A CLOSED FORM FORMULATION FOR THE GENERAL TERM OF A SCALED TRIPLE POWER PRODUCT RECURRENCE SEQUENCE

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Abstract. The closed form for the general term of a scaled triple power product recurrence sequence—with arbitrary initial values—is formulated for the first time, and an open problem offered. Upon removal of a recursion parameter results are shown to collapse correctly, as anticipated, in line with the lower order double power product case studied previously.

1. Introduction

1.1. **Background.** This paper is motivated by analysis of a power product recurrence

$$z_n = s(z_{n-1})^p(z_{n-2})^q, \qquad n \geq 2, \tag{1.1}$$

which produces a sequence $\{z_n\}_{n=0}^{\infty} = \{z_n\}_0^{\infty} = \{z_n(a,b,p,q;s)\}_0^{\infty}$ with first few terms

$$\{z_n(a,b,p,q;s)\}_0^{\infty} = \{a, b, a^q b^p s, a^{pq} b^{p^2+q} s^{p+1}, a^{p^2q+q^2} b^{p^3+2pq} s^{p^2+p+q+1}, \\ a^{p^3q+2pq^2} b^{p^4+3p^2q+q^2} s^{p^3+p^2+p(2q+1)+q+1}, \ldots\}, \tag{1.2}$$

where $z_0 = a$, $z_1 = b$ are initial values and $s \in \mathbb{Z}^+$ is an arbitrary scaling variable. In [2] it was established that, for $p+q \neq 1$, the general $(n+1)$th term of the sequence takes the form

$$z_n(a,b,p,q;s) = a^{\alpha_n^{[2]}(p,q)} b^{\beta_n^{[2]}(p,q)} s^{m_n^{[2]}(p,q)}, \qquad n \geq 0, \tag{1.3}$$

where ($\alpha_0^{[2]}(p,q) = 1$, $\beta_0^{[2]}(p,q) = 0 = m_0^{[2]}(p,q)$ and $\alpha_1^{[2]}(p,q) = 0 = m_1^{[2]}(p,q)$, $\beta_1^{[2]}(p,q) = 1$ being given by $z_0 = a = a^1b^0s^0$, $z_1 = b = a^0b^1s^0$)

$$\alpha_n^{[2]}(p,q) = p^{n-2}qP_{n-2}(-q/p^2), \\ \beta_n^{[2]}(p,q) = p^{n-1}P_{n-1}(-q/p^2), \qquad n \geq 2, \tag{1.4}$$

and

$$m_n^{[2]}(p,q) = [\alpha_n^{[2]}(p,q) + \beta_n^{[2]}(p,q) - 1]/(p+q-1), \qquad n \geq 2, \tag{1.5}$$

with

$$P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-x)^i \tag{1.6}$$

(a so called Catalan polynomial) characterizing the formulation. The origins of such a recursion lie with a very short paper by M. W. Bunder published in the mid-1970s (with multiplying variable s absent). The reader is directed to the authors' article [2], and references therein, that chart recent instances of (1.1) examined—these include scaled ($s > 1$) and non-scaled ($s = 1$) versions in combination with conditions on (and values for) the recurrence parameters p and q.

1.2. **This Paper.** This paper applies the methodology of [2] to the more complex three deep Bunder-type recurrence

$$z_n = s(z_{n-1})^p(z_{n-2})^q(z_{n-3})^r, \qquad n \geq 3, \tag{1.7}$$

for which exponent functions $\alpha_n^{[3]}(p,q,r), \beta_n^{[3]}(p,q,r), \gamma_n^{[3]}(p,q,r)$ and $m_n^{[3]}(p,q,r)$ are sought for the general sequence term

$$z_n(a,b,c,p,q,r;s) = a^{\alpha_n^{[3]}(p,q,r)} b^{\beta_n^{[3]}(p,q,r)} c^{\gamma_n^{[3]}(p,q,r)} s^{m_n^{[3]}(p,q,r)}, \qquad n \geq 0, \tag{1.8}$$

generated by (1.7) with $z_0 = a$, $z_1 = b$, $z_2 = c$. In addition to an increased level of algebraic complexity in considering (1.7) as compared to (1.1), the form of the resulting exponent functions in (1.8) is here based naturally on a parameter

$$c_u(p,q,r) = \sum_{l_1+2l_2+3l_3=u} \binom{l_1+l_2+l_3}{l_1,l_2,l_3} p^{l_1}q^{l_2}r^{l_3} = \sum_{l_1+2l_2+3l_3=u} \frac{(l_1+l_2+l_3)!}{l_1!l_2!l_3!} p^{l_1}q^{l_2}r^{l_3}, \tag{1.9}$$

