EXTRAORDINARY SUBSETS: A GENERALIZATION

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ABSTRACT. For n a positive integer, a subset S of $[n]\,(=\{1,2,3,\ldots,n\})$ is called *extraordinary* if $|S|$ is equal to the smallest element of S. The number of such subsets S, for a given n, is counted by F_n, the nth Fibonacci number.

For positive integers k, n, where $1<k\leq n$, we now investigate those subsets S of $[n]$, where $|S|$ is equal to the kth smallest element of S. We call such subsets S *k-extraordinary*.

1. Extraordinary Subsets

For a positive integer n, let a_n count the number of subsets S of $[n]\,(=\{1,2,3,\ldots,n\})$, where $|S|$ is equal to the smallest element of S. We find that $a_1=1$, for the subset $\{1\}$, and that $a_2=1$, also for the subset $\{1\}$.

For $n\geq 3$, it follows that $a_n=a_{n-1}+a_{n-2}$;

1) If S is counted in a_n with $n\notin S$, then S is counted in a_{n-1}.
2) If S is counted in a_n with $n\in S$, upon removing n from S and then subtracting 1 from each remaining element of S, we obtain the corresponding subset counted in a_{n-2}. [To go in the reverse direction, for each subset S counted in a_{n-2}, increase each element by 1 and then add in the element n.] Consequently, $a_n=F_n$, $n\geq 1$, and there are F_n extraordinary subsets of $[n]$.

Alternately, for $n\geq 1$ and $1\leq k\leq\lfloor\frac{n-1}{2}\rfloor$, the number of extraordinary subsets, where k is the smallest element, is given by $\binom{n-i}{i-1}$. Consequently,

$$a_n=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}=F_n,\ \ n\geq 1.$$

(See [5, Theorem 12.4, pp. 155–156].)

[Note: The idea of an extraordinary subset is introduced in Exercise 50 [1, pp. 263–264] . Further results on these subsets are examined in [3, 4].]

2. k-Extraordinary Subsets

For positive integers n,k, where $1<k\leq n$, let $a_{n,k}$ count the number of subsets S of $[n]$, where $|S|$ is equal to the kth smallest element of S. We use $A_{n,k}$ to denote this collection of subsets of $[n]$, so $|A_{n,k}|=a_{n,k}$. [If we allow k to equal 1, then $a_{n,1}$ is simply $a_n\,(=F_n)$, as shown in Section 1.] When $n=6$ and $k=3$, for instance, we find that $a_{6,3}=7$, for the collection $A_{6,3}$ made up of

$$\{1,2,3\},\{1,2,4,5\},\{1,2,4,6\},\{1,3,4,5\},\{1,3,4,6\},\{2,3,4,5\},\{2,3,4,6\}.$$

For the general case, if $1 < k \leq n$, then

$$\begin{aligned} a_{n,k} &= \binom{k-1}{k-1}\binom{n-k}{0} + \binom{k}{k-1}\binom{n-k-1}{1} + \binom{k+1}{k-1}\binom{n-k-2}{2} + \cdots \\ &\quad + \binom{k-1+\lfloor\frac{n-k}{2}\rfloor}{k-1}\binom{n-k-\lfloor\frac{n-k}{2}\rfloor}{\lfloor\frac{n-k}{2}\rfloor} \\ &= \sum_{i=0}^{\lfloor\frac{n-k}{2}\rfloor} \binom{k-1+i}{k-1}\binom{n-k-i}{i}. \end{aligned}$$

Here, for example,

(i) The summand $\binom{k-1}{k-1}\binom{n-k}{0}$ in $a_{n,k}$ accounts for the unique subset $\{1, 2, 3, \ldots, k\}$ of $[n]$.

(ii) The summand $\binom{k}{k-1}\binom{n-k-1}{1}$ accounts for the subsets S of $[n]$ which include $k-1$ of the elements of $[k]$, the element $k+1$, and one element selected from the $n-k-1$ elements in $\{k+2, k+3, \ldots, n\}$. These are the subsets S of $[n]$ of size $k+1$, where $k+1$ is the kth smallest element of S.

