum to m. Bent diagonals are shown here for an order-6 square:

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Also, half-rows and half-columns of a Franklin square sum to $m/2$, and all 2×2 subsquares sum to an index number related to m (the quartal property). Some Franklin squares also are pandiagonal, wherein the elements on all diagonals in both direction (with wraparound) sum to m. Franklin squares that are not necessarily pandiagonal are called ordinary here. In some of the squares Pasles considers in [18] the just stated Franklin square requirements are relaxed and odd-order Franklin squares are considered. See Pickover [20] for a general treatment of magic squares and related entities.

A square matrix of order n is natural (or normal) if its elements are $0, 1, \ldots, n^2 - 1$ (used here) or $1, 2, \ldots, n^2$ (used by Franklin and some others). Squares that are not necessarily natural are called general here. The squares given by Franklin are natural and even order.

Complete (or most-perfect) magic squares originate in the work of McClintock [11] in 1897. They have the quartal property and the complementary property, wherein two elements $n/2$ positions apart along all diagonals (with wraparound) add to a known index number. Complete natural magic squares of order $n = 4k$ ($k \geq 2$) are constructed (in principle) and enumerated by Ollerenshaw and Brée [16]. They show that complete squares are magic and pandiagonal.

In the next section matrix algebraic definitions are given for magic squares, pandiagonal squares, Franklin squares, and complete squares. The reader may wish to consult these definitions before reading further in this introduction.

Methods for constructing Franklin squares are reviewed by Pasles [17, 18, 19] and the author [15] who gives a systematic construction for natural ones of order $8k$ that reproduces Franklin's original squares for $k = 1, 2$ and that of Jacobs [8] for $k = 3$. Numerical generation and enumeration of all natural order-8 natural Franklin squares is carried out by Schindel, et al. [22]. From an exhaustive numerical search, Hurkens [7] concludes that there are no natural Franklin squares of order 12 and he gives constructions for many higher orders.[1] Ahmed [1] gives an algebraic-geometric method of constructing and enumerating general Franklin squares. This provides a crude upper bound on the number of natural ones of order 8.

Transformation. Here we obtain a transformation of a complete magic square of order $8k$ to a pandiagonal Franklin magic square, thereby confirming a conjecture in [22]. Our transformation involves pre- and post-multiplication of the complete square matrix by a special permutation matrix whose construction is specified. For order 8 our transformation is identical to the row/column permutations used in [22] to transform a pandiagonal Franklin square to a complete square. Verification of our transformation rests on a parameterization constructed for general quartal squares of even order. Also, we show that magic quartal squares are pandiagonal. Thus, a magic Franklin square is pandiagonal. Our parameterizations are specialized to general complete squares and general pandiagonal and ordinary Franklin squares of order-$4k$ $(k \geq 2)$.

The inverse of our transformation generates a pandiagonal quartal magic square from a pandiagonal Franklin square of order-$8k$. Our parameterizations show that all such generated squares are complete only for order 8 as in [22]. For higher orders $8k$ $(k \geq 2)$ the parameterizations indicate that there are many more pandiagonal Franklin squares than complete squares and many more ordinary Franklin squares than pandiagonal ones (as expected). Thus, not all pandiagonal Franklin squares of order $8k$ $(k \geq 2)$ can be obtained from complete ones by our transformation. Since the number of order- $4k$ $(k \geq 2)$ natural complete magic squares is known [16], our transformation gives a lower bound on the number of order-$8k$ natural pandiagonal and ordinary Franklin squares. Also, our parameterizations lead to a crude upper bound on the number of natural squares of each type. In addition, our parameterizations may be useful for finding numerical natural Franklin squares of order $4k$.

Spectra. From our parameterization for general quartal squares, natural ones are found to be rank 3 which agrees with Sylvester's rank inequality. This result applies to all natural complete and Franklin squares and is confirmed by the numerical results for all order-8 Franklin squares [22] as noted in [4]. Using the rank-3 information, we investigate the spectra of general quartal squares and show that magic ones have one pair of eigenvalues $\pm\lambda$. If such square matrices are natural and $\lambda \neq 0$, then they are diagonalizable. This result applies to all natural complete and pandiagonal Franklin squares. Also, we use the spectra information to show that odd matrix powers of pandiagonal Franklin squares are pandiagonal Franklin squares.

2. Definitions

Let us begin with a brief review of definitions of certain special squares regarded as matrices. The definitions that follow involve u - unity vector, u_1, u_2 - half unity vectors, U - unity matrix,

[1] Readers of TFQ may recall that Brown [3] shows that there are no magic squares with distinct entries chosen from the set of Fibonacci numbers $1, 2, 3, 5, 8, \ldots$. However, Freitag [6] gives a representation for order 4 magic squares with entries that are Fibonacci numbers or the sum of two Fibonacci numbers.

U_1 - half unity matrix, I - identity matrix, R - reflection matrix, and K - shifter matrix. For order $n = 4$ these are

$$u_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T, \quad u_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T, \quad u = u_1 + u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T, \tag{2.1}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \; U_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; K = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and similarly for higher even order. The matrix product KM shifts the elements of a square matrix M down one row (bottom row to top) and MK shifts them one column left (first column to last). Matrix powers of K enable multiple row/column shifts. Also, $K^0 = K^n = I$ and $K^{\frac{n}{2}} R K^{\frac{n}{2}} = R$.

A *magic square* matrix M satisfies the following row, column, and two main-diagonal sum conditions:

$$\begin{gathered} Mu = mu, \qquad M^T u = mu, \\ UM = MU = mU, \quad UM^T = M^T U = mU, \end{gathered} \tag{2.2}$$

$$\operatorname{tr}[M] = m, \qquad \operatorname{tr}[RM] = m, \tag{2.3}$$

where m - *magic sum index*, M^T - transpose of M, and $\operatorname{tr}[M]$ - trace of M. If M satisfies (2.2), but not necessarily (2.3), then M is *semi-magic*. If M satisfies (2.3), but not necessarily (2.2), then M is *diagonal-magic*. A square matrix of order n is *natural* if its elements are $0, 1, \ldots n^2 - 1$ for which $m = n(n^2 - 1)/2$. Squares that are not necessarily natural are called *general*. In what follows all special squares (to be defined next) are general unless otherwise noted.

In a *pandiagonal* square matrix M_P, the elements on all diagonals in both directions (with wraparound), sum to m, i.e.

$$\operatorname{tr}\left[K^i M_P\right] = m, \quad \operatorname{tr}\left[K^i R M_P\right] = m, \quad i = 0, 1, \ldots, n-1. \tag{2.4}$$

A pandiagonal square is diagonal-magic by definition.

