LEVEL SIZES OF THE BULGARIAN SOLITAIRE GAME TREE

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Abstract. Bulgarian solitaire is a dynamical system on integer partitions of n which converges to a unique fixed point if $n = 1 + 2 + \cdots + k$ is a triangular number. There are few results about the structure of the game tree, but when k tends to infinity the game tree itself converges to a structure that we are able to analyze. Its level sizes turn out to be a bisection of the Fibonacci numbers. The leaves in this tree structure are enumerated using Fibonacci numbers as well. We also demonstrate to which extent these results apply to the case when k is finite.

1. Introduction

The game of Bulgarian solitaire works as follows: First divide a finite deck of n identical cards into piles. A move consists in removing one card from each pile and forming a new pile. This operation is repeated over and over. We pay attention only to how many piles there are of each positive size, ignoring the locations of the piles. Thus, each position of the game is an integer partition of n whose parts are the pile-sizes, and each move is an operation on the set of partitions inducing a finite dynamical system. See for example [2, 4] for the fascinating history of Bulgarian solitaire.

In terms of Young diagrams of integer partitions, pile-sizes correspond to rows in the diagram and the move is to delete the first column and add it as a new row.

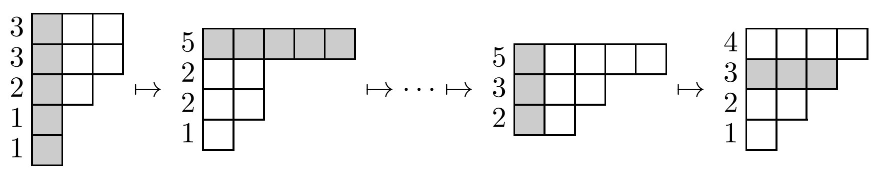

Figure 1. A sequence of positions converging to the fixed point.

The illustrating game for $n = 10$ in Figure 1 converges in twelve moves to the staircase shape of the partition $(4, 3, 2, 1)$, which is a fixed point. In the case when $n = 1 + 2 + \cdots + k$ is a triangular number, all games will converge to the fixed point $(k, k-1, \ldots, 1)$ (see for example [1, 8]).

Let $\mathcal{P}(n)$ be the set of integer partitions of n. Let B be the operation on $\mathcal{P}(n)$ such that $B(\lambda)$ is the partition resulting from a move of Bulgarian solitaire on λ. The *game graph* is a directed graph where $\mathcal{P}(n)$ are the nodes and where there is an edge from $\lambda \in \mathcal{P}(n)$ to $\lambda' \in \mathcal{P}(n)$ if $B(\lambda) = \lambda'$. When the game graph is a tree, we call it a *game tree*.

For non-triangular n, the game has no fixed point. Instead the game graph has cycles, one for each connected component, and the game always reaches some cycle [1]. The number of cycles and their sizes are counted in [1]. The game graph analysis in [6] mainly characterizes and enumerates partitions with no preimage under B (called *Garden of Eden partitions*) and [5] investigates their relation to the cycles.

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In this paper we will consider the case when $n = 1 + 2 + \cdots + k$ is a triangular number. Since all games will converge to the staircase shaped fixed point, the game graph is a directed tree whose leaves are the Garden of Eden partitions and whose root is the staircase partition. The height of this tree is known [3] to be $k(k-1)$, but to our knowledge the level size of the tree at a given level d has not been studied before. Paradoxically, the game tree's being finite is what makes further analysis difficult. We will show how a quasi-infinite version of the game, obtained by fixing the level d and letting $k \to \infty$, makes it possible to study the level sizes.

2. Reversed Bulgarian Solitaire

By reversing all arrows in the directed game tree and labeling as below, we get *reversed Bulgarian solitaire*, starting with the staircase partition. A move in the reversed game is the following:

$$\xrightarrow{i} : \text{Delete row } i. \text{ Add it as the leftmost column.}$$

Such a move can be made if and only if the length of the deleted row is greater than or equal to the height of the Young diagram after the deletion. In other words, if λ is a partition and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N(\lambda)}$ are its parts, the move $\lambda \xrightarrow{i}$ is only possible if $\lambda_i \geq N(\lambda) - 1$.