In general, for $0 \leq i \leq \lfloor\frac{n-k}{2}\rfloor$, the summand $\binom{k-1+i}{k-1}\binom{n-k-i}{i}$ accounts for the subsets S of $[n]$ which include $k-1$ of the elements of $[k-1+i]$, the element $k+i$, and i elements selected from the $n-k-i$ elements in $\{k+i+1, k+i+2, \ldots, n\}$. These are the subsets S of $[n]$ of size k where $k+i$ is the kth smallest element of S.

To further investigate the values of $a_{n,k}$, for $1 \leq k \leq n$, we consider the results in Table 1, where we find $a_{n,k}$, for $1 \leq n \leq 12$ and $1 \leq k \leq n$.

Table 1

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	2	1	1									
4	3	3	1	1								
5	5	5	4	1	1							
6	8	10	7	5	1	1						
7	13	18	16	9	6	1	1					
8	21	33	31	23	11	7	1	1				
9	34	59	62	47	31	13	8	1	1			
10	55	105	119	101	66	40	15	9	1	1		
11	89	185	227	205	151	88	50	17	10	1	1	
12	144	324	426	414	321	213	113	61	19	11	1	1

Once again we see here that $a_{n,1} = a_{n-1,1} + a_{n-2,1}$, for $n \geq 3$. For the results in the first column of Table 1, where $k = 1$, are the Fibonacci numbers. However, for $k > 1$, we do not find that $a_{n,k} = a_{n-1,k} + a_{n-2,k}$.

When $k = 2$, for instance, we find that $a_{7,2} = 18 \neq 10+5 = a_{6,2}+a_{5,2}$. But we do notice that $a_{7,2} = 18 = 10+5+3 = a_{6,2}+a_{5,2}+a_{4,1}$. Likewise, although $a_{8,3} = 31 \neq 16+7 = a_{7,3}+3_{6,3}$, we do find that $a_{8,3} = 31 = 16+7+5+3 = a_{7,3}+a_{6,3}+a_{5,2}+a_{4,1}$. Lastly, we observe that $a_{10,4} = a_{9,4}+a_{8,4}+a_{7,3}+a_{6,2}+a_{5,1} = a_{9,4}+\sum_{i=1}^{4} a_{4+i,i}$. Is there a pattern here? Could it be that for $n \geq k > 1$

$$a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k} a_{n-k-2+i,i}?$$

To help establish this pattern, recall that $a_{n,k}$ counts those subsets S of $[n]$ where the kth smallest element of S equals $|S|$. Consider the case where $n = 9$ and $k = 5$. Among the $a_{9,5} = 31$ subsets S where the fifth smallest element of S equals $|S|$, there are $\binom{4}{4}\binom{3}{0} + \binom{5}{4}\binom{2}{1} = 11$ such subsets which do not include 9. These are precisely the subsets T of $[8]$ where the fifth smallest element of T equals $|T|$. This is $a_{8,5} = 11$. We then partition the remaining $a_{9,5} - a_{8,5}$ subsets counted in the collection $A_{9,5}$ according to the smallest element of $[9]$ that is missing from each of these subsets. For example, consider those subsets U of $[9]$ which contain 9 but do not contain 1. How many such subsets are there? Here the fifth smallest element is either 6 or 7 and the number of such subsets U is $\binom{4}{4}\binom{2}{0} + \binom{5}{4}\binom{1}{1}$, where $\binom{4}{4}\binom{2}{0}$ accounts for the subset $\{2, 3, 4, 5, 6, 9\}$ and $\binom{5}{4}\binom{1}{1}$ for the five subsets that contain four of the elements from $\{2, 3, 4, 5, 6\}$, the element 7, and the element 8 (along with 9). But then $\binom{4}{4}\binom{2}{0} + \binom{5}{4}\binom{1}{1} = 6 = a_{7,5}$.

[Note that we can also set up a one-to-one correspondence between the subsets counted in $A_{7,5}$ with these subsets U in $A_{9,5}$ as follows. Map U in $A_{9,5}$ to the subset U' in $A_{7,5}$ by deleting 9 from U and decreasing each of the remaining elements in U by 1 — or, by taking a subset V' in $A_{7,5}$ and corresponding it with the subset V in $A_{9,5}$, after increasing each element of V' by 1 and then adding in the element 9.]

For the general case, consider $n \geq k > 1$ and the collection $A_{n,k}$, where $|A_{n,k}| = a_{n,k}$.

