ADDITIONAL RESULTS ON SOME RECENT INFINITE SUMS

HELMUT PRODINGER

Abstract. The paper referred to in the title is [1]. We rederive a recursion in a more general setting, without resorting to probabilistic arguments, in a simple and straight forward way. Then we find an explicit form for the exponential generating function in question, from which it is easy to extract coefficients, at least in principle. Since this function is meromorphic, the asymptotic behavior of the numbers in question follows from the behavior at the dominant simple pole.

1. Introduction

The Fibonacci sequence of order k ($k \geq 2$) is defined via the recursion

$$F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)} \quad \text{for} \quad n \geq 2,$$

with $F_1^{(k)} = 1$ and $F_n^{(k)} = 0$ for $n \leq 0$. We choose this compact representation, since we are not interested in negative values of n in this note. The instance $k = 2$ is the classical case.

In the recent paper [1] the sums

$$\alpha_{m,k} := \sum_{n \geq 1} q^{n+k-1} n^m F_n^{(k)}$$

are studied in the instance $q = \frac{1}{2}$. This special choice stems from some probabilistic arguments used in [1]. Here, we will derive a recursion in a most elementary fashion, and thus only require that the series in question converges. This happens if and only if $|q|$ is smaller than the reciprocal of the *generalized golden ratio*, and this is a number that approaches $\frac{1}{2}$ from above when k gets large. Compare the papers [3, 6] for an analysis of these ratios.

After that, we turn to asymptotics via generating functions.

2. The Recursion

From the definition of the Fibonacci sequence of order k we get, by summing on $n \geq 2$:

$$\sum_{n \geq 2} q^{n+k-1} n^m F_n^{(k)} = \sum_{j=1}^{k} \sum_{n \geq 2} q^{n+k-1} n^m F_{n-j}^{(k)}$$

or

$$\begin{aligned}
\alpha_{m,k} - q^k &= \sum_{j=1}^{k} \sum_{n\geq 0} q^{n+j+k-1}(n+j)^m F_n^{(k)} \\
&= \sum_{r=0}^{m} \binom{m}{r} j^{m-r} q^j \sum_{j=1}^{k} \sum_{n\geq 0} q^{n+k-1} n^r F_n^{(k)} \\
&= \sum_{r=0}^{m} \sum_{j=1}^{k} \binom{m}{r} j^{m-r} q^j \alpha_{r,k}. \qquad (2.1)
\end{aligned}$$

For reasons of comparison with [1] we rewrite it:

$$\begin{aligned}
\alpha_{m,k} &= q^k + \sum_{r=0}^{m-1} \sum_{j=1}^{k} \binom{m}{r} j^{m-r} q^j \alpha_{r,k} + \sum_{j=1}^{k} q^j \alpha_{m,k} \\
&= q^k + \sum_{r=0}^{m-1} \sum_{j=1}^{k} \binom{m}{r} j^{m-r} q^j \alpha_{r,k} + \alpha_{m,k} \frac{q^{k+1}-q}{q-1}
\end{aligned}$$

or

$$\frac{1-2q+q^{k+1}}{(1-q)q^k} \alpha_{m,k} = 1 + \sum_{r=0}^{m-1} \sum_{j=1}^{k} \binom{m}{r} j^{m-r} q^{j-k} \alpha_{r,k}.$$

For $q = \frac{1}{2}$, this is Proposition 1.2 from [1].

3. Generating Functions

Now we want to study the behavior of $\alpha_{m,k}$ for fixed k as $m \to \infty$. For that, we introduce the exponential generating functions

$$A_k(z) := \sum_{m\geq 0} \frac{\alpha_{m,k} z^m}{m!}.$$

From (2.1) we get by summing

$$\sum_{m\geq 0} \frac{z^m}{m!} \alpha_{m,k} = q^k \sum_{m\geq 0} \frac{z^m}{m!} + \sum_{r=0}^{m} \sum_{j=1}^{k} \sum_{m\geq 0} \frac{z^m}{m!} \binom{m}{r} j^{m-r} q^j \alpha_{r,k}$$

or

$$\begin{aligned}
A_k(z) &= q^k e^z + \sum_{j=1}^{k} q^j \sum_{r\geq 0} \sum_{m-r\geq 0} \frac{z^r \alpha_{r,k}}{r!} \frac{(jz)^{m-r}}{(m-r)!} \\
&= q^k e^z + \sum_{j=1}^{k} q^j e^{jz} A_k(z),
\end{aligned}$$