rather than the Catalan polynomial of (1.6) to which it reduces as a special case (see Lemma 2.1 later). It is instructive to list the first few instances of our parameter as

$$\begin{aligned}
c_0(p,q,r) &= 1,\\
c_1(p,q,r) &= p,\\
c_2(p,q,r) &= p^2+q,\\
c_3(p,q,r) &= p^3+2pq+r,\\
c_4(p,q,r) &= p^4+3p^2q+2pr+q^2,\\
c_5(p,q,r) &= p^5+4p^3q+3p^2r+3pq^2+2qr,\\
c_6(p,q,r) &= p^6+5p^4q+4p^3r+6p^2q^2+6pqr+q^3+r^2,
\end{aligned} \tag{1.10}$$

and so on, and to see the actual formulation of $c_5(p,q,r)$, for example, in full as

$$\begin{aligned}
c_5(p,q,r) &= \sum_{l_1+2l_2+3l_3=5} \binom{l_1+l_2+l_3}{l_1,l_2,l_3} p^{l_1}q^{l_2}r^{l_3}\\
&= \binom{5}{5,0,0}p^5q^0r^0 + \binom{4}{3,1,0}p^3q^1r^0 + \binom{3}{2,0,1}p^2q^0r^1\\
&\qquad + \binom{3}{1,2,0}p^1q^2r^0 + \binom{2}{0,1,1}p^0q^1r^1\\
&= p^5+4p^3q+3p^2r+3pq^2+2qr.
\end{aligned} \tag{1.11}$$

Noting that $\alpha_n^{[3]}(p,q,r), \beta_n^{[3]}(p,q,r), \gamma_n^{[3]}(p,q,r)$ and $m_n^{[3]}(p,q,r)$ are, for $n = 0,1,2$, defined through its initial values, we finish this introductory section by giving the first few explicit terms of the sequence $\{z_n(a,b,c,p,q,r;s)\}_0^\infty$:

$$\begin{aligned}
&\{z_n(a,b,c,p,q,r;s)\}_0^\infty\\
&\quad = \{a,b,c,a^rb^qc^ps, a^{pr}b^{pq+r}c^{p^2+q}s^{p+1}, a^{(p^2+q)r}b^{p^2q+pr+q^2}c^{p^3+2pq+r}s^{p^2+p+q+1},\\
&\qquad a^{(p^3+2pq+r)r}b^{p^3q+p^2r+2pq^2+2qr}c^{p^4+3p^2q+2pr+q^2}s^{p^3+p^2+p(2q+1)+q+r+1},\ldots\},
\end{aligned} \tag{1.12}$$

emphasizing that our formulation will be seen to be reliant on values of the powers p,q,r in the recurrence equation (1.7) conforming to the condition

$$p+q+r \neq 1. \tag{1.13}$$

2. Analysis

2.1. **Formulation.** Combining (1.7) and (1.8), then

$$\begin{aligned} z_n &= s(z_{n-1})^p(z_{n-2})^q(z_{n-3})^r \\ &= s[a^{\alpha^{[3]}_{n-1}(p,q,r)}b^{\beta^{[3]}_{n-1}(p,q,r)}c^{\gamma^{[3]}_{n-1}(p,q,r)}s^{m^{[3]}_{n-1}(p,q,r)}]^p \\ &\qquad \times \ [a^{\alpha^{[3]}_{n-2}(p,q,r)}b^{\beta^{[3]}_{n-2}(p,q,r)}c^{\gamma^{[3]}_{n-2}(p,q,r)}s^{m^{[3]}_{n-2}(p,q,r)}]^q \\ &\qquad\qquad \times \ [a^{\alpha^{[3]}_{n-3}(p,q,r)}b^{\beta^{[3]}_{n-3}(p,q,r)}c^{\gamma^{[3]}_{n-3}(p,q,r)}s^{m^{[3]}_{n-3}(p,q,r)}]^r, \end{aligned} \tag{2.1}$$