(1) The collection of subsets $S \subseteq [n]$, where $S \in A_{n,k}$ and $n \notin S$ is the same collection of subsets $T \subseteq [n-1]$, where $T \in A_{n-1,k}$, and the number of these subsets T is counted by $a_{n-1,k}$.
(2) The remaining $a_{n,k} - a_{n-1,k}$ subsets in $A_{n,k}$ are then partitioned as follows.

For $1 \leq j \leq k$, let $A_{n,k,j}$ be the collection of subsets S in $A_{n,k}$ which contain n and where the smallest positive integer that is missing from S is j. So each such subset S contains n and $1, 2, 3, \ldots, j-1$ but not j. This then provides the partition

$$A_{n,k} = A_{n-1,k} \cup \left(\cup_{j=1}^{k} A_{n,k,j} \right).$$

For if $S \in A_{n,k,j}$ and $S \in A_{n,k,j'}$, where $j < j'$, then $S \in A_{n,k,j} \Rightarrow j \notin S$, while $S \in A_{n,k,j'} \Rightarrow j \in S$, so $A_{n,k,j} \cap A_{n,k,j'} = \emptyset$.

Further, for each $S \in A_{n,k,j}$, if we remove $1, 2, 3, \ldots, j-1$ and n, then subtract j from the remaining elements, we have the corresponding subset in $A_{n-j-1,k-j+1}$. Consequently,

$$\begin{aligned} |A_{n,k,j}| &= |A_{n-j-1,k-j+1}| = a_{n-j-1,k-j+1}, \text{ so} \\ a_{n,k} &= |A_{n,k}| = |A_{n-1,k}| + |\cup_{j=1}^{k} A_{n,k,j}| \\ &= |A_{n-1,k}| + \sum_{j=1}^{k} |A_{n-j-1,k-j+1}| \\ &= a_{n-1,k} + \sum_{j=1}^{k} a_{n-j-1,k-j+1}. \end{aligned}$$

If we let $i = k - j + 1$, then as j varies from 1 to k, i varies from k to 1, and

$$a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k} a_{n-k-2+i,i}.$$

3. Determining $a_{n,k}$ for Some Specific Values of k

(i) For $k = 1$ we know that $a_{n,k} = a_{n,1} = a_n = F_n$.

(ii) For $k = 2$, we have the recurrence relation

$$\begin{aligned} a_{n,2} &= a_{n-1,2} + a_{n-2,2} + a_{n-3,1} \\ &= a_{n-1,2} + a_{n-2,2} + F_{n-3} \\ &= a_{n-1,2} + a_{n-2,2} + \frac{1}{\sqrt{5}}\left(\alpha^{n-3} - \beta^{n-3}\right), \end{aligned}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. To solve this recurrence relation we use the techniques given in Chapter 7 of [1] or Chapter 10 of [2]. We find that $a_{n,2} = a_{n,2}^{(h)} + a_{n,2}^{(p)}$, where $a_{n,2}^{(h)}$ denotes the homogeneous part of the solution and $a_{n,2}^{(p)}$ the particular part. Since $a_{n,2}^{(h)} = c_1\,\alpha^n + c_2\,\beta^n$, it follows that $a_{n,2}^{(p)} = An\alpha^n + Bn\beta^n$. To determine A we substitute $a_{n,2} = An\alpha^n$ into the recurrence relation $a_{n,2} = a_{n-1,2} + a_{n-2,2} + \frac{1}{\sqrt{5}}\alpha^{n-3}$. This gives us $An\alpha^n = A(n-1)\alpha^{n-1} + A(n-2)\alpha^{n-2} + \frac{1}{\sqrt{5}}\alpha^{n-3}$, which leads to $An\alpha^3 = A(n-1)\alpha^2 + A(n-2)\alpha + \frac{1}{\sqrt{5}}$. Since $\alpha^2 = \alpha + 1$, we find that $A = \frac{3}{10} - \frac{1}{10}\sqrt{5}$, and then a similar calculation yields $B = \frac{3}{10} + \frac{1}{10}\sqrt{5}$. So

$$a_{n,2} = c_1\alpha^n + c_2\beta^n + \left(\frac{3}{10} - \frac{1}{10}\sqrt{5}\right)n\alpha^n + \left(\frac{3}{10} + \frac{1}{10}\sqrt{5}\right)n\beta^n.$$