The reversed game tree for $k = 4$ ($n = 10$) is shown in Figure 2. Recall that edge labels indicate which row is being played (i.e., which row is being deleted in the move)[1]. For reasons of space, in Figure 2 we represent positions as horizontal vectors. In the following, we will always represent positions as vertical vectors, so as to match the notion of Young diagram row lengths in Figure 1. Deleting the first row in the staircase partition leads back to the staircase partition, so in Figure 2 the subtree reached by an initial $\xrightarrow{1}$ move is identical to the whole tree and the edge is a loop in disguise. However, we prefer to keep it a tree.

The reverse of the game in Figure 1 may be written as

$$\begin{matrix}\langle 4\rangle\\ \langle 3\rangle\\ 2\\ 1\end{matrix} \xrightarrow{2} \begin{matrix}\langle 5\rangle\\ \langle 3\rangle\\ \langle 2\rangle\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ 3\\ 1\\ 1\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ 2\\ 2\\ 2\end{matrix} \xrightarrow{1} \begin{matrix}\langle 3\rangle\\ \langle 3\rangle\\ \langle 3\rangle\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ \langle 4\rangle\\ \langle 2\rangle\end{matrix} \xrightarrow{1} \begin{matrix}\langle 5\rangle\\ \langle 3\rangle\\ 1\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ 2\\ 2\\ 1\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 3\rangle\\ \langle 3\rangle\\ 2\\ 2\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ \langle 3\rangle\\ \langle 3\rangle\end{matrix} \xrightarrow{1} \begin{matrix}\langle 4\rangle\\ \langle 4\rangle\\ 1\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 5\rangle\\ 2\\ 2\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}3\\ 3\\ 2\\ 1\\ 1\end{matrix} \tag{2.1}$$

Numbers within the angle brackets $\langle\rangle$ are playable, that is, at least $N(\lambda) - 1$, and we shall refer to them as *bracketed* numbers. Clearly, the start vector representing the staircase partition always has two bracketed numbers.

In our example in Figure 1 we chose a Garden of Eden partition $(3, 3, 2, 1, 1)$ as the starting position. Recall from the introduction that the height of the game tree is $k(k-1)$ when the number of cards n is triangular, $n = 1 + 2 + \cdots + k$. Since the sequence in (2.1) has length $12 = 4 \cdot 3$, this sequence represents the tree height, in other words, λ is not only a Garden of Eden partition but also has the maximum number of moves to the fixed point (and is therefore at the bottommost level in the reversed game tree in Figure 2).

Using the introduced bracket notation, the move rule may now be expressed as follows.

Rule for $\xrightarrow{i}$ *in reversed Bulgarian solitaire* (R1)

1. Delete the bracketed ith number p.
2. Increase all remaining numbers by 1.
3. Add trailing 1's as necessary to make the vector length equal to p.
4. Bracket all numbers greater than or equal to $p - 1$.

[1] If the length of a playable row is repeated in a partition, then playing any of these rows is represented by playing the uppermost row of this length.

