CLOSED FORMS FOR CERTAIN FIBONACCI TYPE SUMS THAT INVOLVE SECOND ORDER PRODUCTS

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Abstract. In this paper, we present closed forms for certain finite sums in which the summand is a product of generalized Fibonacci numbers. We present our results in the form of six theorems that feature a generalized Fibonacci sequence $\{W_n\}$, and an accompanying sequence $\{\overline{W}_n\}$. We add a further layer of generalization to our results with the use of three parameters s, k, and m.

The inspiration for this paper comes from a website of Knott that lists so-called *order 2 summations* involving the Fibonacci and Lucas numbers. Probably the most well-known of these summations is

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

1. Introduction

We begin by establishing the notation for the integer sequences that feature in this paper. Let a and b be integers with $(a,b) \neq (0,0)$. For any non-zero integer p, we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n(a,b,p) = W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b, \tag{1.1}$$

and

$$\overline{W}_n(a,b,p) = \overline{W}_n = W_{n-1} + W_{n+1}.$$

Setting $\Delta = p^2 + 4$, we leave to the reader the simple task of showing that

$$\overline{\overline{W}}_n = \Delta W_n. \tag{1.2}$$

For $(a,b,p) = (0,1,1)$, we have $\{W_n\} = \{F_n\}$, and $\{\overline{W}_n\} = \{L_n\}$, which are the Fibonacci and Lucas sequences, respectively. Taking $(a,b) = (0,1)$, we write $\{W_n(p)\} = \{U_n\}$, and $\{\overline{W}_n(p)\} = \{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas sequences, respectively.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the Binet forms for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

Note that the closed forms for all the sequences that we consider in this paper can be obtained from (1.3) and (1.4). In the sequel, we require the constant $e_W = AB = b^2 - pab - a^2$. It is immediate that $e_F = 1$ and $e_L = -5$.

The motivation for this paper comes from the many formulas in Section 9.4 of [1] that Knott calls *order 2 summations*. These formulas are finite sums for the Fibonacci numbers, probably the most well-known being

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Some other formulas that appear in [1] are

$$\begin{aligned}
\sum_{i=1}^{2n} F_i F_{i-1} &= F_{2n}^2, \\
\sum_{i=1}^{2n+1} F_i F_{i-1} &= F_{2n+1}^2 - 1, \\
\sum_{i=0}^{n-1} F_{2i+1}^2 &= \frac{1}{5}\left(F_{4n} + 2n\right), \\
\sum_{i=0}^{n} F_{2i}^2 &= \frac{1}{5}\left(F_{4n+2} - 2n - 1\right).
\end{aligned}$$

For each formula in [1], Knott also records the Lucas counterpart.

In this paper, we present similar formulas that involve the more general sequence $\{W_n\}$, so that each of our finite sums can be specialized to both the Fibonacci and Lucas numbers. Furthermore, the formulas that we present involve several parameters. This adds another level of generalization, thus enabling us to capture most of the formulas in Section 9.4 of [1] as special cases.

The title of our paper contains the description *second order products*. By this we mean that, in each of our formulas, either the summand or the closed form involves a product of two terms from Fibonacci/generalized Fibonacci sequences. In the formulas that Knott lists, it is the summand that is a product of two terms from the Fibonacci/Lucas sequences.

In Section 2, we give our main results in the form of six theorems, and in Section 3 we provide a sample proof. Finally, in Section 4, we clarify the extent to which our results imply the order 2 summations listed in Section 9.4 of Knott [1].

2. The Main Results

In all that follows, $n \geq 1$ is an integer. We employ some conventional notation to present our results. Throughout we take i to be the dummy variable, so for instance $[W_{si}]_{n_1}^{n_2}$ means $W_{sn_2} - W_{sn_1}$.