in turn giving individual recurrences for the exponent functions of (1.8) as

$$\begin{aligned} \alpha^{[3]}_n(p,q,r) &= p\alpha^{[3]}_{n-1}(p,q,r) + q\alpha^{[3]}_{n-2}(p,q,r) + r\alpha^{[3]}_{n-3}(p,q,r), \\ \beta^{[3]}_n(p,q,r) &= p\beta^{[3]}_{n-1}(p,q,r) + q\beta^{[3]}_{n-2}(p,q,r) + r\beta^{[3]}_{n-3}(p,q,r), \\ \gamma^{[3]}_n(p,q,r) &= p\gamma^{[3]}_{n-1}(p,q,r) + q\gamma^{[3]}_{n-2}(p,q,r) + r\gamma^{[3]}_{n-3}(p,q,r), \\ m^{[3]}_n(p,q,r) &= pm^{[3]}_{n-1}(p,q,r) + qm^{[3]}_{n-2}(p,q,r) + rm^{[3]}_{n-3}(p,q,r) + 1, \end{aligned} \tag{2.2}$$

which, defining a matrix

$$\mathbf{F}_n(p,q,r) = \begin{pmatrix} \alpha^{[3]}_n(p,q,r) & \beta^{[3]}_n(p,q,r) & \gamma^{[3]}_n(p,q,r) & m^{[3]}_n(p,q,r) \\ \alpha^{[3]}_{n-1}(p,q,r) & \beta^{[3]}_{n-1}(p,q,r) & \gamma^{[3]}_{n-1}(p,q,r) & m^{[3]}_{n-1}(p,q,r) \\ \alpha^{[3]}_{n-2}(p,q,r) & \beta^{[3]}_{n-2}(p,q,r) & \gamma^{[3]}_{n-2}(p,q,r) & m^{[3]}_{n-2}(p,q,r) \end{pmatrix}, \tag{2.3}$$

we capture as

$$\mathbf{F}_n(p,q,r) = \mathbf{H}(p,q,r)\mathbf{F}_{n-1}(p,q,r) + \mathbf{K}, \tag{2.4}$$

where

$$\mathbf{H}(p,q,r) = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Application of (2.4) on itself $n-3$ times yields

$$\mathbf{F}_n(p,q,r) = \mathbf{H}^{n-2}(p,q,r)\mathbf{S} + \mathbf{T}_n(p,q,r)\mathbf{K}, \tag{2.6}$$

with

$$\mathbf{S} = \mathbf{F}_2(p,q,r) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

absorbing the initial values of the sequence $\{z_n(a,b,c,p,q,r;s)\}_0^\infty$, and (denoting the 3-square identity matrix as $\mathbf{I}_3$)

$$\mathbf{T}_n(p,q,r) = \mathbf{H}^{n-3}(p,q,r) + \mathbf{H}^{n-4}(p,q,r) + \cdots + \mathbf{H}^2(p,q,r) + \mathbf{H}(p,q,r) + \mathbf{I}_3; \tag{2.8}$$

with reference to (2.6), our objective is to find closed forms for the matrix entries of $\mathbf{H}^{n-2}(p,q,r)$ and $\mathbf{T}_n(p,q,r)$, which latter has, directly from (2.8), the compact form

$$\mathbf{T}_n(p,q,r) = [\mathbf{H}(p,q,r) - \mathbf{I}_3]^{-1}[\mathbf{H}^{n-2}(p,q,r) - \mathbf{I}_3], \tag{2.9}$$

and necessarily requires $p+q+r \neq 1$ (as in (1.13)) to ensure $\mathbf{H}(p,q,r) - \mathbf{I}_3$ is non-singular.

2.2. **Algebraic Details.** First, noting that

$$[\mathbf{I}_3 - t\mathbf{H}(p,q,r)]^{-1} = \sum_{u\geq 0}[t\mathbf{H}(p,q,r)]^u, \tag{2.10}$$

then

$$\mathbf{H}^u(p,q,r) = [t^u]\{[\mathbf{I}_3 - t\mathbf{H}(p,q,r)]^{-1}\}, \qquad u \geq 0. \tag{2.11}$$

It is straightforward to find that

$$[\mathbf{I}_3 - t\mathbf{H}(p,q,r)]^{-1} = \frac{1}{1-pt-qt^2-rt^3}\begin{pmatrix} 1 & (q+rt)t & rt \\ t & 1-pt & rt^2 \\ t^2 & (1-pt)t & 1-pt-qt^2 \end{pmatrix} \tag{2.12}$$

which, writing

$$F(t;p,q,r) = (1-pt-qt^2-rt^3)^{-1} \tag{2.13}$$

and

$$\mathbf{X}(p,q,r) = \begin{pmatrix} 0 & q & r \\ 1 & -p & 0 \\ 0 & 1 & -p \end{pmatrix},$$

$$\mathbf{Y}(p,q,r) = \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & r \\ 1 & -p & -q \end{pmatrix}, \tag{2.14}$$

means that (2.12) may be expressed as

$$[\mathbf{I}_3 - t\mathbf{H}(p,q,r)]^{-1} = F(t;p,q,r)[\mathbf{I}_3 + \mathbf{X}(p,q,r)t + \mathbf{Y}(p,q,r)t^2]. \tag{2.15}$$

