DETERMINANTS OF RISING POWERS OF SECOND ORDER LINEAR RECURRENCE ENTRIES BY MEANS OF THE DESNANOT-JACOBI IDENTITY

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Abstract. We apply the Desnanot-Jacobi identity to give an alternative proof of the determinants whose entries are rising powers of the Fibonacci numbers given by Prodinger. We then generalize the determinants to include entries that are rising powers of the terms in a second order linear recurrence.

1. Introduction

In 1966, Carlitz [2] gave the following curious formula of the determinant whose entries are powers of the Fibonacci numbers:

$$|F^{r}_{n+i+j}|_{0\le i,j\le r} = (-1)^{(n+1)\binom{r+1}{2}}(F_1^r F_2^{r-1}\cdots F_r)^2 \cdot \prod_{i=0}^{r}\binom{r}{i}. \tag{1.1}$$

Recently, Tangboonduangjit and Thanatipanonda [4] have proved this result again where the indices of the entries are slightly more general and whose method of proof is different from the one provided by Carlitz. Another recent work which is related to the formula (1.1) is by Prodinger [3]. He considered the determinants whose entries are the rising powers of the Fibonacci numbers $F_m^{\langle r\rangle}$ defined by

$$F_m^{\langle r\rangle} = F_m F_{m+1}\cdots F_{m+r-1}.$$

In particular he proved the following formula:

$$|F^{\langle r\rangle}_{n+i+j}|_{0\le i,j\le r} = (-1)^{n\binom{r+1}{2}+\binom{r+2}{3}}(F_1F_2\cdots F_r)^{r+1}. \tag{1.2}$$

Prodinger's proof employed the LU-decomposition of the matrix whose entries are the rising powers of the Fibonacci numbers. In this work, we generalize the result of Prodinger further by making the dimension of the matrix to be independent from the rising power. This results in adding one more parameter to the formula (1.2) and we simply prove the result by induction. We then generalize the entries of the determinant to include the rising powers of the terms of a second order linear recurrence with constant coefficients. We used *Maple*©programs to facilitate some computations in this work. Thanatipanonda has included particular *Maple*©codes associated with this work at his personal website [5].

2. Main Result

For a $d\times d$ matrix, let $A_k(i,j)$ be the determinant of a $k\times k$ submatrix whose first entry is at the position of the ith row and the jth column of the original matrix. Then the Desnanot-Jacobi identity states [1]:

$$A_d(1,1)A_{d-2}(2,2) = A_{d-1}(1,1)A_{d-1}(2,2) - A_{d-1}(2,1)A_{d-1}(1,2).$$

This determinant identity is particularly useful when the determinants $A_d(i,j)$ have explicit formulas for all d, i, and j, which turn out to be the case in this work. We now state the main result.

Theorem 2.1. *Let* $D(n,r,d) = |F_{n+i+j}^{\langle r\rangle}|_{0\le i,j\le d-1}$ *for integers* n, r, *and* d *with* $r \ge 0$ *and* $d > 0$. *Then*

$$D(n,r,d) = (-1)^{n\binom{d}{2}+\binom{d+1}{3}} \prod_{i=1}^{d-1} (F_i F_{r+1-i})^{d-i} \cdot \prod_{i=d-1}^{2(d-1)} F_{n+i}^{\langle r+1-d\rangle}.$$

Proof. The proof is by induction on d. For the base case $d = 1$, we easily verify that $F_n^{\langle r\rangle} = D(n,r,1)$. For the case $d = 2$, we have

$$\begin{aligned}
|F_{n+i+j}^{\langle r\rangle}|_{0\le i,j\le 1} &= F_n^{\langle r\rangle} F_{n+2}^{\langle r\rangle} - F_{n+1}^{\langle r\rangle} F_{n+1}^{\langle r\rangle} \\
&= F_{n+1}^{\langle r-1\rangle} F_{n+2}^{\langle r-1\rangle} (F_n F_{n+r+1} - F_{n+r} F_{n+1}) \\
&= F_{n+1}^{\langle r-1\rangle} F_{n+2}^{\langle r-1\rangle} \cdot (-1)^{n+1} F_r F_1 \\
&= D(n,r,2),
\end{aligned}$$

where we apply Vajda's well-known identity (see, for example, [4]):

$$F_n F_{n+i+j} - F_{n+i} F_{n+j} = (-1)^{n+1} F_i F_j \tag{2.1}$$

in the third equality. For the induction step, we assume that the result is true for all square matrices of order no greater than d. Then, by the Desnanot-Jacobi identity and the induction hypotheses, we have