which leads to

$$A_k(z) = \frac{(1-qe^z)q^k e^z}{1-2qe^z+q^{k+1}e^{(k+1)z}}.$$

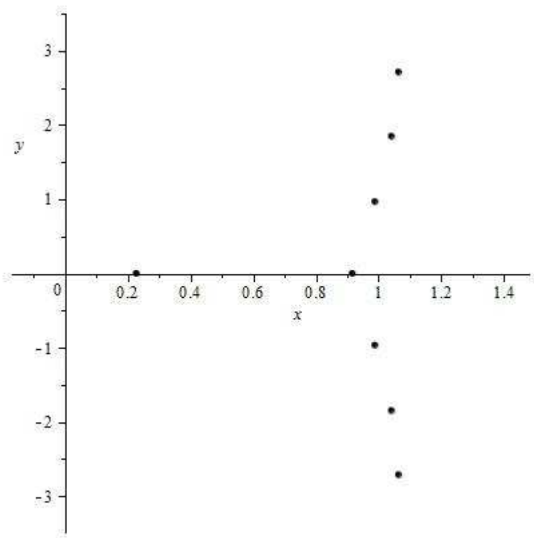
From this explicit form it is easy to extract coefficients:

$$\alpha_{m,k} = m![z^m]\frac{(1-qe^z)q^k e^z}{1-2qe^z+q^{k+1}e^{(k+1)z}}.$$

We report the first few instances for $q=\frac{1}{2}$, extending the list from [1]:

$$\alpha_{0,m}=1, \qquad \alpha_{1,m}=2^{k+1}-k-1,$$
$$\alpha_{2,m}=2^{2k+3}-(4k+3)2^{k+1}+k^2+2k-1,$$
$$\alpha_{3,m}=3\,2^{3k+4}-3(3k+2)2^{2k+3}+(12k^2+18k+1)2^{k+1}-k^3-3k^2+3k-1.$$

Now we turn to asymptotics. The function $A_k(z)$ is meromorphic, with only simple poles. The asymptotics of such functions is very well understood, see [2]. We must investigate the zeroes of the denominator $1-2qe^z+q^{k+1}e^{(k+1)z}$. In the following diagram, we depicted them for $q=0.4$ and $k=7$.



There is a dominant (=closest to the origin) zero, that we call $\rho_k(q)$; in the example it is about 0.22717. There is a second real solution (0.9163 in the example), which is canceled by a zero of the numerator. Some information of such dominant zeroes can be found e.g. in [3, 6]. For us, a crude approximation, obtained by *bootstrapping*, will be enough. For an explanation of the method, we refer to [4, 5]. We write $x=qe^z$, and write the equation as

$$x=\frac{1+x^{k+1}}{2}.$$

A first approximation is $x\sim\frac{1}{2}$; plugging this into the RHS, this leads to the improved approximation $\frac{1}{2}\big(1+\frac{1}{2^{k+1}}\big)$. In terms of the variable z, this means

$$\rho_k \sim \log\Big(\frac{1}{2q}\big(1+\frac{1}{2^{k+1}}\big)\Big) \sim -\log(2q)+\frac{1}{2^{k+1}}.$$

This evaluates to 0.22705 for our running example $q=0.4$, $k=7$.

The dominant part of $A_k(z)$ is thus,

$$A_k(z)=\frac{(1-qe^z)q^k e^z}{1-2qe^z+q^{k+1}e^{(k+1)z}} \sim \frac{C_k(q)}{1-z/\rho_k(q)},$$

with

$$\begin{aligned} C_k(q) &= \frac{(1-qe^{\rho})q^k e^{\rho}}{-2qe^{\rho}+(k+1)e^{(k+1)\rho}}\frac{1}{-\rho} \\ &= \frac{(1-qe^{\rho})q^k e^{\rho}}{-2qe^{\rho}+(k+1)[-1+2qe^{\rho}]}\frac{1}{-\rho} \\ &= \frac{(1-qe^{\rho})q^k e^{\rho}}{\rho(k+1-2kqe^{\rho})}, \end{aligned}$$

where we wrote $\rho = \rho_k(q)$ for convenience.

Consequently we get the desired asymptotics for $m \to \infty$ and fixed k:

$$\alpha_{m,k} \sim m! C_k(q)\big(\rho_k(q)\big)^{-m}.$$

References

[1] S. D. Dafnis and A. N. Philippou, *Infinite sums of weighted Fibonacci numbers of order k*, The Fibonacci Quarterly, **54.2** (2016), 149–153.

[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[3] K. Hare, H. Prodinger, and J. Shallit, *Three series for the generalized golden mean*, The Fibonacci Quarterly, **52.4** (2014), 307–313.

[4] D. E. Knuth, *The average time for carry propagation*, Nederl. Akad. Wetensch. Indag. Math., **40** (1978), 238–242.

[5] H. Prodinger, Analytic methods in *Handbook of Enumerative Combinatorics*, Miklos Bona, Ed., CRC Press, 2015, 173–252.

[6] H. Prodinger, *Two families of series of the generalized golden ratio*, The Fibonacci Quarterly, **53.1** (2015), 74–77.

MSC2010: 11B39, 11B37

Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa
E-mail address: `hproding@sun.ac.za`