Thus, by (2.11),

$$\begin{aligned}\mathbf{H}^u(p,q,r) &= [t^u]\{F(t;p,q,r)\}\mathbf{I}_3 \\ &\quad + [t^{u-1}]\{F(t;p,q,r)\}\mathbf{X}(p,q,r) + [t^{u-2}]\{F(t;p,q,r)\}\mathbf{Y}(p,q,r) \\ &= c_u(p,q,r)\mathbf{I}_3 + c_{u-1}(p,q,r)\mathbf{X}(p,q,r) + c_{u-2}(p,q,r)\mathbf{Y}(p,q,r),\end{aligned} \tag{2.16}$$

since the parameter $c_u(p,q,r)$ has the property that the sequence of functions $\{c_u(p,q,r)\}_{u=0}^{\infty}$ has $F(t;p,q,r)$ as its ordinary generating function, with

$$[t^u]\{F(t;p,q,r)\} = c_u(p,q,r), \qquad u \geq 0, \tag{2.17}$$

in standard fashion (for completeness, the reader unfamiliar to this type of observation is referred to Appendix A for a formal argument). Setting $u = n-2$ in (2.16) gives

$$\mathbf{H}^{n-2}(p,q,r) = c_{n-2}(p,q,r)\mathbf{I}_3 + c_{n-3}(p,q,r)\mathbf{X}(p,q,r) + c_{n-4}(p,q,r)\mathbf{Y}(p,q,r) \tag{2.18}$$

and, writing down (directly from (2.12))

$$[\mathbf{H}(p,q,r) - \mathbf{I}_3]^{-1} = \frac{1}{p+q+r-1}\mathbf{Z}(p,q,r) \tag{2.19}$$

where

$$\mathbf{Z}(p,q,r) = \begin{pmatrix} 1 & q+r & r \\ 1 & 1-p & r \\ 1 & 1-p & 1-p-q \end{pmatrix}, \tag{2.20}$$

then $\mathbf{T}_n(p,q,r)$ (2.9) is, using (2.18) and (2.19),

$$\begin{aligned}\mathbf{T}_n(p,q,r) &= [(c_{n-2}(p,q,r)-1)\mathbf{Z}(p,q,r) + c_{n-3}(p,q,r)\mathbf{Z}(p,q,r)\mathbf{X}(p,q,r) \\ &\quad + c_{n-4}(p,q,r)\mathbf{Z}(p,q,r)\mathbf{Y}(p,q,r)]/(p+q+r-1).\end{aligned} \tag{2.21}$$

Suppose the matrix $\mathbf{T}_n(p,q,r)$ has a general form

$$\mathbf{T}_n(p,q,r) = \begin{pmatrix} T_1(p,q,r,n) & T_2(p,q,r,n) & T_3(p,q,r,n) \\ T_4(p,q,r,n) & T_5(p,q,r,n) & T_6(p,q,r,n) \\ T_7(p,q,r,n) & T_8(p,q,r,n) & T_9(p,q,r,n) \end{pmatrix}. \tag{2.22}$$

We need only calculate those functions $T_1(p,q,r,n), T_4(p,q,r,n)$ and $T_7(p,q,r,n)$, since

$$\mathbf{T}_n(p,q,r)\mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & T_1(p,q,r,n) \\ 0 & 0 & 0 & T_4(p,q,r,n) \\ 0 & 0 & 0 & T_7(p,q,r,n) \end{pmatrix}, \tag{2.23}$$

and they alone contribute to $\mathbf{F}_n(p,q,r)$ (2.6). To that end the product matrices of (2.21) are noted to be

$$\mathbf{Z}(p,q,r)\mathbf{X}(p,q,r) = \begin{pmatrix} q+r & \cdot & \cdot \\ 1-p & \cdot & \cdot \\ 1-p & \cdot & \cdot \end{pmatrix},$$