$$\begin{aligned}
&|F_{n+i+j}^{\langle r\rangle}|_{0\le i,j\le d} = \frac{D(n,r,d)D(n+2,r,d) - D(n+1,r,d)^2}{D(n+2,r,d-1)} \\
&= \frac{1}{(-1)^{(n+2)\binom{d-1}{2}+\binom{d}{3}}} \cdot \frac{\left[\prod_{i=1}^{d-1}(F_i F_{r+1-i})^{d-i}\right]^2}{\prod_{i=1}^{d-2}(F_i F_{r+1-i})^{d-1-i}} \\
&\quad \cdot \frac{\prod_{i=d-1}^{2d-2}\left(F_{n+i}^{\langle r+1-d\rangle} F_{n+2+i}^{\langle r+1-d\rangle} - F_{n+1+i}^{\langle r+1-d\rangle} F_{n+1+i}^{\langle r+1-d\rangle}\right)}{\prod_{i=d-2}^{2d-4} F_{n+2+i}^{\langle r+2-d\rangle}} \\
&= \frac{1}{(-1)^{(n+2)\binom{d-1}{2}+\binom{d}{3}}} \cdot F_{d-1}F_{r+2-d} \prod_{i=1}^{d-2} F_i F_{r+1-i} \cdot \prod_{i=1}^{d-1}(F_i F_{r+1-i})^{d-i} \\
&\quad \cdot \frac{\left[\prod_{i=d}^{2d-2} F_{n+i}^{\langle r+1-d\rangle} F_{n+1+i}^{\langle r+1-d\rangle}\right] \cdot \left(F_{n+d-1}^{\langle r+1-d\rangle} F_{n+2d}^{\langle r+1-d\rangle} - F_{n+d}^{\langle r+1-d\rangle} F_{n+2d-1}^{\langle r+1-d\rangle}\right)}{\prod_{i=d}^{2d-2} F_{n+i}^{\langle r+2-d\rangle}} \\
&= \frac{1}{(-1)^{(n+2)\binom{d-1}{2}+\binom{d}{3}}} \prod_{i=1}^{d-1}(F_i \cdot F_{r+1-i})^{d+1-i} \cdot \frac{\prod_{i=d}^{2d-2} F_{n+i}^{\langle r+1-d\rangle} F_{n+1+i}^{\langle r+1-d\rangle}}{\prod_{i=d}^{2d-2} F_{n+i}^{\langle r+2-d\rangle}} \\
&\quad \cdot F_{n+d}^{\langle r-d\rangle} F_{n+2d}^{\langle r-d\rangle} \cdot (F_{n+d-1}F_{n+r+d} - F_{n+r}F_{n+2d-1}).
\end{aligned}$$

Applying Vajda's identity (2.1) to the last expression, we have

$$\begin{aligned}
&|F^{\langle r\rangle}_{n+i+j}|_{0\le i,j\le d}\\
&= \frac{1}{(-1)^{(n+2)\binom{d-1}{2}+\binom{d}{3}}} \prod_{i=1}^{d-1}(F_iF_{r+1-i})^{d+1-i}\cdot \frac{\prod_{i=d}^{2d-2} F^{\langle r+1-d\rangle}_{n+1+i}}{\prod_{i=d}^{2d-2} F_{n+r-d+1+i}}\cdot F^{\langle r-d\rangle}_{n+d}F^{\langle r-d\rangle}_{n+2d}\cdot(-1)^{n+d}F_dF_{r+1-d}\\
&= \frac{(-1)^{n+d}}{(-1)^{(n+2)\binom{d-1}{2}+\binom{d}{3}}}\left[F_dF_{r+1-d}\prod_{i=1}^{d-1}(F_iF_{r+1-i})^{d+1-i}\right]\left[\prod_{i=d}^{2d-2} F^{\langle r-d\rangle}_{n+1+i}\right]F^{\langle r-d\rangle}_{n+d}F^{\langle r-d\rangle}_{n+2d}\\
&= (-1)^{n\binom{d+1}{2}+\binom{d+2}{3}}\prod_{i=1}^{d}(F_iF_{r+1-i})^{d+1-i}\prod_{i=d-1}^{2d-1} F^{\langle r-d\rangle}_{n+1+i}\\
&= D(n,r,d+1).
\end{aligned}$$

This completes the proof by induction. □

Note that by letting $d = r+1$ in Theorem 2.1 above, we obtain the original result of Prodinger, namely the identity (1.2).

3. Generalization to Second Order Linear Recurrence

In this section, we let W_n and U_n denote the second order linear recurrences with constant coefficients defined by

$$W_0 = a,\ W_1 = b, \quad \text{and} \quad W_n = c_1W_{n-1} + c_2W_{n-2} \quad \text{for any integer } n \ne 0,1$$

and

$$U_0 = 0,\ U_1 = 1, \quad \text{and} \quad U_n = c_1U_{n-1} + c_2U_{n-2} \quad \text{for any integer } n \ne 0,1,$$

where a, b, c_1 and c_2 are any constants.

Theorem 3.1. *Let $E(n,r,d) = |W^{\langle r\rangle}_{n+i+j}|_{0\le i,j\le d-1}$ for integers $n, r,$ and d with $r \ge 0$ and $d > 0$. Then*

$$E(n,r,d) = (-1)^{n\binom{d}{2}+\binom{d+1}{3}} c_2^{(n+d-2)\binom{d}{2}}\Delta^{\binom{d}{2}}\cdot\prod_{i=1}^{d-1}(U_iU_{r+1-i})^{d-i}\cdot\prod_{i=d-1}^{2(d-1)} W^{\langle r+1-d\rangle}_{n+i},$$

where $\Delta = \begin{vmatrix} W_1 & W_2\\ W_0 & W_1\end{vmatrix} = b^2 - c_1ab - c_2a^2.$

Proof. The proof is similar to that of Theorem 2.1. However, instead of using Vajda's identity, we use the following identity:

$$W_nW_{n+i+j} - W_{n+i}W_{n+j} = (-1)\cdot(-c_2)^n\cdot\Delta\cdot U_iU_j. \tag{3.1}$$

which can be found in the recent paper by Tangboonduangjit and Thanatipanonda [4]. □

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