In a *quartal* square matrix Q, the sum of all 2×2 subsquares (including broken ones) is $2r$, i.e.

$$(I + K)\, Q\, (I + K) = 2rU, \tag{2.5}$$

where r is an *auxiliary index*. The following identities for a quartal square matrix Q follow from (2.5) as given in [16] (in element form) and [13] (in matrix form):

$$Q(I + K) = K^{2i} Q(I + K), \quad (I + K)Q = (I + K)QK^{2i}, \quad i = 0, 1, \ldots, \frac{n}{2} - 1. \tag{2.6}$$

A *complete* (or most-perfect) square matrix C of order $n = 4k$ is quartal and has the *complementary* property wherein the sum of any element of C and its counterpart element $n/2$ positions along the same diagonal in both directions (with wraparound) is r, i.e.

$$C + K^{\frac{n}{2}} C K^{\frac{n}{2}} = rU, \tag{2.7}$$

where r can be shown to be the same r as in (2.5). It is known [16] that a complete square C can be transformed to another complete square C' by the shifting transformation

$$C' = K^i C K^j \tag{2.8}$$

which can be used to transform C to standard form with the element $C'_{1,1} = 0$. This transformation also applies to quartal and pandiagonal squares. Next, we give two theorems that follow from the definitions.

Theorem 2.1. *A complete square is magic and pandiagonal. For a semi-magic quartal square*

$$m = \frac{n}{2} r. \tag{2.9}$$

Proofs by Ollerenshaw and Brée [16] can be verified by matrix operations on the relevant defining equations. The pandiagonal property follows directly from (2.7). Also, we found the following (apparently) new and useful result.

Theorem 2.2. *A quartal magic square Q is pandiagonal.*

Proof. On applying U to (2.5) with (2.2), (2.9) follows. The trace of $K^i\times$ (2.5) then gives

$$\mathrm{tr}\left[K^i Q\right] + 2\mathrm{tr}\left[K^{i+1} Q\right] + \mathrm{tr}\left[K^{i+2} Q\right] = 4m, \quad i = 0, 1, \ldots, n-1. \tag{2.10}$$

On solving this system of linear equations together with $(2.3)_1$, one obtains $(2.4)_1$ with $M_P = Q$. Similarly, the trace of $K^i R\times$ (2.5) and $(2.3)_2$ lead to $(2.4)_2$, hence, Q is pandiagonal[2]. □

A *Franklin square* matrix F (as considered here) must satisfy three requirements. First, it must be quartal. Second, the sum of its upper and lower half-columns and right and left half-rows must be $m/2$, i.e.,

$$F u_1 = F u_2 = F^T u_1 = F^T u_2 = \frac{m}{2} u \tag{2.11}$$

which makes F semi-magic and requires that $n = 4k$ for a natural Franklin square. Third, the elements on its *bent diagonals* must sum to m. A square has four families of bent diagonals as illustrated in the Introduction. The bent-diagonal sum conditions can be expressed in matrix form by introducing the matrix X with elements X_{ij} defined as

$$X_{ij} = \begin{cases} 1, & \text{if } i \le n/2 \text{ and } j = i; \\ 1, & \text{if } i > n/2 \text{ and } j = n+1-i; \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Then, the following theorem provides a convenient way to enforce or check the bent diagonal sum conditions.

Theorem 2.3. *If X and $F^T X$ are pandiagonal, then F satisfies the bent-diagonal sum conditions.*

Proof. To illustrate this, consider the order-4 matrix F with element F_{ij} and form

$$\begin{aligned} FX &= \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{11}+F_{14} & F_{12}+F_{13} & 0 & 0 \\ F_{21}+F_{24} & F_{22}+F_{23} & 0 & 0 \\ F_{31}+F_{34} & F_{32}+F_{33} & 0 & 0 \\ F_{41}+F_{44} & F_{42}+F_{43} & 0 & 0 \end{bmatrix}, \\ F^T X &= \begin{bmatrix} F_{11} & F_{21} & F_{31} & F_{41} \\ F_{12} & F_{22} & F_{32} & F_{42} \\ F_{13} & F_{23} & F_{33} & F_{43} \\ F_{14} & F_{24} & F_{34} & F_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{11}+F_{41} & F_{21}+F_{31} & 0 & 0 \\ F_{12}+F_{42} & F_{22}+F_{32} & 0 & 0 \\ F_{13}+F_{43} & F_{23}+F_{33} & 0 & 0 \\ F_{14}+F_{44} & F_{24}+F_{34} & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.13}$$

in which all diagonals in both directions (with wraparound) sum to the four bent-diagonal sums of F. The equivalence of these pandiagonal conditions on FX and $F^T X$ to the bent-diagonal sum conditions are easily seen to extend to higher even-order F. □

[2] The proof given in [16] is for a complete square and uses (2.7).

The pandiagonal condition on FX and F^TX can be enforced or checked using (2.4). A simpler form of the bent-diagonal sum conditions for a semi-magic quartal F is developed in the next section. As shown by Pasles [18], the lowest even-order for existence of a natural Franklin square is 8. He also considers squares with different Franklin conditions and odd-order Franklin squares. A Franklin square of order $8k$ can be transformed to standard form with the element $F_{1,1} = 0$ using permutations of the type given by Ahmed [1] and Hurkens [7]. Franklin squares also may be pandiagonal. If they are not necessarily pandiagonal, they are called ordinary.

3. Quartal Square Parameterization

We construct a general parameterization for even-order quartal squares and apply it to obtain two useful identities. In a later section this parameterization is specialized to obtain general parameterizations for complete and Franklin squares.

The following quartal square matrix Q_n of even order n is constructed by applying the quartal property to construct the 2×2 square in its upper left corner and proceeding sequentially to construct 2×2 squares down and to the right:

$$Q_n = \left[\begin{array}{cccc} \gamma & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_2 & 2r-\gamma-\alpha_2-\beta_2 & \gamma-\alpha_3+\beta_2 & 2r-\gamma-\alpha_4-\beta_2 \\ \beta_3 & \gamma+\alpha_2-\beta_3 & -\gamma+\alpha_3+\beta_3 & \gamma+\alpha_4-\beta_3 \\ \beta_4 & 2r-\gamma-\alpha_2-\beta_4 & \gamma-\alpha_3+\beta_4 & 2r-\gamma-\alpha_4-\beta_4 \\ \beta_5 & \gamma+\alpha_2-\beta_5 & -\gamma+\alpha_3+\beta_5 & \gamma+\alpha_4-\beta_5 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{n-1} & \gamma+\alpha_2-\beta_{n-1} & -\gamma+\alpha_3+\beta_{n-1} & \gamma+\alpha_4-\beta_{n-1} \\ \beta_n & 2r-\gamma-\alpha_2-\beta_n & \gamma-\alpha_3+\beta_n & 2r-\gamma-\alpha_4-\beta_n \end{array}\right.$$

$$\left.\begin{array}{cccc} \alpha_5 & \cdots & \alpha_{n-1} & \alpha_n \\ \gamma-\alpha_5+\beta_2 & \cdots & \gamma-\alpha_{n-1}+\beta_2 & 2r-\gamma-\alpha_n-\beta_2 \\ -\gamma+\alpha_5+\beta_3 & \cdots & -\gamma+\alpha_{n-1}+\beta_3 & \gamma+\alpha_n-\beta_3 \\ \gamma-\alpha_5+\beta_4 & \cdots & \gamma-\alpha_{n-1}+\beta_4 & 2r-\gamma-\alpha_n-\beta_4 \\ -\gamma+\alpha_5+\beta_5 & \cdots & -\gamma+\alpha_{n-1}+\beta_5 & \gamma+\alpha_n-\beta_5 \\ \vdots & \ddots & \vdots & \vdots \\ -\gamma+\alpha_5+\beta_{n-1} & \cdots & -\gamma+\alpha_{n-1}+\beta_{n-1} & \gamma+\alpha_n-\beta_{n-1} \\ \gamma-\alpha_5+\beta_n & \cdots & \gamma-\alpha_{n-1}+\beta_n & 2r-\gamma-\alpha_n-\beta_n \end{array}\right]. \tag{3.1}$$