From $a_{2,2} = 1$ and $a_{3,2} = 1$ we learn that $c_1 = \frac{\sqrt{5}}{25}$ and $c_2 = -\frac{\sqrt{5}}{25}$. So, for $n \geq 2$,

$$\begin{aligned} a_{n,2} &= \frac{\sqrt{5}}{25}\alpha^n - \frac{\sqrt{5}}{25}\beta^n + \left(\frac{3}{10} - \frac{1}{10}\sqrt{5}\right)n\alpha^n + \left(\frac{3}{10} + \frac{1}{10}\sqrt{5}\right)n\beta^n \\ &= \frac{1}{5}F_n + \frac{3}{10}nL_n - \frac{1}{2}nF_n, \end{aligned}$$

where L_n denotes the nth Lucas number.

(iii) Continuing for $k = 3$, we now consider the recurrence relation

$$\begin{aligned} a_{n,3} &= a_{n-1,3} + a_{n-2,3} + a_{n-3,2} + a_{n-4,1} \\ &= a_{n-1,3} + a_{n-2,3} + \left[\frac{1}{5}F_{n-3} + \frac{3}{10}(n-3)L_{n-3} - \frac{1}{2}(n-3)F_{n-3}\right] + F_{n-4}. \end{aligned}$$

Once again the homogeneous part of the solution has the form $c_1\alpha^n + c_2\beta^n$, but now the form of the particular part of the solution is given by $a_{n,3}^{(p)} = A_1 n\alpha^n + A_2 n^2\alpha^n + B_1 n\beta^n + B_1 n^2\beta^n$.

To determine A_1, A_2 we substitute $a_{n,3} = A_1 n\alpha^n + A_2 n^2\alpha^n$ into the recurrence relation $a_{n,3} = a_{n-1,3} + a_{n-2,3} + \frac{1}{5\sqrt{5}}\alpha^{n-3} + \frac{3}{10}n\alpha^{n-3} - \frac{9}{10}\alpha^{n-3} - \frac{1}{2\sqrt{5}}n\alpha^{n-3} + \frac{3}{2\sqrt{5}}\alpha^{n-3} + \frac{1}{\sqrt{5}}\alpha^{n-4}$. Upon dividing through by α^{n-4} and simplifying, this leads to

$$\begin{aligned} A_1 n\alpha^4 + A_2 n^2\alpha^4 &= A_1(n-1)\alpha^3 + A_2(n-1)^2\alpha^3 + A_1(n-2)\alpha^2 \\ &\quad + A_2(n-2)^2\alpha^2 + \frac{1}{5\sqrt{5}}\alpha + \frac{3}{10}n\alpha - \frac{9}{10}\alpha - \frac{1}{2\sqrt{5}}n\alpha + \frac{3}{2\sqrt{5}}\alpha + \frac{1}{\sqrt{5}}. \end{aligned}$$

When this expression is expanded, we compare the coefficients for n and n^0 (the constant terms) to learn that

$$\begin{aligned} 10A_2 + 4\sqrt{5}A_2 &= \frac{1}{10}\sqrt{5} - \frac{1}{10} \quad \text{and} \\ 5A_1 + 2\sqrt{5}A_1 - 8A_2 - 3\sqrt{5}A_2 &= \frac{2}{5} - \frac{2\sqrt{5}}{25}. \end{aligned}$$

Solving these equations simultaneously, we arrive at $A_1 = \frac{19}{100} - \frac{7\sqrt{5}}{100}$ and $A_2 = -\frac{3}{20} + \frac{\sqrt{5}}{100}$. Then a similar calculation yields $B_1 = \frac{19}{100} + \frac{7\sqrt{5}}{100}$ and $B_2 = -\frac{3}{20} - \frac{\sqrt{5}}{100}$. So $a_{n,3} = c_1\alpha^n + c_2\beta^n + \left(\frac{19}{100} - \frac{7\sqrt{5}}{100}\right)n\alpha^n + \left(-\frac{3}{20} + \frac{7\sqrt{5}}{100}\right)n^2\alpha^n + \left(\frac{19}{100} + \frac{7\sqrt{5}}{100}\right)n\beta^n + \left(-\frac{3}{20} - \frac{7\sqrt{5}}{100}\right)n^2\beta^n$. From $a_{3,3} = 1$ and $a_{4,3} = 1$ it follows that $c_1 = -\frac{1}{125}\sqrt{5}$ and $c_2 = \frac{1}{125}\sqrt{5}$. Consequently,