Theorem 2.1. *Let $s \neq 0$ and m be integers. Then*

$$\sum_{i=1}^{n} W_{2si+m} = \begin{cases} \frac{1}{U_s} U_{sn} W_{s(n+1)+m}, & s \text{ even;} \\ \frac{1}{V_s}\left[W_{si} V_{s(i+1)+m}\right]_0^n, & s \text{ odd and } n \text{ even;} \\ \frac{1}{V_s} V_{sn} W_{s(n+1)+m}, & s \text{ odd and } n \text{ odd.} \end{cases}$$

Theorem 2.2. *Let $s \neq 0$ and m be integers. Then*

$$\sum_{i=1}^{n} (-1)^i W_{2si+m} = \begin{cases} \frac{(-1)^n}{U_s} U_{sn} W_{s(n+1)+m}, & s \text{ odd;} \\ \frac{1}{V_s}\left[W_{si} V_{s(i+1)+m}\right]_0^n, & s \text{ even and } n \text{ even;} \\ \frac{-1}{V_s} V_{sn} W_{s(n+1)+m}, & s \text{ even and } n \text{ odd.} \end{cases}$$

We have arranged the different cases in Theorems 2.1 and 2.2 in order to highlight the comparisons between the various outcomes.

We now define an additional four finite sums whose closed forms we present in this section. In the definitions that follow, $s \neq 0$ is an integer, and k and m are integers.

$$T_1(n) = T_1(n,s,k,m) = \sum_{i=1}^{n} W_{si+k}W_{si+m},$$

$$T_2(n) = T_2(n,s,k,m) = \sum_{i=1}^{n} (-1)^i W_{si+k}W_{si+m},$$

$$T_3(n) = T_3(n,s,k,m) = \sum_{i=1}^{n} W_{si+k}\overline{W}_{si+m},$$

$$T_4(n) = T_4(n,s,k,m) = \sum_{i=1}^{n} (-1)^i W_{si+k}\overline{W}_{si+m}.$$

As for Theorems 2.1 and 2.2, we have arranged the different cases in Theorems 2.3 and 2.4 in order to highlight the comparisons between the various outcomes. We have done likewise for Theorems 2.5 and 2.6.

Theorem 2.3. *For $T_1(n)$ as defined above,*

$$T_1(n) = \begin{cases} \frac{1}{\Delta}\left(\frac{1}{U_s}\left[W_{si}\overline{W}_{s(i+1)+k+m}\right]_0^n - (-1)^m e_W V_{k-m} n\right), & s \text{ even}; \\ \frac{1}{V_s}\left[W_{si}W_{s(i+1)+k+m}\right]_0^n, & s \text{ odd and } n \text{ even}; \\ \frac{1}{V_s}\left(W_{sn+k}W_{s(n+1)+m} - W_{s+k}W_m\right), & s \text{ odd and } n \text{ odd}. \end{cases}$$

In Theorem 2.3, let $(s,k,m) = (1,1,5)$ and take $W_n = F_n$. We then have

$$\sum_{i=1}^{n} F_{i+1}F_{i+5} = F_nF_{n+7},\ n \text{ even}.$$

Theorem 2.4. *For $T_2(n)$ as defined above,*

$$T_2(n) = \begin{cases} \frac{1}{\Delta}\left(\frac{1}{U_s}\left[(-1)^i W_{si}\overline{W}_{s(i+1)+k+m}\right]_0^n - (-1)^m e_W V_{k-m} n\right), & s \text{ odd}; \\ \frac{1}{V_s}\left[W_{si}W_{s(i+1)+k+m}\right]_0^n, & s \text{ even and } n \text{ even}; \\ \frac{-1}{V_s}\left(W_{sn+k}W_{s(n+1)+m} + W_{s+k}W_m\right), & s \text{ even and } n \text{ odd}. \end{cases}$$

In Theorem 2.4, let $(s,k,m) = (2,1,5)$ and take $W_n = F_n$. We then obtain

$$\sum_{i=1}^{n} (-1)^i F_{2i+1}F_{2i+5} = \frac{1}{3}F_{2n}F_{2n+8},\ n \text{ even}.$$

Theorem 2.5. *For $T_3(n)$ as defined above,*

$$T_3(n) = \begin{cases} \frac{1}{U_s}\left[W_{si}W_{s(i+1)+k+m}\right]_0^n + (-1)^m e_W U_{k-m} n, & s \text{ even}; \\ \frac{1}{V_s}\left[W_{si}\overline{W}_{s(i+1)+k+m}\right]_0^n, & s \text{ odd and } n \text{ even}; \\ \frac{1}{V_s}\left(W_{sn+k}\overline{W}_{s(n+1)+m} - W_{s+k}\overline{W}_m\right), & s \text{ odd and } n \text{ odd}. \end{cases}$$