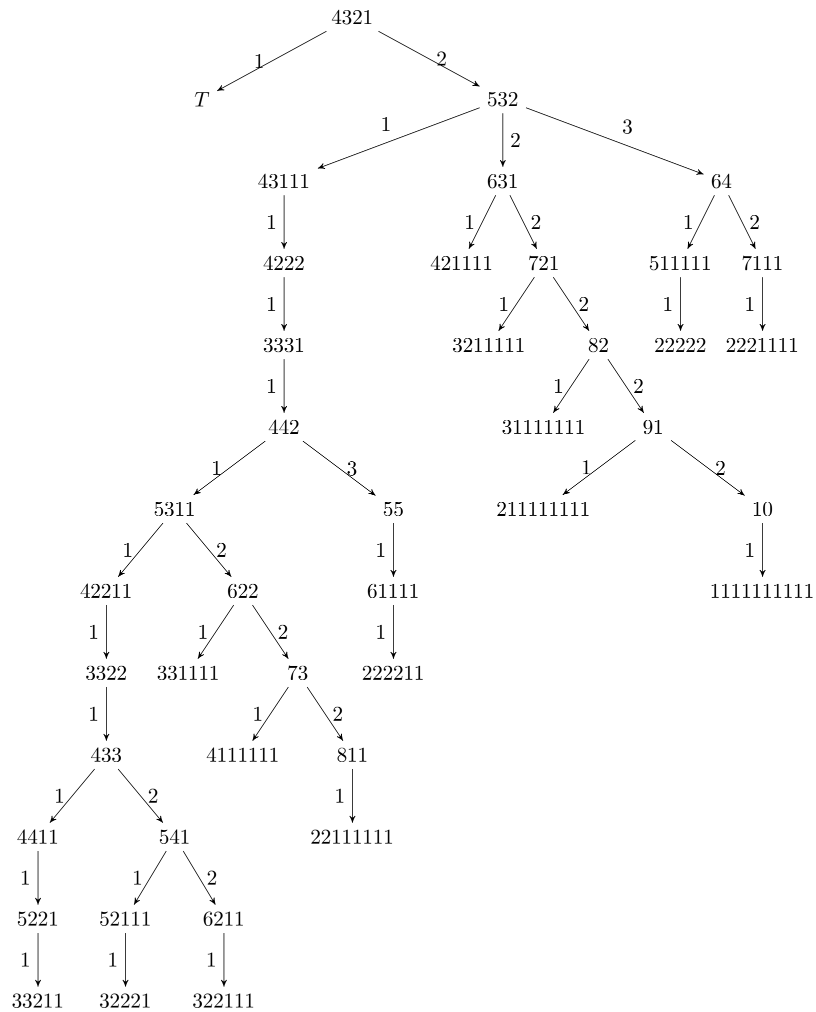

FIGURE 2. Reversed game tree T for $k = 4$.

We will denote a sequence of played rows by $[\mathtt{r}_1, \mathtt{r}_2, \ldots]$ or simply $[\mathtt{r}_1\mathtt{r}_2 \ldots]$. Such a sequence may also be referred to as a *play sequence*. Also, we will use exponents to mean repeated play of a row number: $\mathtt{r}^s = \underbrace{\mathtt{r}, \mathtt{r}, \ldots, \mathtt{r}}_{s \text{ times}}$. Thus, the play sequence in (2.1) is denoted $[\mathtt{2}, \mathtt{1}^{11}]$.

In our small example, the partitions in the game soon lost all similarity with the staircase shape. Now we will investigate what happens as k grows. To this end, instead of writing down the lengths of the rows, we will note the difference: row length minus original row length in the staircase (where we append zeros as necessary to compute the difference).

This is what the sequence (2.1) looks like in this *difference notation*:

$$\begin{matrix}\langle 0\rangle\\ \langle 0\rangle\\ 0\\ 0\end{matrix} \xrightarrow{2} \begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\\ -1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ 0\\ 1\\ 0\\ -1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ -1\\ 0\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle -1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\\ 0\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ -1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ -1\\ 0\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ -1\\ 0\\ 0\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle -1\rangle\\ \langle 0\rangle\\ 0\\ 1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ \langle 0\rangle\\ \langle 1\rangle\\ -1\end{matrix} \xrightarrow{1} \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\\ -1\\ 0\end{matrix} \xrightarrow{1} \begin{matrix}\langle 1\rangle\\ -1\\ 0\\ 0\end{matrix} \xrightarrow{1} \begin{matrix}-1\\ 0\\ 0\\ 0\\ 1\end{matrix}$$

To see more clearly what happens as k grows, below follows an example for a larger k.

Example 2.1. *For $k = 8$ ($n = 36$), the play sequence* $[\mathtt{2}, \mathtt{1}^6]$ *yields*

$$\begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{array} \xrightarrow{2} \begin{array}{c}\langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\\ 0\\ 0\\ 0\\ 0\\ -1\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ 0\\ 0\\ 0\\ 0\\ 0\\ -1\\ 0\\ 1\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\\ 0\\ 0\\ 0\\ -1\\ 0\\ 1\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\\ 0\\ 0\\ -1\\ 0\\ 1\\ 0\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\\ 0\\ -1\\ 0\\ 1\\ 0\\ 0\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\\ -1\\ 0\\ 1\\ 0\\ 0\\ 0\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ -1\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\end{array}$$

Each zero travels upwards, one step in each move, until it expires (either is deleted or is transformed into a 1). As we will see, it takes at least $k/2$ moves for the bottom zero in the start vector to expire, and neither the -1 below it or the numbers further down have any influence on the game so far (not being bracketed). Therefore, all game trees for $k \geq 8$ are identical up to and including level four. This observation brings us to introduce a quasi-infinite version of the game.