$$\begin{aligned} a_{n,3} &= -\frac{1}{125}\sqrt{5}\alpha^n + \frac{1}{125}\sqrt{5}\beta^n + \left(\frac{19}{100} - \frac{7\sqrt{5}}{100}\right)n\alpha^n \\ &\quad + \left(-\frac{3}{20} + \frac{7\sqrt{5}}{100}\right)n^2\alpha^n + \left(\frac{19}{100} + \frac{7\sqrt{5}}{100}\right)n\beta^n + \left(-\frac{3}{20} - \frac{7\sqrt{5}}{100}\right)n^2\beta^n \\ &= -\frac{1}{25}F_n + \frac{19}{100}nL_n - \frac{7}{20}nF_n - \frac{3}{20}n^2L_n + \frac{7}{20}n^2F_n. \end{aligned}$$

(iv) To determine $a_{n,4}$, we consider the recurrence relation

$$\begin{aligned} a_{n,4} &= a_{n-1,4} + a_{n-2,4} + a_{n-3,3} + a_{n-4,2} + a_{n-5,1} \\ &= a_{n-1,4} + a_{n-2,4} + \Big[-\frac{1}{25}F_{n-3} + \frac{19}{100}(n-3)L_{n-3} \\ &\quad - \frac{7}{20}(n-3)F_{n-3} - \frac{3}{20}(n-3)^2L_{n-3} + \frac{7}{20}(n-3)^2F_{n-3}\Big] \\ &\quad + \left[\frac{1}{5}F_{n-4} + \frac{3}{10}(n-4)L_{n-4} - \frac{1}{2}(n-4)F_{n-4}\right] + F_{n-5}. \end{aligned}$$

Here the solution has the form $a_{n,4} = c_1\alpha^n + c_2\beta^n + (A_1n + A_2n^2 + A_3n^3)\alpha^n + (B_1n + B_2n^2 + B_3n^3)\beta^n$. Calculations, somewhat more complicated but comparable to those that were performed in (ii) and (iii), yield

$$\begin{aligned} A_1 &= \frac{9}{100} - \frac{59}{1500}\sqrt{5}, \quad & A_2 &= -\frac{17}{100} + \frac{39}{500}\sqrt{5}, \quad & A_3 &= \frac{3}{50} - \frac{2}{75}\sqrt{5} \\ B_1 &= \frac{9}{100} + \frac{59}{1500}\sqrt{5}, \quad & B_2 &= -\frac{17}{100} - \frac{39}{500}\sqrt{5}, \quad & B_3 &= \frac{3}{50} + \frac{2}{75}\sqrt{5}. \end{aligned}$$

The initial conditions $a_{4,4} = 1$ and $a_{5,4} = 1$ then lead to $c_1 = -\frac{1}{125}\sqrt{5}$ and $c_2 = \frac{1}{125}\sqrt{5}$. So, for $n \geq 4$,

$$\begin{aligned} a_{n,4} &= -\frac{1}{125}\sqrt{5}\alpha^n + \frac{1}{125}\sqrt{5}\beta^n \\ &\quad + \left[\left(\frac{9}{100} - \frac{59}{1500}\sqrt{5}\right)n + \left(-\frac{17}{100} + \frac{39}{500}\sqrt{5}\right)n^2 + \left(\frac{3}{50} - \frac{2}{75}\sqrt{5}\right)n^3\right]\alpha^n \\ &\quad + \left[\left(\frac{9}{100} + \frac{59}{1500}\sqrt{5}\right)n + \left(-\frac{17}{100} - \frac{39}{500}\sqrt{5}\right)n^2 + \left(\frac{3}{50} + \frac{2}{75}\sqrt{5}\right)n^3\right]\beta^n \\ &= \left(-\frac{2}{15}n^3 + \frac{39}{100}n^2 - \frac{59}{300}n - \frac{1}{25}\right)F_n + \left(\frac{3}{50}n^3 - \frac{17}{100}n^2 + \frac{9}{100}n\right)L_n. \end{aligned}$$

4. Sums of Consecutive Column Entries

(i) For $k = 1$, it follows that for $n \geq 1$, $\sum_{i=1}^{n} a_{i,1} = \sum_{i=1}^{n} F_i = F_{n+2} - 1$. (See [5, Theorem 5.1, pp. 69–70].)