Q_n has $2n-1$ free parameters, namely the integers γ and $\alpha_i,\ \beta_i,\ (i = 2, 3, \dots n)$. The quartal property of Q_n can be verified by (2.5) for any specific even-order n. As clear from its construction, any even-order quartal square can be represented by Q_n. Of course Q_n is natural only for a limited set of its free parameters. Examination of (3.1) for odd n shows that only a trivial one-parameter Q_n involving γ and r is possible.

From (3.1) it is not difficult to verify the following useful identities for a quartal matrix Q:

$$(I+K)\,Qu_1 = \frac{n}{2}ru, \quad (I+K)\,Qu_2 = \frac{n}{2}ru, \tag{3.2}$$

$$(I+K)\,QX = 2rU_1, \quad (I+R)\,QX = 2rU_1. \tag{3.3}$$

Theorem 3.1. *A semi-magic quartal square Q of order $4k$ satisfies the bent-diagonal sum conditions if*

$$tr[QX] = \frac{n}{2}r = m, \quad tr\left[Q^TX\right] = \frac{n}{2}r = m. \tag{3.4}$$

Proof. For a semi-magic quartal square Q, Theorem 2.1 and the identities (3.3) give

$$\mathrm{tr}\,[QX] + \mathrm{tr}\,[KQX] = 2m, \quad \mathrm{tr}\,[QX] + \mathrm{tr}\,[RQX] = 2m,$$

$$\left.\begin{array}{l} \mathrm{tr}\left[K^{2i}QX\right] = \mathrm{tr}\,[QX] = \mathrm{tr}\left[K^{2i+1}QX\right] = \mathrm{tr}\,[KQX] = m \\ \mathrm{tr}\left[K^{2i}RQX\right] = \mathrm{tr}\,[RQX] = \mathrm{tr}\left[K^{2i+1}RQX\right] = \mathrm{tr}\,[KRQX] = m \end{array}\right\} \; i = 0, 1, \ldots \frac{n}{2} - 1, \tag{3.5}$$

and similarly for $Q^T X$ since Q^T is quartal too. Therefore, by (2.4), (3.4), and (3.5), QX and $Q^T X$ are pandiagonal and therefore, by Theorem 2.3, Q satisfies the bent-diagonal sum conditions. □

This theorem provides a simple way of enforcing or checking the bent-diagonal sum conditions on a square that is known to be semi-magic and quartal.

4. Complete to Franklin Transformation

We consider the possible transformation of a complete magic square matrix C to a pandiagonal Franklin square matrix $\hat{F}$ as given by

$$\hat{F} = ZCZ, \tag{4.1}$$

where the permutation matrix Z remains to be determined. It is required to be symmetric and rotationally symmetric, i.e.

$$Z^T = Z^{-1} = Z, \quad RZR = Z. \tag{4.2}$$

It follows from these requirements that $\hat{F}$ is magic since C is magic. Furthermore, $\hat{F}$ is natural when C is natural since Z is a permutation matrix. Other requirements will be placed on Z in what follows to make $\hat{F}$ satisfy the pandiagonal Franklin square conditions which are considered next.

Quartal Property. We require that Z be of the form

$$Z = \sum_{i=1}^{n/2} I_i' K^{2i} = \sum_{i=1}^{n/2} K^{2i} I_i'', \quad \sum_{i=1}^{n/2} I_i' = \sum_{i=1}^{n/2} I_i'' = I, \tag{4.3}$$

where I_i' and I_i'' are diagonal matrices with elements 0 and 1. Then, by (4.3) and the identities (2.6), we form

$$(I+K)ZC(I+K) = (I+K)\sum_{i=1}^{n/2} I_i' K^{2i} C(I+K)$$

$$= (I+K)\sum_{i=1}^{n/2} I_i' C(I+K) = (I+K)C(I+K) = 2rU. \tag{4.4}$$

Thus, according to (2.5), $\tilde{C} = ZC$ is quartal and so

$$(I+K)\tilde{C}Z(I+K) = \sum_{i=1}^{n/2} (I+K)\tilde{C}K^{2i} I_i''(I+K)$$

$$= (I+K)\tilde{C}\sum_{i=1}^{n/2} I_i''(I+K) = (I+K)\tilde{C}(I+K) = 2rU. \tag{4.5}$$

Therefore,

$$(I+K)\,ZCZ(I+K) = (I+K)\,\hat{F}(I+K) = 2rU, \tag{4.6}$$

hence, $\hat{F}$ is quartal and, by Theorem 2.2, $\hat{F}$ is pandiagonal. Therefore, the identities (3.2) and (3.3) apply to $\hat{F}$.

Half-Row/Column Sums. To make $\hat{F}$ satisfy the half-row/column sum conditions (2.11), we first apply (4.1) to the complementary condition (2.7) with (2.9) to obtain

$$\hat{F} + \left[ZK^{\frac{n}{2}}Z\right]\hat{F}\left[ZK^{\frac{n}{2}}Z\right] = \frac{2}{n}mU. \tag{4.7}$$

On post-multiplying this by u_1 and requiring that

$$\left[ZK^{\frac{n}{2}}Z\right]u_1 = u_1, \tag{4.8}$$

we have

$$\left(I + ZK^{\frac{n}{2}}Z\right)\hat{F}u_1 = mu. \tag{4.9}$$

Let

$$\hat{F}u_1 = \frac{m}{2}u + v \tag{4.10}$$

and substitute this into (4.9) and (3.2) with (2.9) to obtain

$$\begin{aligned}\left(I + ZK^{\frac{n}{2}}Z\right)v &= 0,\\ (I+K)\,v &= 0.\end{aligned} \tag{4.11}$$

The only solution to both of these equations is $v = 0$, hence, $\hat{F}u_1 = \frac{m}{2}u$ and, since $\hat{F}$ is magic, $\hat{F}u_2 = \frac{m}{2}u$. A similar argument leads to $\hat{F}^T u_1 = \hat{F}^T u_2 = \frac{m}{2}u$. Thus, $\hat{F}$ satisfies the half-row/column sum conditions (2.11) and is semi-magic.