$$\mathbf{Z}(p,q,r)\mathbf{Y}(p,q,r) = \begin{pmatrix} r & \cdot & \cdot \\ r & \cdot & \cdot \\ 1-p-q & \cdot & \cdot \end{pmatrix}, \tag{2.24}$$

yielding

$$\begin{aligned} T_1(p,q,r,n) &= [c_{n-2}(p,q,r) + (q+r)c_{n-3}(p,q,r) + rc_{n-4}(p,q,r) - 1]/(p+q+r-1), \\ T_4(p,q,r,n) &= [c_{n-2}(p,q,r) + (1-p)c_{n-3}(p,q,r) + rc_{n-4}(p,q,r) - 1]/(p+q+r-1), \\ T_7(p,q,r,n) &= [c_{n-2}(p,q,r) + (1-p)c_{n-3}(p,q,r) \\ &\qquad + \ (1-p-q)c_{n-4}(p,q,r) - 1]/(p+q+r-1). \end{aligned} \tag{2.25}$$

Having obtained those necessary elements of $\mathbf{T}_n(p,q,r)\mathbf{K}$, then completion of the r.h.s. of (2.6) requires $\mathbf{H}^{n-2}(p,q,r)\mathbf{S}$ which, from (2.18), is simply

$$\mathbf{H}^{n-2}(p,q,r)\mathbf{S} = c_{n-2}(p,q,r)\mathbf{S} + c_{n-3}(p,q,r)\mathbf{X}(p,q,r)\mathbf{S} + c_{n-4}(p,q,r)\mathbf{Y}(p,q,r)\mathbf{S}, \tag{2.26}$$

and is yielded by (2.7) and (2.14). This result and (2.23) now deliver $\mathbf{F}_n(p,q,r)$ (2.6) as

$$\mathbf{F}_n(p,q,r) = \begin{pmatrix} rc_{n-3}(p,q,r) & qc_{n-3}(p,q,r) + rc_{n-4}(p,q,r) & c_{n-2}(p,q,r) & T_1(p,q,r,n) \\ rc_{n-4}(p,q,r) & qc_{n-4}(p,q,r) + rc_{n-5}(p,q,r) & c_{n-3}(p,q,r) & T_4(p,q,r,n) \\ rc_{n-5}(p,q,r) & qc_{n-5}(p,q,r) + rc_{n-6}(p,q,r) & c_{n-4}(p,q,r) & T_7(p,q,r,n) \end{pmatrix}, \tag{2.27}$$

so that, by (2.3), we simply read off the required exponent functions of (1.8) as

$$\begin{aligned} \alpha_n^{[3]}(p,q,r) &= rc_{n-3}(p,q,r), \\ \beta_n^{[3]}(p,q,r) &= qc_{n-3}(p,q,r) + rc_{n-4}(p,q,r), \\ \gamma_n^{[3]}(p,q,r) &= c_{n-2}(p,q,r), \\ m_n^{[3]}(p,q,r) &= T_1(p,q,r,n) \\ &= [\alpha_n^{[3]}(p,q,r) + \beta_n^{[3]}(p,q,r) + \gamma_n^{[3]}(p,q,r) - 1]/(p+q+r-1), \end{aligned} \tag{2.28}$$

remarking that in order to align the form of some of the second and third row terms of (2.27) with those of the first—in accordance with (2.3)—we have utilized as appropriate the linear relation

$$c_u(p,q,r) = pc_{u-1}(p,q,r) + qc_{u-2}(p,q,r) + rc_{u-3}(p,q,r), \qquad n \geq 3, \tag{2.29}$$

that is readily shown to hold (Appendix B). Defining $c_{-1}(p,q,r) = 0$ then (2.28) is valid, and generates terms of the sequence $\{z_n(a,b,c,p,q,r;s)\}_0^\infty$, for all $n \geq 3$ (that is, beyond the sequence initial values), as the reader is invited to check. Note also that, for completeness and self-consistency within (2.27),

$$T_1(p,q,r,n-1) = T_4(p,q,r,n) = T_7(p,q,r,n+1), \tag{2.30}$$

these relations being immediate from (2.29).