3. Quasi-infinite Reversed Bulgarian Solitaire

In quasi-infinite difference notation, the start vector should now be an infinite sequence of zeros. However, we will write only two zeros and assume that there are infinitely many invisible non-bracketed zeros below them. Now, we can write the game $[\mathtt{21111}\ldots]$ as

$$\begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{2} \begin{array}{c}\langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ 0\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{1} \begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{1} \cdots .$$

With this notation, the game looks very different. In *quasi-infinite reversed Bulgarian solitaire* the start vector is $\begin{smallmatrix}\langle 0\rangle\\ \langle 0\rangle\end{smallmatrix}$ and the rules are a straightforward extension of the rules (R1) to quasi-infinite difference notation.

Rule for $\xrightarrow{i}$ in reversed quasi-infinite difference notation

1. Delete the bracketed ith number.
2. Increase all numbers above it by 1 and make them bracketed.
3. Bracket the new ith number (if there is one) if it differs at most 1 from the old one. (R2)
4. If a zero was played, add zeros at the end so that there are two, and make them bracketed.

In the quasi-infinite version, the game $[\mathtt{2345}]$ is represented as

$$\begin{array}{c}\langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{2} \begin{array}{c}\langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{3} \begin{array}{c}\langle 2\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{4} \begin{array}{c}\langle 3\rangle\\ \langle 2\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array} \xrightarrow{5} \begin{array}{c}\langle 4\rangle\\ \langle 3\rangle\\ \langle 2\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array}$$

Note that the vectors are always strictly decreasing down to the zeros. This property follows by induction as it is invariant under the move rules.

Proposition 3.1. *The quasi-infinite reversed Bulgarian solitaire has the following properties.*

(i) *Once* `1` *is played, only* `1` *can be played (until the start vector is reached).*
(ii) *For $r \geq 2$, the play sequence* $[\mathtt{234}\ldots\mathtt{r}\,\mathtt{1}^{\mathtt{r}}]$ *leads to the start vector. Together with* $[\mathtt{1}]$, *these are the only play sequences with this property.*

Proof. (i): It follows from (R2) that the position vector following a $\xrightarrow{1}$ move either has no bracketed number (so it is a leaf), one bracketed number (so another $\xrightarrow{1}$ must be played to reach the start vector), or is the start vector (in case a zero was played).

(ii): Playing [`234`...`r`], we reach

$$\begin{array}{c}\langle r{-}1\rangle\\ \langle r{-}2\rangle\\ \vdots\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 0\rangle\end{array},$$

and then [$\mathtt{1}^{\mathtt{r}}$] deletes the top number r times, yielding the start vector. From property (i) it is clear that the start vector cannot be obtained in any other way. □

Figure 3 shows what the quasi-infinite game tree looks like for the first three moves. As