Bent-Diagonal Sums. On post-multiplying (4.7) by X and requiring that

$$\left[ZK^{\frac{n}{2}}Z\right]X = X\left[ZK^{\frac{n}{2}}Z\right], \tag{4.12}$$

with (2.9), we obtain

$$\hat{F}X + \left(ZK^{\frac{n}{2}}Z\right)\hat{F}X\left(ZK^{\frac{n}{2}}Z\right) = \frac{4}{n}mU_1 \tag{4.13}$$

and similarly for $\hat{F}^T$. Thus,

$$\mathrm{tr}\left[\hat{F}X\right] = \mathrm{tr}\left[\hat{F}^T X\right] = m. \tag{4.14}$$

Therefore, since $\hat{F}$ is quartal and semi-magic, by Theorem 3.1, $\hat{F}$ satisfies the bent-diagonal sum conditions. Since all the Franklin conditions are satisfied, $\hat{F}$ is a pandiagonal Franklin magic square subject to the noted conditions on Z.

Construction of Z. To construct Z that satisfies the previously imposed conditions, we start with order 8 and then show how to construct Z successively for order $16, 24, \ldots$. For order 8 we take Z as

$$Z_8 = \begin{bmatrix} I_2 & O_2 & O_2 & O_2 \\ O_2 & O_2 & I_2 & O_2 \\ O_2 & I_2 & O_2 & O_2 \\ O_2 & O_2 & O_2 & I_2 \end{bmatrix}, \quad I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad O_2 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.15}$$

where I_2 and O_2 are submatrices of Z_8. The inverse transformation corresponding to (4.1), namely

$$\hat{M} = Z\hat{F}Z \tag{4.16}$$

with $Z = Z_8$ is equivalent to the row/column permutations used by Schindel, et al. [22] to transform an order-8 pandiagonal Franklin magic square $\hat{F}_8$ to a complete magic square $\hat{M} = \hat{C}_8$. The inverse transformation (4.16) is discussed further below.

It is not difficult to show that Z_8 satisfies (4.2) and (4.3). On noting that

$$Z_8K^4Z_8 = \begin{bmatrix} O_2 & I_2 & O_2 & O_2 \\ I_2 & O_2 & O_2 & O_2 \\ O_2 & O_2 & O_2 & I_2 \\ O_2 & O_2 & I_2 & O_2 \end{bmatrix}, \quad X_8 = \begin{bmatrix} I_2 & O_2 & O_2 & O_2 \\ O_2 & I_2 & O_2 & O_2 \\ O_2 & R_2 & O_2 & O_2 \\ R_2 & O_2 & O_2 & O_2 \end{bmatrix}, \quad R_2 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.17}$$

it can be verified that Z_8 satisfies (4.8) and (4.12).

For order $n = 8k$ ($k = 2, 3, \ldots$), we construct Z_n sequentially by adding rows and columns onto Z_{n-8} as follows:

$$Z_n = \begin{bmatrix} I_2 & O_2 & O & O_2 & O_2 \\ O_2 & O_2 & O & I_2 & O_2 \\ O & O & Z_{n-8} & O & O \\ O_2 & I_2 & O & O_2 & O_2 \\ O_2 & O_2 & O & O_2 & I_2 \end{bmatrix}, \quad n = 16, 24, \ldots, \tag{4.18}$$

where O are submatrices of proper dimensions with 0 elements. Again, Z_n satisfies (4.2) and (4.3). On noting that $Z_nK^{\frac{n}{2}}Z_n$ is of the same form as $Z_8K^4Z_8$ in (4.17), it can be verified that Z_n satisfies (4.8) and (4.12). Thus, Z_n is a suitable Z matrix for transforming a complete magic square C_n to a pandiagonal Franklin magic square matrix $\hat{F}_n$ by (4.1) for order $n = 8k$. It appears that no suitable Z_n exists for $n = 8k + 4$. Also, it is possible that other suitable Z_n may exist for $n = 8k$.

Inverse Transformation. We return to the inverse transformation (4.16) for an order-$8k$ pandiagonal Franklin square $\hat{F}$ transformed to a square $\hat{M}$. It follows from the preceding results that since $\hat{F}$ is magic, quartal, and pandiagonal, then so is $\hat{M}$. However, the complementary property (2.7) for $\hat{M}$ does not follow from the Franklin square conditions for $\hat{F}$ except for $n = 8$ as shown by the parameterizations given in the next section.

Other transformations of magic squares have been given previously. In particular, a regular (or associative) magic square can be transformed to a pandiagonal magic square by the Planck transformation [21] as given in matrix form by Nordgren [13]. Also, he shows that a Franklin square can be transformed to a (nonquartal) pandiagonal magic square in two ways [15].

5. Parameterizations

On placing restrictions on the general parameterization for a quartal square (3.1), we obtain general parameterizations for complete and Franklin squares of order $4k$. A study of their transformations confirms the foregoing general results for order $8k$. In addition, our parameterizations provide a means of generating natural squares of each type by numerical search on the free parameters. We recall that parameterizations are available for general order-3 and order-4 magic squares as discussed by Loly, et al. [9] and Nordgren [14].

Complete Squares. To obtain a general parameterization for a complete magic squares C_n of order $n = 4k$, we enforce the complementary condition (2.7) on Q_n from (3.1) which leads to

$$
\begin{aligned}
\alpha_{\frac{n}{2}+i} &= \begin{cases} 2r-\gamma-\alpha_i-\alpha_{\frac{n}{2}+1}, & i=2,4,\ldots,\frac{n}{2}, \\ \gamma-\alpha_i+\alpha_{\frac{n}{2}+1}, & i=3,5,\ldots,\frac{n}{2}-1, \end{cases} \\
\beta_{\frac{n}{2}+1} &= r-\alpha_{\frac{n}{2}+1}, \\
\beta_{\frac{n}{2}+i} &= \begin{cases} r-\gamma-\beta_i+\alpha_{\frac{n}{2}+1}, & i=2,4,\ldots,\frac{n}{2}, \\ r+\gamma-\beta_i-\alpha_{\frac{n}{2}+1}, & i=3,5,\ldots,\frac{n}{2}-1. \end{cases}
\end{aligned}
\tag{5.1}
$$

Thus, C_n has n free parameters, namely $\gamma, \alpha_2, \alpha_3, \ldots \alpha_{\frac{n}{2}+1}, \beta_2, \beta_3, \ldots \beta_{\frac{n}{2}}$.