In Theorem 2.5, let $(s,k,m) = (1,3,5)$ and take $W_n = F_n$. This yields

$$\sum_{i=1}^{n} F_{i+3}L_{i+5} = F_{n+3}L_{n+6} - 33,\ n \text{ odd}.$$

Theorem 2.6. *For $T_4(n)$ as defined above,*

$$T_4(n) = \begin{cases} \frac{1}{U_s}\left[(-1)^i W_{si} W_{s(i+1)+k+m}\right]_0^n + (-1)^m e_W U_{k-m} n, & s \text{ odd;} \\ \frac{1}{V_s}\left[W_{si}\overline{W}_{s(i+1)+k+m}\right]_0^n, & s \text{ even and } n \text{ even;} \\ \frac{-1}{V_s}\left(W_{sn+k}\overline{W}_{s(n+1)+m} + W_{s+k}\overline{W}_m\right), & s \text{ even and } n \text{ odd.} \end{cases}$$

In Theorem 2.6, let $(s,k,m) = (2,1,5)$ and take $W_n = F_n$. We then see that

$$\sum_{i=1}^{n} (-1)^i F_{2i+1} L_{2i+5} = \frac{1}{3} F_{2n} L_{2n+8}, \ n \text{ even.}$$

3. A Sample Proof

In this section, we give a detailed proof of Theorem 2.5. The method that we use employs the Binet forms (1.3) and (1.4), and serves as a template for the proofs of the other theorems in Section 2. Some of the algebra that follows is lengthy and detailed, so we find it appropriate to declare that we have checked each step carefully with the use of the computer algebra system ©Mathematica 9.0.

Expressing $W_{si+k}\overline{W}_{si+m}$ in terms of the Binet forms, then expanding and summing the finite geometric progressions that arise, we see that

$$\begin{aligned} &\sum_{i=1}^{n} W_{si+k}\overline{W}_{si+m} \\ &= \frac{1}{\alpha-\beta}\left(\frac{A^2\alpha^{k+m+2s}\left(\alpha^{2sn}-1\right)}{\left(\alpha^{2s}-1\right)} - \frac{B^2\beta^{k+m+2s}\left(\beta^{2sn}-1\right)}{\left(\beta^{2s}-1\right)}\right) \\ &\quad + AB(-1)^{s+m} U_{k-m} \sum_{i=0}^{n-1} (-1)^{is}. \end{aligned} \tag{3.1}$$

Now, since $\beta^{2s} = \alpha^{-2s}$, the middle line in (3.1) can be expressed as

$$\frac{A^2\alpha^{k+m+2s}\left(\alpha^{2sn}-1\right) + B^2\beta^{k+m}\left(\beta^{2sn}-1\right)}{(\alpha-\beta)\left(\alpha^{2s}-1\right)}. \tag{3.2}$$

Multiplying both the numerator and denominator of (3.2) by $\alpha^{-s} = (-1)^s\beta^s$, we see that (3.2) becomes

$$\frac{A^2\alpha^{k+m+s}\left(\alpha^{2sn}-1\right) + B^2(-1)^s\beta^{k+m+s}\left(\beta^{2sn}-1\right)}{(\alpha-\beta)\left(\alpha^s-(-1)^s\beta^s\right)}. \tag{3.3}$$

Recalling that $e_W = AB$, we then rewrite (3.1) as

$$\begin{aligned} &\sum_{i=1}^{n} W_{si+k}\overline{W}_{si+m} \\ &= \frac{A^2\alpha^{k+m+s}\left(\alpha^{2sn}-1\right) + B^2(-1)^s\beta^{k+m+s}\left(\beta^{2sn}-1\right)}{(\alpha-\beta)\left(\alpha^s-(-1)^s\beta^s\right)} \\ &\quad + e_W(-1)^{s+m} U_{k-m} \sum_{i=0}^{n-1} (-1)^{is}. \end{aligned} \tag{3.4}$$