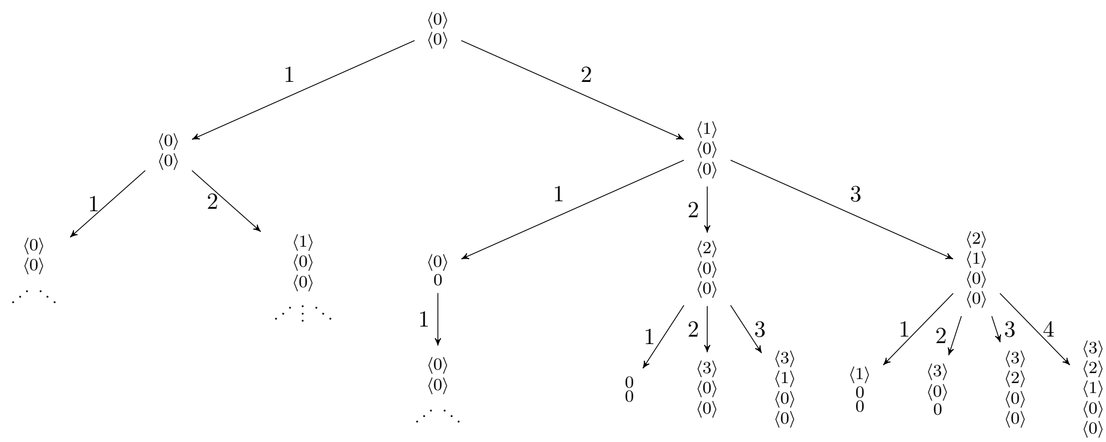

FIGURE 3. Quasi-infinite game tree for the first three moves. Every instance of a node $\begin{smallmatrix}\langle 0\rangle\\ \langle 0\rangle\end{smallmatrix}$ represents a new copy of the entire tree.

suggested by Proposition 3.1, this tree is recursive in several ways and we will use that to compute its *growth function*, $g = 1 + 2x + 5x^2 + 13x^3 + \cdots$, the generating function for the size of its levels. Figure 3 shows only eight nodes on level three, but since the start vector $\begin{smallmatrix}\langle 0\rangle\\ \langle 0\rangle\end{smallmatrix}$ reappears after a `1` move, a copy of the whole tree hangs below it. This copy adds five nodes to the eight nodes shown on level three. Since a `1` move leads to the start vector, the set of all sequences starting with `1` contribute the term xg to the growth function g.

The start vector also reappears after the play sequence [`211`], so another tree copy hangs below that node. Note that the reappearance does not mean that we are back to the staircase; this partition is different at the bottom. However, the quasi-infinite analysis is based on the assumption that k is large enough that this difference will not affect the game continuation in terms of number of playable rows. The level size of the quasi-infinite game tree will coincide with the finite case until a certain depth. This depth will be determined in Section 5.

We will now identify all other occurrences of the start vector in the game tree. By Proposition 3.1(ii) other games that lead to the start vector are [$\mathtt{231}^{\mathtt{3}}$], [$\mathtt{2341}^{\mathtt{4}}$], etc. The contribution to the growth function of these recursions is $x^3g + x^5g + \cdots$.

Another recursiveness appears when we consider play sequences not including any `1` move. For these sequences, the game goes on as if the top row did not exist. The subtree of these play sequences must therefore be isomorphic to the full tree, but only after the first move 2, which makes 3 playable. This gives us a term xg, corresponding to $[2(\ldots \geq 2)]$, where $(\ldots\geq r)$ denotes a (possibly empty) sequence of numbers r or greater.

The neglected top row is always playable, so the type $[2(\ldots \geq 2)1]$ will contribute x^2g to the growth function.

Is it possible to end by more than one 1? Yes, if the two top rows are neglected until the end. Thus the type $[23(\ldots \geq 3)11]$ contributes x^4g. Similarly for three or more top rows one gets $x^6g + x^8g + \cdots$.

Table 3 presents the play sequences we have covered so far and their respective contribution to the growth function g.

Table 3. Play sequences and their corresponding contribution to the growth function g.

Sequence	Contribution	Sequence	Contribution
$[1\ldots]$	xg	$[2(\ldots \geq 2)]$	xg
$[21^2]$	x^3g	$[2(\ldots \geq 2)1]$	x^2g
$[231^3]$	x^5g	$[23(\ldots \geq 3)1^2]$	x^4g
$[2341^4]$	x^7g	$[234(\ldots \geq 4)1^3]$	x^6g
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[234\ldots j, 1^j]$	$x^{2j-1}g$	$[234\ldots j(\ldots \geq j)1^{j-1}]$	$x^{2(j-1)}g$

By virtue of Proposition 3.1(i), these sequences do in fact cover all possible play sequences. Adding all terms, together with the fact that level 0 (the root) has size 1, we obtain

$$g(x) = (2x + x^2 + x^3 + \cdots)g(x) + 1 = \left(x + \frac{x}{1-x}\right) g(x) + 1. \tag{3.1}$$

We are now ready to formulate the main result, which involves the classical Fibonacci sequence ($F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$).