As a check on the analysis leading to the formulations of (2.28) we see, for example, that $z_6(a,b,c,p,q,r;s)$ agrees with the computer output of (1.12). From (1.8) we write, using (in order) (2.28) and (1.10),

$$\begin{aligned} &z_6(a,b,c,p,q,r;s)\\ &\quad= a^{\alpha_6^{[3]}(p,q,r)} b^{\beta_6^{[3]}(p,q,r)} c^{\gamma_6^{[3]}(p,q,r)} s^{m_6^{[3]}(p,q,r)}\\ &\quad= a^{rc_3(p,q,r)} b^{qc_3(p,q,r)+rc_2(p,q,r)} c^{c_4(p,q,r)} s^{T_1(p,q,r,6)}\\ &\quad= a^{r(p^3+2pq+r)} b^{q(p^3+2pq+r)+r(p^2+q)} c^{p^4+3p^2q+2pr+q^2} s^{T_1(p,q,r,6)}\\ &\quad= a^{(p^3+2pq+r)r} b^{p^3q+p^2r+2pq^2+2qr} c^{p^4+3p^2q+2pr+q^2} s^{T_1(p,q,r,6)}\\ &\quad= a^{(p^3+2pq+r)r} b^{p^3q+p^2r+2pq^2+2qr} c^{p^4+3p^2q+2pr+q^2} s^{p^3+p^2+p(2q+1)+q+r+1}, \end{aligned} \tag{2.31}$$

as in (1.12), since

$$\begin{aligned} &T_1(p,q,r,6)\\ &\quad= [\alpha_6^{[3]}(p,q,r) + \beta_6^{[3]}(p,q,r) + \gamma_6^{[3]}(p,q,r) - 1]/(p+q+r-1)\\ &\quad= [rc_2(p,q,r) + (q+r)c_3(p,q,r) + c_4(p,q,r) - 1]/(p+q+r-1)\\ &\quad= [r(p^2+q) + (q+r)(p^3+2pq+r) + p^4 + 3p^2q + 2pr + q^2 - 1]/(p+q+r-1)\\ &\quad\ \vdots\\ &\quad= p^3 + p^2 + p(2q+1) + q + r + 1, \end{aligned} \tag{2.32}$$

after some algebraic manipulation; many terms of the sequence $\{z_n(a,b,c,p,q,r;s)\}_0^\infty$ have been verified similarly, via computations, for $n \geq 3$.

2.3. **The Case $r = 0$.** It is clear from computer output that setting $r = 0$ in the recursion (1.7) causes the sequence general term (1.8) to collapse in some sense to that of (1.3) according to

$$\begin{aligned} \beta_n^{[3]}(p,q,0) &= \alpha_{n-1}^{[2]}(p,q),\\ \gamma_n^{[3]}(p,q,0) &= \beta_{n-1}^{[2]}(p,q), \end{aligned} \tag{2.33}$$

together with

$$m_n^{[3]}(p,q,0) = m_{n-1}^{[2]}(p,q), \tag{2.34}$$

which all hold for $n \geq 1$ and are intuitively obvious. It is anticipated that $\alpha_n^{[3]}(p,q,0) = 0$, too (as given by (2.28)), and indeed this can be established as follows by way of confirmation, for we see, using (1.5) and (2.28), that (2.34) reads

$$\begin{aligned} 0 &= m_n^{[3]}(p,q,0) - m_{n-1}^{[2]}(p,q)\\ &= \alpha_n^{[3]}(p,q,0) - \alpha_{n-1}^{[2]}(p,q) + \beta_n^{[3]}(p,q,0) - \beta_{n-1}^{[2]}(p,q) + \gamma_n^{[3]}(p,q,0) \end{aligned} \tag{2.35}$$

after a little rearrangement, and in turn (deploying (2.33))

$$\begin{aligned} 0 &= \alpha_n^{[3]}(p,q,0) - \beta_n^{[3]}(p,q,0) + \beta_n^{[3]}(p,q,0) - \gamma_n^{[3]}(p,q,0) + \gamma_n^{[3]}(p,q,0) \\ &= \alpha_n^{[3]}(p,q,0). \end{aligned} \tag{2.36}$$

We finish our analysis by establishing (2.33) and (2.34) rigorously, so as to substantiate them as results. First we prove the following.