For order 8, with $\gamma = 0$ (for C_8 in standard form), (5.1) and (3.1) give the following parameterization for a complete magic square:

$$
C_8 = \left[\begin{array}{cccc}
0 & \alpha_2 & \alpha_3 & \alpha_4 \\
\beta_2 & 2r-\alpha_2-\beta_2 & \beta_2-\alpha_3 & 2r-\alpha_4-\beta_2 \\
\beta_3 & \alpha_2-\beta_3 & \beta_3+\alpha_3 & \alpha_4-\beta_3 \\
\beta_4 & 2r-\alpha_2-\beta_4 & \beta_4-\alpha_3 & 2r-\alpha_4-\beta_4 \\
r-\alpha_5 & -r+\alpha_5+\alpha_2 & r-\alpha_5+\alpha_3 & -r+\alpha_5+\alpha_4 \\
r+\alpha_5-\beta_2 & r-\alpha_5-\alpha_2+\beta_2 & r+\alpha_5-\alpha_3-\beta_2 & r-\alpha_5-\alpha_4+\beta_2 \\
r-\alpha_5-\beta_3 & -r+\alpha_5+\alpha_2+\beta_3 & r-\alpha_5+\alpha_3-\beta_3 & -r+\alpha_5+\alpha_4+\beta_3 \\
r+\alpha_5-\beta_4 & r-\alpha_5-\alpha_2+\beta_4 & r+\alpha_5-\alpha_3-\beta_4 & r-\alpha_5-\alpha_4+\beta_4
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
\alpha_5 & 2r-\alpha_5-\alpha_2 & \alpha_5-\alpha_3 & 2r-\alpha_5-\alpha_4 \\
-\alpha_5+\beta_2 & \alpha_5+\alpha_2-\beta_2 & -\alpha_5+\beta_2+\alpha_3 & \alpha_5+\alpha_4-\beta_2 \\
\alpha_5+\beta_3 & 2r-\alpha_5-\alpha_2-\beta_3 & \alpha_5+\beta_3-\alpha_3 & 2r-\alpha_5-\alpha_4-\beta_3 \\
-\alpha_5+\beta_4 & \alpha_5+\alpha_2-\beta_4 & -\alpha_5+\beta_4+\alpha_3 & \alpha_5+\alpha_4-\beta_4 \\
r & r-\alpha_2 & r-\alpha_3 & r-\alpha_4 \\
r-\beta_2 & -r+\alpha_2+\beta_2 & r+\alpha_3-\beta_2 & -r+\alpha_4+\beta_2 \\
r-\beta_3 & r-\alpha_2+\beta_3 & r-\alpha_3-\beta_3 & r-\alpha_4+\beta_3 \\
r-\beta_4 & -r+\alpha_2+\beta_4 & r+\alpha_3-\beta_4 & -r+\alpha_4+\beta_4
\end{array}\right]
\tag{5.2}
$$

which has 7 free parameters. Its magic properties follow from Theorem 2.1 and can be verified directly as can the complementary property (2.7) and the quartal property (2.5).

Transformed Pandiagonal Franklin Squares. By the transformation (4.1) with (4.15) applied to C_8 of (5.2) we obtain the following parameterization for an order-8 pandiagonal Franklin square:

$$
\hat{F}_8 = \left[\begin{array}{cccc}
0 & \alpha_2 & \alpha_5 & 2r-\alpha_5-\alpha_2 \\
\beta_2 & 2r-\alpha_2-\beta_2 & -\alpha_5+\beta_2 & \alpha_5+\alpha_2-\beta_2 \\
r-\alpha_5 & -r+\alpha_5+\alpha_2 & r & r-\alpha_2 \\
r+\alpha_5-\beta_2 & r-\alpha_5-\alpha_2+\beta_2 & r-\beta_2 & -r+\alpha_2+\beta_2 \\
\beta_3 & \alpha_2-\beta_3 & \alpha_5+\beta_3 & 2r-\alpha_5-\alpha_2-\beta_3 \\
\beta_4 & 2r-\alpha_2-\beta_4 & -\alpha_5+\beta_4 & \alpha_5+\alpha_2-\beta_4 \\
r-\alpha_5-\beta_3 & -r+\alpha_5+\alpha_2+\beta_3 & r-\beta_3 & r-\alpha_2+\beta_3 \\
r+\alpha_5-\beta_4 & r-\alpha_5-\alpha_2+\beta_4 & r-\beta_4 & -r+\alpha_2+\beta_4
\end{array}\right.
$$

$$\begin{matrix} \alpha_3 & \alpha_4 & \alpha_5-\alpha_3 & 2r-\alpha_5-\alpha_4 \\ \beta_2-\alpha_3 & 2r-\alpha_4-\beta_2 & -\alpha_5+\alpha_3+\beta_2 & \alpha_5+\alpha_4-\beta_2 \\ r+\alpha_3-\alpha_5 & -r+\alpha_5+\alpha_4 & r-\alpha_3 & r-\alpha_4 \\ r+\alpha_5-\alpha_3-\beta_2 & r-\alpha_5-\alpha_4+\beta_2 & r+\alpha_3-\beta_2 & -r+\alpha_4+\beta_2 \\ \beta_3+\alpha_3 & \alpha_4-\beta_3 & \alpha_5-\alpha_3+\beta_3 & 2r-\alpha_5-\alpha_4-\beta_3 \\ \beta_4-\alpha_3 & 2r-\alpha_4-\beta_4 & -\alpha_5+\alpha_3+\beta_4 & \alpha_5+\alpha_4-\beta_4 \\ r-\alpha_5+\alpha_3-\beta_3 & -r+\alpha_5+\alpha_4+\beta_3 & r-\alpha_3-\beta_3 & r-\alpha_4+\beta_3 \\ r+\alpha_5-\alpha_3-\beta_4 & r-\alpha_5-\alpha_4+\beta_4 & r+\alpha_3-\beta_4 & -r+\alpha_4+\beta_4 \end{matrix} \Bigg], \tag{5.3}$$

where the Franklin square conditions can be verified directly and $\hat{F}_8$ satisfies the diagonal magic sum conditions (2.3) which makes $\hat{F}_8$ pandiagonal by Theorem 2.2. This verifies the transformation (4.1) for order 8. It also has been verified in the same manner for orders 16 and 24. For order $n = 8k$, the transformed $\hat{F}_n$ has n free parameters (for $\gamma \neq 0$), the same as C_n from which it was transformed.

Ordinary Franklin Squares. We construct a general parameterization for an ordinary Franklin square F_n of order $n = 4k$ ($k \geq 2$) by enforcing the half-row/column sum conditions and the bent diagonal sum conditions on Q_n of (3.1). From these conditions, namely (2.11) and (3.4) of Theorem 3.1, applied to Q_n we find that

$$\begin{aligned} \alpha_{\frac{n}{2}} &= \frac{n}{4}r - \gamma - \sum_{i=2}^{n/2-1} \alpha_i, \quad \alpha_{n-1} = \gamma + \sum_{i=1}^{n/4-1} \alpha_{2i+1} - \sum_{i=n/4}^{n/2-2} \alpha_{2i+1}, \\ \alpha_n &= \frac{n}{4}r - \gamma - \sum_{i=1}^{n/4-1} \alpha_{2i+1} - \sum_{i=n/4}^{n/2-2} \alpha_{2i+2}, \end{aligned} \tag{5.4}$$

and similar formulas for $\beta_{\frac{n}{2}}$, β_{n-1}, and β_n. These formulas determine 6 of the $2n-1$ free parameters of Q leaving $2n-7$ free parameters for F_n. For $n = 4$, we find that

$$\alpha_2 = \alpha_4 = \beta_2 = \beta_4 = r - \gamma, \quad \alpha_3 = \beta_3 = \gamma, \tag{5.5}$$

so there is only one free parameter γ and no natural F_4 as also shown by Pasles [18].