Theorem 3.2. *The generating function $g(x) = g_0 + g_1x + g_2x^2 + \cdots$ for the level sizes in the quasi-infinite reversed Bulgarian solitaire is*

$$g(x) = \frac{1-x}{1-3x+x^2} = 1 + 2x + 5x^2 + 13x^3 + \cdots$$

where $g_m = F_{2m+1}$ is a bisection of the Fibonacci sequence and has the recurrence $g_m = 3g_{m-1} - g_{m-2}$ for $m \geq 2$.

Proof. The formula is a consequence of (3.1) and identifying coefficients yields the recurrence. This is the sequence A001519 in The On-Line Encyclopedia of Integer Sequences (OEIS) [7]. □

4. Height and Leaf Count

In the non-reversed game, the *height* of a position is defined as the number of moves until the fixed point is reached (for the first time). This corresponds to disregarding the left subtree in the reversed game (for example in Figure 2), i.e., considering the root and the subtree $[2\ldots]$ only.

For the quasi-infinite tree in Figure 3, let $h(x) = h_0 + h_1x + h_2x^2 + \cdots$ be the height generating function which counts the level sizes of the root node together with the subtree $[2\ldots]$. (As pointed out before, in this subtree occurrences of the quasi-infinite start vector are not proper staircases, being different at the bottom.) It satisfies

$$g(x) = xg(x) + h(x).$$

Using Theorem 3.2 we get

$$h(x) = \frac{(1-x)^2}{1-3x+x^2} = 1 + x + 3x^2 + 8x^3 + \cdots$$

and identifying coefficients gives the recurrence

$$h_0 = 1, \quad h_1 = 1, \quad h_2 = 3, \quad h_m = 3h_{m-1} - h_{m-2}, m \geq 3.$$

We have $h_i = F_{2i}, i \geq 1$, so this is also a bisection of the Fibonacci sequence. It is A088305 in OEIS [7].

Recall that the *leaves* of a game tree are called Garden of Eden partitions. We now turn our attention to their enumeration. In Figure 2, we find one leaf on level 3, three leaves on level 4 and so on until level 12 where there are three leaves. Thus, $x^3 + 3x^4 + x^5 + \cdots + 3x^{12}$ is the generating function that counts the leaves on each level in this tree. As before, we have disregarded the left subtree.

For the quasi-infinite tree in Figure 3, disregarding the left subtree, let $\ell(x) = \ell_0 + \ell_1 x + \ell_2 x^2 + \cdots$ be the generating function for the number of leaves on each level. We have $\ell(x) = x^3 + 3x^4 + \cdots$, where the first term counts [221] and the second term [2221], [2231], and [2321]. If we include the left subtree, we get another generating function $L(x) = L_0 + L_1 x + L_2 x^2 + \cdots$ and as with g and h they are related by $L(x) = xL(x) + \ell(x)$.

All games are listed in Table 3 and we can count those that start with 2 and end in a leaf (no bracketed numbers). Games starting with [21^2], [231^3], [2341^4], etc., may continue in any way to a leaf, so their contribution to the growth function is $x^3 L + x^5 L + \cdots$. The game [$2(\ldots \geq 2)1$] ends in a leaf unless it is either [21] or of the type [$23(\ldots \geq 3)1$], so the contribution is $x^2 g - x^2 - x^3 g$. In the same way, the game type [$23(\ldots \geq 3)11$] contributes $x^4 g - x^4 - x^5 g$, etc. Adding all contributions, we get

$$\ell(x) = (x^3 + x^5 + \cdots)L(x) + (x^2 - x^3 + x^4 - x^5 + \cdots)g(x) - (x^2 + x^4 + x^6 + \cdots).$$

Substituting $L(x) = \ell(x)/(1-x)$, using the expression for $g(x)$ in Theorem 3.2, and solving for $\ell(x)$ yields

$$\ell(x) = \frac{x^3 - x^4}{(1-x-x^2)(1-3x+x^2)} = x^3 + 3x^4 + 9x^5 + 25x^6 + \cdots. \tag{4.1}$$

The coefficient sequence $(\ell_1, \ell_2, \ldots)$ is essentially A094292 in OEIS [7] (but with an additional initial zero) and has the explicit formula

$$\ell_i = \frac{F_{2i-2} - F_{i-1}}{2},$$

as is easily verified, for example by decomposing $\ell(x)$ in (4.1) into partial fractions and identifying coefficients in each part.