Lemma 2.1. *For $u \geq 0$,*

$$c_u(p,q,0) = p^u P_u(-q/p^2).$$

Proof. From (1.9) we write, for $u \geq 0$ (and adopting the convention that $0^0 = 1$),

$$\begin{aligned} c_u(p,q,0) &= \sum_{l_1+2l_2+3l_3=u} \binom{l_1+l_2+l_3}{l_1,l_2,l_3} p^{l_1} q^{l_2} 0^{l_3} \\ &= \sum_{l_1+2l_2+3\cdot 0=u} \binom{l_1+l_2+0}{l_1,l_2,0} p^{l_1} q^{l_2} 0^0 + \sum_{\substack{l_1+2l_2+3l_3=u \\ (l_3>0)}} \binom{l_1+l_2+l_3}{l_1,l_2,l_3} p^{l_1} q^{l_2} 0^{l_3} \\ &= \sum_{l_1+2l_2=u} \binom{l_1+l_2}{l_1,l_2} p^{l_1} q^{l_2} \\ &= \sum_{l_2=0}^{\lfloor u/2 \rfloor} \binom{u-l_2}{u-2l_2,l_2} p^{u-2l_2} q^{l_2} \\ &= p^u \sum_{l_2=0}^{\lfloor u/2 \rfloor} \binom{u-l_2}{l_2} (q/p^2)^{l_2}, \end{aligned} \tag{L.1}$$

$= p^u P_u(-q/p^2)$ by comparison with (1.6). □

With reference to (1.4) then, using Lemma 2.1, equations (2.33) are now immediate from (2.28) as

$$\begin{aligned} \beta_n^{[3]}(p,q,0) &= qc_{n-3}(p,q,0) = q \cdot p^{n-3} P_{n-3}(-q/p^2) = \alpha_{n-1}^{[2]}(p,q), \\ \gamma_n^{[3]}(p,q,0) &= c_{n-2}(p,q,0) = p^{n-2} P_{n-2}(-q/p^2) = \beta_{n-1}^{[2]}(p,q), \end{aligned} \tag{2.37}$$

themselves combining to give

$$\begin{aligned} m_n^{[3]}(p,q,0) &= [\alpha_n^{[3]}(p,q,0) + \beta_n^{[3]}(p,q,0) + \gamma_n^{[3]}(p,q,0) - 1]/(p+q+0-1) \\ &= [0 + \alpha_{n-1}^{[2]}(p,q) + \beta_{n-1}^{[2]}(p,q) - 1]/(p+q-1) \\ &= [\alpha_{n-1}^{[2]}(p,q) + \beta_{n-1}^{[2]}(p,q) - 1]/(p+q-1) \\ &= m_{n-1}^{[2]}(p,q) \end{aligned} \tag{2.38}$$

by (1.5), which is (2.34).

2.4. **A Final Remark and Open Problem.** As a final remark we note that, on appealing to the $s = 1$ version of (1.7) mentioned in an earlier paper [1], those functional exponents of the resulting sequence term (1.8) appear to correspond to elements of an extended Horadam type recurrence with particular 0-1 initial values. Denoting by $\{w_n(w_0, w_1, w_2; p, -q, -r)\}_0^\infty$ the recurrence sequence generated by the linear order three recursion $w_n = pw_{n-1}+qw_{n-2}+rw_{n-3}$ ($n \geq 3$) with initial values w_0, w_1, w_2, then we may write (directly from [1, Eq. (2.6), p. 175])

$$\begin{aligned}\alpha_n^{[3]}(p,q,r) &= w_n(1,0,0;p,-q,-r),\\ \beta_n^{[3]}(p,q,r) &= w_n(0,1,0;p,-q,-r),\\ \gamma_n^{[3]}(p,q,r) &= w_n(0,0,1;p,-q,-r),\end{aligned}\tag{2.39}$$

with $(p+q+r-1)m_n^{[3]}(p,q,r) = w_n(1,0,0;p,-q,-r)+w_n(0,1,0;p,-q,-r)+w_n(0,0,1;p,-q,-r)-1$ in consequence; these interesting observations are empiric ones, based on extensive computations, which remain to be proved and pose an open problem for any reader to tackle.

3. Summary

A three-deep power product recurrence has been examined for the first time, and a closed form for the resulting sequence general term found using a technique deployed previously to examine the two-deep version (for which results are recoverable by contraction, as has been demonstrated); in doing so a potential difficulty articulated in [2]—namely, that of generating an appropriate form for arbitrary exponentiation of the matrix $\mathbf{H}(p,q,r)$ (2.5)—has been resolved. With this in mind, what is evidently a successful methodology should in principle be applicable to a general ρ-deep power product recurrence (subject to ρ initial values) to deliver the sequence general term closed form in $2\rho+1$ variables (or 2ρ if the recursion scalar s is absent).

Note that solution pathologies for those values of recursion parameters p, q, r that violate the constraint (1.13) (and sum to unity) are not explored here.

Appendix A

Here we illustrate how (2.17) is established.