For $n = 8$, with $\gamma = 0$ (for F in standard form), (5.4) and (3.1) give the following parameterization for an ordinary Franklin square:

$$F_8 = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & 2r-\alpha_2-\alpha_3 \\ \beta_2 & 2r-\alpha_2-\beta_2 & \beta_2-\alpha_3 & \alpha_2+\alpha_3-\beta_2 \\ \beta_3 & \alpha_2-\beta_3 & \beta_3+\alpha_3 & 2r-\alpha_2-\alpha_3-\beta_3 \\ 2r-\beta_2-\beta_3 & \beta_2+\beta_3-\alpha_2 & 2r-\alpha_3-\beta_2-\beta_3 & -2r+\alpha_2+\alpha_3+\beta_2+\beta_3 \\ \beta_5 & \alpha_2-\beta_5 & \beta_5+\alpha_3 & 2r-\alpha_2-\alpha_3-\beta_5 \\ \beta_6 & 2r-\alpha_2-\beta_6 & \beta_6-\alpha_3 & \alpha_2+\alpha_3-\beta_6 \\ \beta_3-\beta_5 & \alpha_2-\beta_3+\beta_5 & \alpha_3+\beta_3-\beta_5 & 2r-\alpha_2-\alpha_3-\beta_3+\beta_5 \\ 2r-\beta_3-\beta_6 & \beta_3+\beta_6-\alpha_2 & 2r-\alpha_3-\beta_3-\beta_6 & -2r+\alpha_2+\alpha_3+\beta_3+\beta_6 \end{bmatrix}$$

$$\left.\begin{matrix} \alpha_5 & \alpha_6 & \alpha_3-\alpha_5 & 2r-\alpha_3-\alpha_6 \\ \beta_2-\alpha_5 & 2r-\alpha_6-\beta_2 & -\alpha_3+\alpha_5+\beta_2 & \alpha_3+\alpha_6-\beta_2 \\ \beta_3+\alpha_5 & \alpha_6-\beta_3 & \alpha_3-\alpha_5+\beta_3 & 2r-\alpha_3-\alpha_6-\beta_3 \\ 2r-\alpha_5-\beta_2-\beta_3 & -\alpha_6+\beta_2+\beta_3 & 2r-\alpha_3+\alpha_5-\beta_2-\beta_3 & -2r+\alpha_3+\alpha_6+\beta_2+\beta_3 \\ \beta_5+\alpha_5 & \alpha_6-\beta_5 & \alpha_3-\alpha_5+\beta_5 & 2r-\alpha_3-\alpha_6-\beta_5 \\ \beta_6-\alpha_5 & 2r-\alpha_6-\beta_6 & -\alpha_3+\alpha_5+\beta_6 & \alpha_3+\alpha_6-\beta_6 \\ \alpha_5+\beta_3-\beta_5 & \alpha_6-\beta_3+\beta_5 & \alpha_3-\alpha_5-\beta_5+\beta_3 & 2r-\alpha_3-\alpha_6+\beta_5-\beta_3 \\ 2r-\alpha_5-\beta_3-\beta_6 & -\alpha_6+\beta_3+\beta_6 & 2r-\alpha_3+\alpha_5-\beta_3-\beta_6 & -2r+\alpha_3+\alpha_6+\beta_3+\beta_6 \end{matrix}\right] \tag{5.6}$$

which has 8 free parameters, one more than the pandiagonal Franklin square of (5.3). Thus, as expected, there are many more order-8 ordinary Franklin squares than pandiagonal ones. For order-8 natural Franklin squares this ratio is 2 to 1 in the numerical results of Schindel, et al. [22].

Pandiagonal Franklin Squares. The diagonal magic sum conditions (2.3) on an ordinary F_n of order $n = 4k$ $(k \geq 2)$, as parameterized above, are satisfied if

$$4 \sum_{i=1}^{n/4-1} (\alpha_{2i+1} + \beta_{2i+1}) - (n-8)\,\gamma = m \tag{5.7}$$

in which case F_n is pandiagonal according to Theorem 2.2 and has $2n-8$ free parameters.

From (5.6) or (5.7), F_8 satisfies the diagonal magic sum conditions (2.3) if

$$\alpha_3 + \beta_3 = [F_8]_{13} + [F_8]_{31} = r \tag{5.8}$$

in which case, by Theorem 2.2, F_8 is pandiagonal. The element condition of (5.8) is satisfied by $\hat{F}_8$ of (5.3) and it provides a simple way of identifying order-8 Franklin squares (in standard form) that are pandiagonal. Also, $[F_8]_{33} = r$ for this case.

On enforcing (5.8) on F_8 of (5.6) we obtain the pandiagonal Franklin square $\hat{F}_8$ of (5.3) with slightly different numbering of the free parameters. In view of this equivalence, it follows that any order-8 pandiagonal Franklin square $\hat{F}_8$ can be transformed to a complete square C_8 by (4.16) as can be verified directly and is done numerically by Schindel, et al. [22] for natural $\hat{F}_8$. Thus, we have a one-to-one correspondence between the sets of (natural or general) $\hat{F}_8$ and C_8.

For higher-order Franklin squares, (5.7) again determines one of the parameters, leaving $2n-8$ free parameters for a pandiagonal Franklin square $\tilde{F}_n$ of order $n = 4k$. For $n = 4k$ $(k \geq 3)$ the number of free parameters $2n-8$ in $\tilde{F}_n$ exceeds the number of free parameters n in a general complete square C_n and its transformed pandiagonal Franklin square $\hat{F}_n$ $(n = 8k)$; their numbers being equal for $n = 8$. Thus, for $n = 8k$ $(k \geq 2)$ there are pandiagonal Franklin magic squares that cannot be transformed to complete magic squares. For example, the order-16 pandiagonal Franklin square given by Morris [12] (attributed to Franklin) does not transform to a complete square by (4.16) with Z_{16} from (4.18).