(ii) For $k = 2$, consider the n equations

$$a_{i,2} = a_{i-1,2} + a_{1-2,2} + F_{i-3}, \quad 3 \leq i \leq n+2.$$

Summing these n equations we find that

$$\begin{aligned} \sum_{i=1}^{n} a_{i,2} = \sum_{i=2}^{n} a_{i,2} &= a_{n+2,2} - a_{2,2} - \sum_{i=0}^{n-1} F_i \\ &= a_{n+2,2} - 1 - [F_{n+1} - 1] = a_{n+2,2} - F_{n+1} \\ &= a_{n+2,2} - a_{n+1,1}. \end{aligned}$$

(iii) When $k = 3$, the $n-1$ equations

$$a_{i,3} = a_{i-1,3} + a_{i-2,3} + a_{i-3,2} + F_{i-4}, \;\; 4 \leq i \leq n+2,$$

can be rewritten as

$$a_{i-2,3} = a_{i,3} - a_{i-1,3} - a_{i-3,2} - F_{i-4}, \;\; 4 \leq i \leq n+2.$$

Upon summing and simplifying we find that

$$\begin{aligned} \sum_{i=1}^{n} a_{i,3} = \sum_{i=2}^{n} a_{i,3} = \sum_{i=3}^{n} a_{i,3} &= a_{n+2,3} - a_{3,3} - \sum_{i=1}^{n-1} a_{i,2} - \sum_{i=0}^{n-2} F_i \\ &= a_{n+2,3} - 1 - [a_{n+1,2} - F_n] - [F_n - 1] = a_{n+2,3} - a_{n+1,2}. \end{aligned}$$

(iv) Continuing for $k = 4$, the system of $n-2$ equations

$$a_{i,4} = a_{i-1,4} + a_{i-2,4} + a_{i-3,3} + a_{i-4,2} + F_{i-5}, \;\; 5 \leq i \leq n+2$$

provides the corresponding system

$$a_{i-2,4} = a_{i,4} - a_{i-1,4} - a_{i-3,3} - a_{i-4,2} - F_{i-5}, \;\; 5 \leq i \leq n+2.$$

Upon summing these $n-2$ equations we arrive at the following.

$$\begin{aligned}\sum_{i=3}^{n} a_{i,4} = \sum_{i=4}^{n} a_{i,4} &= a_{n+2,4} - a_{4,4} - \sum_{i=2}^{n-1} a_{i,3} - \sum_{i=1}^{n-2} a_{i,2} - \sum_{i=0}^{n-3} F_i \\ &= a_{n+2,4} - 1 - [a_{n+1,3} - a_{n,2}] - [a_{n,2} - F_{n-1}] - [F_{n-1} - 1] \\ &= a_{n+2,4} - a_{n+1,3}\end{aligned}$$

(v) When $k=5$ we have the following system of $n-3$ equations — namely,

$$a_{i,5} = a_{i-1,5} + a_{i-2,5} + a_{i-3,4} + a_{i-4,3} + a_{i-5,2} + F_{i-6}, \;\; 6 \le i \le n+2.$$

This system can then be rewritten as

$$a_{i-2,5} = a_{i,5} - a_{i-1,5} - a_{i-3,4} - a_{i-4,3} - a_{i-5,2} - F_{i-6}, \;\; 6 \le i \le n+2.$$

When we sum these $n-3$ equations we find that

$$\begin{aligned}&\sum_{i=4}^{n} a_{i,5} = \sum_{i=5}^{n} a_{i,5} = a_{n+2,5} - a_{5,5} - \sum_{i=3}^{n-1} a_{i,4} - \sum_{i=2}^{n-2} a_{i,3} - \sum_{i=1}^{n-3} a_{i,2} - \sum_{i=0}^{n-4} F_i \\ &= a_{n+2,5} - 1 - [a_{n+1,4} - a_{n,3}] - [a_{n,3} - a_{n-1,2}] - [a_{n-1,2} - a_{n-2,1}] - [F_{n-2} - 1] \\ &= a_{n+2,5} - a_{n+1,4}.\end{aligned}$$