Since $L(x) = \ell(x)/(1-x)$, (4.1) yields $L(x) = x^3/((1-x-x^2)(1-3x+x^2))$. It is well-known that dividing a generating function by $1-x$ gives the generating function for the cumulative sum. Thus, $L_m = \sum_{i=3}^{m} \ell_i$ is the number of leaves up to and including level m. The sequence $(L_0, L_1, \ldots)$ is A056014 in OEIS and by decomposing $L(x)$ into partial fractions it is easily verified that

$$L_m = \sum_{i=3}^{m} \ell_i = \frac{F_{2m-1} - F_{m+1}}{2}.$$

5. The Finite Case $n = k(k+1)/2$

We will now investigate what happens when we keep k fixed and increase the tree level d. In other words, we will see how far the results in the quasi-infinite case apply to the finite case, when the staircase partition has k rows. Playing 2 lengthens the top row by one and shortens the bottom row by one, but the other $k-2$ rows are unaffected. These will correspond to $k-2$ zeros in our representation. As we have seen in Section 3, only the two uppermost zeros can be involved in a move, so the number of zeros decreases by at most two every move. Therefore, up to and including level $\lfloor k/2 \rfloor$, the level sizes are the same as in the quasi-infinite game tree.

In the finite game tree, at level $\lfloor k/2 \rfloor$ the first $\lceil k/2 \rceil$ vector components are unaffected. For example, for $k = 6$, the first three components of the vectors at level three are as in Figure 3. The fourth component, however, is not zero but a negative number. We know that the sixth component at level 1 is -1 (cf. Example 2.1), which means that the fifth component at level 2 and the fourth component at level 3 are also negative. Thus, the play sequences [2345], [2344], [2334] and [2234] are illegal for $k = 6$ (k even). For $k = 7$ (k odd) the situation is similar. The first four vector components are the same as in the quasi-infinite game tree, but the fifth component in each vector at level 3 is negative and that makes [2345] illegal. This observation generalizes to the following theorem.

Theorem 5.1. *When $n = k(k+1)/2$, the sizes of levels $0, 1, \ldots, \lfloor k/2 \rfloor$ in the reversed Bulgarian solitaire game tree coincide with those of the quasi-infinite game tree but the next level, $\lfloor k/2 \rfloor + 1$, has fewer elements, 1 less for odd k and $1 + k/2$ less for even k.*

Thus, when $n = k(k+1)/2$, for $1 \leq m \leq \lfloor k/2 \rfloor$ the number of partitions with m moves of Bulgarian solitaire to the staircase partition is the Fibonacci number F_{2m}.

As we saw in Section 1, the game tree for the finite case has a finite height, $k(k-1)$. One may ask the question whether it is a subtree of the quasi-infinite tree. In other words, are all play sequences in the finite reversed game also legal in the quasi-infinite reversed game? The answer is no. The minimal counter-example is the play sequence [211113] in the reversed game for $k = 4$ (which leads from the staircase partition $(4, 3, 2, 1)$ to the partition $(5, 5)$, see Figure 2). This play sequence is not legal in the quasi-infinite reversed game since [21111] leads to the start vector in which the third row is not playable.

6. Discussion

The current paper concerns Bulgarian solitaire in the case where the number of cards n is triangular, $n = k(k+1)/2$ for some positive integer k. Theorem 5.1 gives the number of partitions with m moves to the staircase partition for $1 \leq m \leq \lfloor k/2 \rfloor + 1$. A natural open question is the extension of this result to *all* m, i.e., to $m \leq k(k-1)$. Recall that the maximal play sequence length in this case is $k(k-1)$.

As mentioned in the introduction, when n is not triangular, the game graph has one cycle per connected component and the game eventually reaches a cycle of partitions. Note that the fixed point partition for triangular n also constitutes a cycle (of length one). The maximal sequence length to a cycle partition for any n is conjectured by Griggs and Ho [3].

For non-triangular n, one may also ask the question corresponding to the one addressed in the current paper: *How many* partitions are there with a given sequence length to a cycle partition? Hopkins and Jones [5] has data on the number of leaves (Garden of Eden partitions) in the game graph by distance to the cycle, but a full answer to this question is yet to be found.

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