6. Spectra

Since the number of nonzero eigenvalues of a matrix cannot exceed its rank, we first determine the rank of our parameterized complete and Franklin squares. According to Maple© (in Scientific WorkPlace©), all of the order-8 squares parameterized above are rank 3. This agrees with the numerical results for all order-8 natural Franklin squares generated in [22] as noted in [4]. To see why this is so, consider the reduced row echelon form of the general order-8 quartal square Q_8 from (3.1) for which Maple© gives

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\alpha_4-\alpha_2}{\gamma-\alpha_3} & \frac{\alpha_5-\alpha_3}{\gamma-\alpha_3} & \frac{\alpha_6-\alpha_2}{\gamma-\alpha_3} & \frac{\alpha_7-\alpha_3}{\gamma-\alpha_3} & \frac{\alpha_8-\alpha_2}{\gamma-\alpha_3} \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{\alpha_4-\alpha_2}{\gamma-\alpha_3} & \frac{\gamma-\alpha_5}{\gamma-\alpha_3} & -\frac{\alpha_6-\alpha_2}{\gamma-\alpha_3} & \frac{\gamma-\alpha_7}{\gamma-\alpha_3} & -\frac{\alpha_8-\alpha_2}{\gamma-\alpha_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.1}$$

If Q_8 is natural, none of the elements in (6.1) can be 0 and $\operatorname{rank}[Q_8] = 3$. A detailed analysis of the Gaussian elimination procedure used to derive (6.1) when applied to Q_n from (3.1) results in a reduced row echelon form similar to (6.1). Therefore, natural Q_n, C_n, $\hat{F}_n$, and F_n are rank 3, since they all are quartal.

For certain general quartal squares, $\operatorname{rank}[Q_n] = 2$. This includes the case $\gamma = \alpha_i = \eta$ for i odd, and $\alpha_i = \mu$ for i even. Also, $\operatorname{rank}[Q_n] = 1$ if $\gamma = \alpha_i = r = 0$. The reduced column echelon form of Q_n replaces α_i by β_i in (6.1) and leads to the same conclusions.

A bound on $\operatorname{rank}[Q_n]$ can be obtained from the quartal property condition (2.5) and Sylvester's rank inequality for order-n square matrices A, B, C extended to the form

$$\operatorname{rank}[A] + \operatorname{rank}[B] + \operatorname{rank}[C] - 2n \le \operatorname{rank}[ABC]. \tag{6.2}$$

With

$$A = C = I + K, \quad B = Q_n, \quad ABC = 2rU, \tag{6.3}$$

and noting that for even n

$$\operatorname{rank}[I+K] = n-1, \quad \operatorname{rank}[U] = 1, \tag{6.4}$$

(2.5) and (6.2) lead to

$$\operatorname{rank}[Q_n] \le 3 \tag{6.5}$$

which agrees with the foregoing results.

Next we consider the eigenvalues and eigenvectors of quartal squares.

Theorem 6.1. *A diagonal-magic quartal square Q_n has the eigenvalue m and one pair of eigenvalues $\pm\lambda$. If $\lambda \neq 0$ and this Q_n, is natural, then it is diagonalizable.*

Proof. As noted by Mattingly [10], the eigenvalue m follows from the semi-magic condition $(2.2)_1$ and all other eigenvalues are less than m in absolute value. Since, as shown above, $\operatorname{rank}[Q_n] \le 3$, Q_n has at most 3 nonzero eigenvalues m, λ_1, λ_2 which must sum to $\operatorname{tr}[Q_n]$. From the diagonal-magic condition (2.3), $\operatorname{tr}[Q_n] = m$, hence, $\lambda_1 = \lambda,\ \lambda_2 = -\lambda$ for a diagonal-magic quartal square Q_n.

It is not difficult to find the eigenvectors for five 0 eigenvalues of Q_8 of (3.1) to be the columns of the matrix

$$\begin{bmatrix} \alpha_4-\alpha_2 & \alpha_5-\alpha_3 & \alpha_6-\alpha_2 & \alpha_7-\alpha_3 & \alpha_8-\alpha_2 \\ -(\alpha_3-\gamma) & 0 & -(\alpha_3-\gamma) & 0 & -(\alpha_3-\gamma) \\ -(\alpha_4-\alpha_2) & \gamma-\alpha_5 & -(\alpha_6-\alpha_2) & \gamma-\alpha_7 & -(\alpha_8-\alpha_2) \\ \alpha_3-\gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha_3-\gamma & 0 & 0 & 0 \\ 0 & 0 & \alpha_3-\gamma & 0 & 0 \\ 0 & 0 & 0 & \alpha_3-\gamma & 0 \\ 0 & 0 & 0 & 0 & \alpha_3-\gamma \end{bmatrix} \tag{6.6}$$

and this can be generalized to higher order Q_n without difficulty by following the same pattern for the elements. Thus, if Q_n is natural, these $n-3$ simple eigenvectors are linearly independent. If $\lambda \neq 0$, then the three nonzero eigenvalues are distinct and the spectral matrix is diagonal. □

This theorem applies to all complete squares and all pandiagonal Franklin squares. When $\lambda_1 = -\lambda_2 = 0$ these two eigenvalues may have generalized eigenvectors in which case Q_n would not be diagonalizable. Since an ordinary Franklin square F need only be semi-magic, it may have a pair of equal eigenvalues which may have generalized eigenvectors in which case F would not be diagonalizable. Numerical examples of nondiagonalizable general C_8 and F_8 have been found for both such cases by the author.

7. Matrix Powers of Franklin Squares

Theorem 7.1. *An odd matrix power of a pandiagonal Franklin square is a pandiagonal Franklin square.*

Proof. The sum conditions (2.11) on the half rows/columns of a Franklin square F together with (2.2) lead to

$$F^i u_1 = F^{i-1} F u_1 = \frac{m}{2} F^{i-1} u = \frac{m^i}{2} u \tag{7.1}$$

and similarly for $F^i u_2$, $\left[F^i\right]^T u_1$, and $\left[F^i\right]^T u_2$. Thus, F^i satisfies the appropriate sum conditions corresponding to (2.11) on its half rows/columns and F^i is semi-magic.

In order to verify the diagonal magic sum conditions on F^i, we recall that the Jordan form of F and its matrix power F^i can be written as

$$F = SDS^{-1}, \quad F^i = SD^iS^{-1}, \tag{7.2}$$

where for magic F with eigenvalues $m, \pm\lambda$ (as noted above) and i odd

$$D = \begin{bmatrix} m & 0 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad D^i = \begin{bmatrix} m^i & 0 & 0 & 0 & \cdots \\ 0 & \lambda^i & 0 & 0 & \cdots \\ 0 & 0 & -\lambda^i & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{7.3}$$

or if $\lambda = 0$ and there are two generalized eigenvectors, then

$$D = \begin{bmatrix} m & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad D^i = \begin{bmatrix} m^i & m^{i-1} & m^{i-2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7.4}$$

In both of these cases

$$\mathrm{tr}\,[F] = m, \quad \mathrm{tr}\left[F^i\right] = m^i, \ (i \text{ odd}). \tag{7.5}$$

With $S^{-1}RS$ represented by a matrix with elements s_{ij}, we have

$$\mathrm{tr}\,[RF] = \mathrm{tr}\left[DS^{-1}RS\right] = ms_{11} + \lambda\left(s_{22} - s_{33}\right) = m, \tag{7.6}$$

hence,

$$s_{11} = 1, \quad s_{22} - s_{33} = 0 \text{ or } \lambda = 0. \tag{7.7}$$

Therefore,

$$\mathrm{tr}\left[RF^i\right] = \mathrm{tr}\left[D^iS^{-1}RS\right] = m^i s_{11} + \lambda^i\left(s_{22} - s_{33}\right) = m^i. \tag{7.8}$$

Thus, F^i satisfies the appropriate diagonal sum conditions corresponding to (2.3), hence, F^i is magic. The quartal condition on F^i can be established from (2.5) and (7.3) in the same manner as for (7.8), where again the coefficients of λ and λ^i vanish.