(vi) So now we assume that for $2 \le k \le r$, $\sum_{i=k}^{n} a_{i,k} = a_{n+2,k} - a_{n+1,k-1}$, and consider the following system of $n-(k-2) = n-(r+1-2) = n-r+1$ equations.

$$\begin{aligned}a_{i,r+1} = a_{i-1,r+1} &+ a_{i-2,r+1} + a_{i-3,r} + a_{i-4,r-1} \\ &- \cdots - a_{i-r,3} - a_{i-(r+1),2} - F_{i-(r+2)}, \quad r+2 \le i \le n+2.\end{aligned}$$

Summing these $n-r+1$ equations and simplifying then leads us to

$$\begin{aligned}\sum_{i=r}^{n} a_{i,r+1} = \sum_{i=r+1}^{n} a_{i,r+1} &= a_{n+2,r+1} - a_{r+1,r+1} - \sum_{i=r-1}^{n-1} a_{i,r} \\ &- \sum_{i=r-2}^{n-2} a_{i,r-1} - \ldots - \sum_{i=2}^{n-(r-2)} a_{i,3} - \sum_{i=1}^{n-(r-1)} a_{i,2} - \sum_{i=0}^{n-r} F_i \\ &= a_{n+2,r+1} - 1 - [a_{n+1,r} - a_{n,r-1}] - [a_{n,r-1} - a_{n-1,r-2}] \\ &- \ldots - [a_{n-r+4,3} - a_{n-r+3,2}] - [a_{n-r+3,2} - a_{n-r+2,1}] \\ &- [F_{n-r+2} - 1] = a_{n+2,r+1} - a_{n+1,r}.\end{aligned}$$

5. The Sum of All the Row Entries

We see from Table 1 that for $1 \le n \le 12$, $\sum_{k=1}^{n} a_{n,k} = 2^{n-1}$. Assuming that this pattern continues for all $1 \le n \le r-1$, we consider the following for the row where $n = r$.

$$\begin{aligned}
a_{r,1} &= a_{r-1,1} + a_{r-2,1} \\
a_{r,2} &= a_{r-1,2} + a_{r-2,2} + a_{r-3,1} \\
a_{r,3} &= a_{r-1,3} + a_{r-2,3} + a_{r-3,2} + a_{r-4,1} \\
&\vdots \\
a_{r,i} &= a_{r-1,i} + a_{r-2,i} + a_{r-3,i-1} + \cdots + a_{r-i-1,1} \\
&\vdots \\
a_{r,r-2} &= a_{r-1,r-2} + a_{r-2,r-2} + a_{r-3,r-3} + \cdots + a_{1,1} \\
a_{r,r-1} &= a_{r-1,r-1}.
\end{aligned}$$

Upon adding the entries in the given columns we now find that

$$\begin{aligned}
\sum_{k=1}^{r} a_{r,k} &= \sum_{k=1}^{r-1} a_{r,k} + 1 \\
&= \sum_{k=1}^{r-1} a_{r-1,k} + \sum_{k=1}^{r-2} a_{r-2,k} + \sum_{k=1}^{r-3} a_{r-3,k} + \cdots + \sum_{k=1}^{1} a_{1,k} + 1 \\
&= (2^{r-2} + 2^{r-3} + 2^{r-4} + \cdots + 2^0) + 1 = 2^{r-1}.
\end{aligned}$$

Consequently we find that for a given value of $n \ge 1$, exactly half of the subsets of $[n]$ are k-extraordinary for some (unique) k, where $1 \le k \le n$. This suggests that we could have arrived at this result by counting those subsets of $[n]$ which are not k-extraordinary for each $1 \le k \le n$. The first such subset would be $\emptyset$, the null set. For $k = 1$, only $\{1\}$ is 1-extraordinary and the $n-1$ subsets $\{m\}$, for $2 \le m \le n$, are not 1-extraordinary – nor are they k-extraordinary for $k > 1$ since a k-extraordinary subsets needs to contain at least k elements. In general, for $1 \le i \le n-1$, the $\binom{n-1}{i}$ subsets of size i which do not contain i cannot be i-extraordinary because here the ith smallest element is greater than i. Nor can such a subset be k-extraordinary for $k \ne i$. If $k < i$ the subset has too many elements, while if $k > i$, then it has too few. Therefore, it follows from the binomial theorem that the number of subsets of $[n]$ which are not k-extraordinary for some $1 \le k \le n$ is $\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$.

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