Proof. Consider

$$\begin{aligned}\sum_{u\geq 0} c_u(p,q,r)t^u &= \sum_{u\geq 0}\left(\sum_{l_1+2l_2+3l_3=u}\binom{l_1+l_2+l_3}{l_1,l_2,l_3}p^{l_1}q^{l_2}r^{l_3}\right)t^u\\ &= \sum_{u\geq 0}\sum_{l_1+2l_2+3l_3=u}\binom{l_1+l_2+l_3}{l_1,l_2,l_3}p^{l_1}q^{l_2}r^{l_3}t^{l_1+2l_2+3l_3}\\ &= \sum_{l_1,l_2,l_3\geq 0}\binom{l_1+l_2+l_3}{l_1,l_2,l_3}(pt)^{l_1}(qt^2)^{l_2}(rt^3)^{l_3}\\ &= \sum_{k\geq 0}\sum_{l_1+l_2+l_3=k}\binom{k}{l_1,l_2,l_3}(pt)^{l_1}(qt^2)^{l_2}(rt^3)^{l_3}\\ &= \sum_{k\geq 0}(pt+qt^2+rt^3)^k\\ &= (1-pt-qt^2-rt^3)^{-1}\\ &= F(t;p,q,r),\end{aligned}\tag{P.1}$$

as defined in (2.13). □

Appendix B

Here we establish the recurrence (2.29), noting that (2.17) and (2.13) combine as

$$\sum_{u\geq 0} c_u(p,q,r)t^u = F(t;p,q,r) = \frac{1}{(1-pt-qt^2-rt^3)}, \tag{B.1}$$

which is used within a short proof along with (1.10).

If (2.29) is correct then we should find that

$$\sum_{u\geq 0}[pc_{u+2}(p,q,r) + qc_{u+1}(p,q,r) + rc_u(p,q,r) - c_{u+3}(p,q,r)]t^u = 0, \tag{B.2}$$

since the l.h.s. comprises an infinite sum of zeros; this is achieved in routine fashion, as follows.

Proof. Consider

$$\begin{aligned}
&\sum_{u\geq 0}[pc_{u+2}(p,q,r) + qc_{u+1}(p,q,r) + rc_u(p,q,r) - c_{u+3}(p,q,r)]t^u \\
&\quad = p\sum_{u\geq 2} c_u(p,q,r)t^{u-2} + q\sum_{u\geq 1} c_u(p,q,r)t^{u-1} + r\sum_{u\geq 0} c_u(p,q,r)t^u - \sum_{u\geq 3} c_u(p,q,r)t^{u-3} \\
&\quad = \frac{p}{t^2}\left(\sum_{u\geq 0} c_u(p,q,r)t^u - c_0(p,q,r) - c_1(p,q,r)t\right) + \frac{q}{t}\left(\sum_{u\geq 0} c_u(p,q,r)t^u - c_0(p,q,r)\right) \\
&\qquad + r\sum_{u\geq 0} c_u(p,q,r)t^u - \frac{1}{t^3}\left(\sum_{u\geq 0} c_u(p,q,r)t^u - c_0(p,q,r) - c_1(p,q,r)t - c_2(p,q,r)t^2\right) \\
&\quad = \left(\frac{p}{t^2} + \frac{q}{t} + r - \frac{1}{t^3}\right)\sum_{u\geq 0} c_u(p,q,r)t^u - \frac{p}{t^2}(1+pt) - \frac{q}{t} + \frac{1}{t^3}[1+pt+(p^2+q)t^2] \\
&\quad = \frac{(pt+qt^2+rt^3-1)}{t^3}\sum_{u\geq 0} c_u(p,q,r)t^u + \frac{1}{t^3} \\
&\quad = \frac{(pt+qt^2+rt^3-1)}{t^3}\cdot\frac{1}{(1-pt-qt^2-rt^3)} + \frac{1}{t^3} \\
&\quad = -\frac{1}{t^3} + \frac{1}{t^3} \\
&\quad = 0,
\end{aligned} \tag{P.2}$$

as required. □

References

[1] P. J. Larcombe and O. D. Bagdasar, *On a result of Bunder involving Horadam sequences: A proof and generalization*, The Fibonacci Quarterly, **51.2** (2013), 174–176.

[2] P. J. Larcombe and E. J. Fennessey, *A scaled power product recurrence examined using matrix methods*, Bulletin of the I.C.A., **78** (2016), 41–51.

CLOSED FORM FOR THE GENERAL TERM OF A SCALED TRIPLE POWER PRODUCT

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