To proceed, we require two identities that we state together. These identities are

$$\begin{aligned} &A^2\alpha^{k+m+s}\left(\alpha^{2sn}-1\right) \pm B^2\beta^{k+m+s}\left(\beta^{2sn}-1\right) \\ &= \left(A\alpha^{sn}-B\beta^{sn}\right)\left(A\alpha^{sn+s+k+m} \mp B\beta^{sn+s+k+m}\right) \\ &\quad - (A-B)\left(A\alpha^{s+k+m} \mp B\beta^{s+k+m}\right) \qquad \text{if } sn \text{ is even.} \end{aligned} \tag{3.5}$$

The interested reader can prove each of the identities in (3.5) by expanding both sides and noting that $\alpha^{sn}\beta^{sn} = (\alpha\beta)^{sn} = (-1)^{sn} = 1$.

Now let s be even. Then with the use of (3.5) with the plus option on the left, we see via the Binet forms that the right side of (3.4) matches the case of Theorem 2.5 that corresponds to s even. Next, let s be odd and n be even. Accordingly, with the use of (3.5) with the minus option on the left, we see that the right side of (3.4) matches the case of Theorem 2.5 that corresponds to s odd and n even.

We next establish the final case of Theorem 2.5 in which s and n are both odd. To this end, we express the right side of (3.4) as a fraction with denominator $(\alpha-\beta)(\alpha^s+\beta^s)$. The numerator of this fraction is

$$\begin{aligned} &A^2\alpha^{k+m+s}\left(\alpha^{2sn}-1\right) - B^2\beta^{k+m+s}\left(\beta^{2sn}-1\right) \\ &- AB(\alpha\beta)^m\left(\alpha^s+\beta^s\right)\left(\alpha^{k-m}-\beta^{k-m}\right). \end{aligned} \tag{3.6}$$

According to the final case of Theorem 2.5, we need to prove that the expression in (3.6) is equal to

$$\begin{aligned} &\left(A\alpha^{sn+k}-B\beta^{sn+k}\right)\left(A\alpha^{sn+s+m}+B\beta^{sn+s+m}\right) \\ &-\left(A\alpha^{s+k}-B\beta^{s+k}\right)\left(A\alpha^m+B\beta^m\right). \end{aligned} \tag{3.7}$$

The outcome after subtracting (3.7) from (3.6), expanding the result, and then factoring is

$$AB\left(1+(\alpha\beta)^{ns}\right)\left(\alpha^{m+s}\beta^k-\alpha^k\beta^{m+s}\right). \tag{3.8}$$

Since $\alpha\beta = -1$ and ns is odd, the expression in (3.8) reduces to zero. This completes the proof of Theorem 2.5.

4. Concluding Comments

With one exception, our main results yield, as special cases, all the order 2 summations listed in Section 9.4 of Knott [1]. Regarding this claim, it is clear that we need to address three formulas that Knott lists.

The first of the formulas in question is

$$\sum_{i=1}^{2n-1}(2n-i)F_i^2 = F_{2n}^2. \tag{4.1}$$

Since formulas like (4.1) are outside the scope of this paper, none of our main results implies (4.1).

The remaining two formulas in question are

$$\sum_{i=0}^{n}(-1)^{r(1+i)}F_{r(1+i)}^{2} = \frac{1}{5F_r}\Big((-1)^{r(n+1)}F_{(2n+3)r} - (2n+3)F_r\Big), \tag{4.2}$$

$$\sum_{i=0}^{n}(-1)^{r(1+i)}L_{r(1+i)}^{2} = \frac{1}{F_r}\Big((-1)^{r(n+1)}F_{(2n+3)r} + (2n+1)F_r\Big). \tag{4.3}$$

Each of (4.2) and (4.3) cleverly captures both alternating non-alternating sums. Since none of our main results has this property, then clearly no single result of ours yields (4.2) or (4.3). However, for r even, the summands in (4.2) and (4.3) are special cases of the summand of T_1. Accordingly, for r even, set $s = k = m = r$ in T_1. We then see that (4.2) and (4.3) follow from the first case of Theorem 2.3. Likewise, for r odd, (4.2) and (4.3) follow from the first case of Theorem 2.4.

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References

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