With (7.2), the bent-diagonal condition (3.4) for a Franklin magic square F reads

$$\operatorname{tr}[FX] = \operatorname{tr}\left[DS^{-1}XS\right] = m. \tag{7.9}$$

By the same argument used to derive (7.8) we find that

$$\operatorname{tr}\left[F^iX\right] = \operatorname{tr}\left[\left(F^i\right)^T X\right] = m^i \tag{7.10}$$

which are the appropriate bent-diagonal conditions corresponding to (3.4) for a quartal square. Therefore, F^i is a Franklin magic square and, by Theorem 2.2, it is pandiagonal. □

In a similar manner it can be shown that the product of an odd number of pandiagonal Franklin squares is a pandiagonal Franklin square. Theorem 7.1 has been confirmed numerically for the order-16 pandiagonal Franklin square given by Morris [12] (attributed to Franklin). Numerical results also show that odd powers of nonpandiagonal Franklin squares are not Franklin squares in general.

A simpler proof of Theorem 7.1 is possible for a pandiagonal Franklin square $\hat{F}$ of order-$8k$ transformed from a complete magic square C by (4.1). It follows from(4.1) and (4.2) that

$$\hat{F}^i = [ZCZ]^i = ZC^iZ, \tag{7.11}$$

where C^i is a complete magic square for odd i as shown by Nordgren [13]. Therefore, it follows from (4.1) and (7.11) that $\hat{F}^i$ is a pandiagonal Franklin magic square for odd i. Matrix powers of magic squares also are considered by Cook, et al. [5].[3]

8. Number of Franklin Squares

As noted by Schindel, et al. [22], only one third of their 1,105,920 numerically generated natural order-8 Franklin squares are pandiagonal and can be transformed to complete magic squares.[4] This number corresponds to the number 368,640 of order-8 natural complete magic squares enumerated by Ollerenshaw and Brée [16]. Thus, our transformation of order-8 natural complete magic squares to natural pandiagonal Franklin magic squares (the inverse of the transformation in [22]) produces the same one third of the known natural Franklin squares of order-8.

As indicated by our parameterizations and noted in [22], for higher orders one expects there to be even more Franklin squares that cannot be transformed to or from complete magic squares. Since the number of complete magic squares is determined in [16], our transformation provides a lower bound on the number of pandiagonal and ordinary Franklin squares of order-$8k$.

Ahmed [1] determines the number of general order-8 Franklin squares with magic sum equal to that of natural squares to be 2.29... E14 which is an upper bound on the number of natural ones. In our parameterization for a general order-8 Franklin square there are 9 free parameters (when γ is retained) each of which can take up to 64 values $(0, 1, \ldots, 63)$ for equivalence with a natural magic square, giving $64 \times 63 \times \cdots \times 56 = 9.99\ldots$E15 possible squares as an upper bound on the number of general and natural ones. This number can be reduced somewhat by a more detailed consideration of allowable values for other elements of F_8 in its parameterization (5.6). Similar results can be obtained from our parameterizations

[3] Unfortunately, the purported Franklin square given by them does not meet the Franklin square conditions.

[4] The number 1,105,920 was verified by Amela [2] using a different method of construction.

for higher-order complete and Franklin squares. As already noted, Hurkens [7] found that there are no natural Franklin squares of order-12. The actual number of natural Franklin squares of order-$4k$ $(k \geq 4)$ remains to be determined, perhaps along the lines of Ollerenshaw and Brée [16].

References

[1] M. Ahmed, *How many squares are there, Mr Franklin?*, The American Mathematical Monthly, **111.5** (2004), 394–410.

[2] M. Amela, *Structured 8 by 8 Franklin squares*, `http://www.region.com.ar/amela/franklinsquares`.

[3] J. Brown, *Reply to exploring Fibonacci magic squares*, The Fibonacci Quarterly, **3.2** (1969), 146.

[4] I. Cameron, A. Rogers, and P. Loly, *Signatura of magic and latin integer squares: Isentropic clans and indexing*, Discussiones Mathematicae Probability and Statistics, **33** (2013), 121–149.

[5] C. Cook, M. Bacon, and R. Hillman, *The "magicness" of powers of some magic squares*, The Fibonacci Quarterly, **48.4** (2010), 298–306.

[6] H. Freitag, *A magic square involving Fibonacci numbers*, The Fibonacci Quarterly, **6.1** (1968), 77–80.

[7] C. Hurkens, *Plenty of Franklin magic squares, but none of order 12* (2007), `http://www.win.tue.nl/bs/spor/2007-06.pdf`.

[8] C. Jacobs, *A reexamination of the Franklin square*, The Mathematics Teacher, **64.1** (1971), 55–62.

[9] P. Loly, I. Cameron, W. Trump, and D. Schindel, *Magic square spectra*, Linear Algebra and its Applications, **430** (2009), 2659–2680.

[10] R. Mattingly, *Even order regular magic squares are singular*, The American Mathematical Monthly, **107.9** (2000), 777–782.

[11] E. McClintock, *On the most perfect forms of magic squares, with methods for their production*, American Journal of Mathematics, **19** (1897), 99–120.

[12] D. Morris, *Best Franklin squares - Franklin's construction method Part 5* (2005), `http://www.bestfranklinsquares.com`.

[13] R. Nordgren, *On properties of special magic squares*, Linear Algebra and its Applications, **437** (2012), 2009–2025.

[14] R. Nordgren, *Spectra of order-4 special magic squares*, Linear Algebra and its Applications, **458** (2014), 80–98.

[15] R. Nordgren, *How Franklin (may have) made his squares*, (to appear).

[16] K. Ollerenshaw and D. Brée, *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration*, The Institute of Mathematics and its Applications, Southend-on-Sea, UK, 1998.

[17] P. Pasles, *The lost squares of Dr. Franklin: Ben Franklin's missing squares and the secret of the magic circle*, The American Mathematical Monthly, **108.6** (2001), 489–511.

[18] P. Pasles, *A bent for magic*, Mathematics Magazine, **79.1** (2006), 3–13.

[19] P. Pasles, *Benjamin Franklin's Numbers*, Princeton University Press, Princeton, NJ, 2008.

[20] C. Pickover, *The Zen of Magic Squares, Circles, and Stars*, Princeton University Press, Princeton, NJ, 2002.

[21] C. Planck, *Pandiagonal magic squares of orders 6 and 10 without minimal numbers*, The Monist, **29** (1919), 307–316.

[22] D. Schindel, M. Rempel, and P. Loly, *Enumerating the bent diagonal squares of Dr. Benjamin Franklin FRS*, Proceedings of the Royal Society A, **462** (2006), 2271–2279.

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