ON MODULI FOR WHICH CERTAIN SECOND-ORDER LINEAR RECURRENCES CONTAIN A COMPLETE SYSTEM OF RESIDUES MODULO m

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ABSTRACT. Let $u(a,b)$ denote the Lucas sequence defined by the second-order recursion relation $u_{n+2} = au_{n+1} + bu_n$ with initial terms $u_0 = 0$ and $u_1 = 1$, where a and b are integers. The positive integer m is said to be nondefective if $u(a,b)$ contains a complete system of residues modulo m. All possibilities for m to be nondefective are found when $b = \pm 1$. This paper generalizes results of S. A. Burr for the Fibonacci sequence $u(1,1)$.

1. Introduction

Let $(w) = w(a,b)$ denote the sequence satisfying the second-order linear recursion relation

$$w_{n+2} = aw_{n+1} + bw_n, \tag{1.1}$$

where the parameters a and b and the initial terms w_0 and w_1 are all integers. We distinguish two special recurrences, the Lucas sequence of the first kind (LSFK) $(u) = u(a,b)$ with initial terms $u_0 = 0$ and $u_1 = 1$, and the Lucas sequence of the second kind (LSSK) $(v) = v(a,b)$ with initial terms $v_0 = 2$ and $v_1 = a$.

The positive integer m is said to be *defective* with respect to $w(a,b)$, or simply defective if the recurrence $w(a,b)$ is given, if $w(a,b)$ contains an incomplete system of residues modulo m. Otherwise, m is said to be *nondefective* with respect to $w(a,b)$. In [4], Burr found all nondefective integers m with respect to the Fibonacci sequence. This result will be given in Theorem 2.4. In this paper, we will generalize Burr's result by finding all nondefective integers m with respect to the LSFK $u(a,\pm 1)$. In Theorem 2.6, we find all nondefective integers m with respect to the LSFK $u(a,1)$. In Theorem 2.9, we will prove a similar result with respect to the LSFK $u(a,-1)$.

Associated with the recurrence $w(a,b)$ is the characteristic polynomial

$$f(x) = x^2 - ax - b \tag{1.2}$$

with characteristic roots α and β and discriminant $D = a^2 + 4b = (\alpha - \beta)^2$. By the Binet formulas,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n. \tag{1.3}$$

Throughout this paper, p will denote a prime and m will denote a positive integer. It was shown in [7, pp. 344–345] that $w(a,b)$ is purely periodic modulo m if $\gcd(b,m) = 1$. We will usually take b to equal to ± 1, so that $w(a,b)$ will then automatically be purely periodic modulo m. From here on, we assume that $\gcd(p,b) = \gcd(m,b) = 1$.

Before proceeding further, we will need the following definitions and results. The *period length* $\lambda_w(m)$ of $w(a,b)$ modulo m is the least positive integer r such that $w_{n+r} \equiv w_n \pmod{m}$ for all $n \geq 0$. The *restricted period length* $h_w(m)$ of $w(a,b)$ modulo m is the least positive integer t such that $w_{n+t} \equiv Mw_n \pmod{m}$ for all $n \geq 0$ and some fixed residue M modulo

m such that $\gcd(M, m) = 1$. Here $M = M_w(m)$ is called the *multiplier* of $w(a, b)$ modulo m. Since the LSFK $u(a, b)$ is purely periodic modulo m and has initial terms $u_0 = 0$ and $u_1 = 1$, it is easily seen that $h_u(m)$ is the least positive integer r such that $u_r \equiv 0 \pmod{m}$. It is proven in [7, pp. 354–355] that $h_w(m) \mid \lambda_w(m)$. Let $E_w(m) = \frac{\lambda_w(m)}{h_w(m)}$. Then by [7, pp. 354–355], $E_w(m)$ is the multiplicative order of the multiplier $M_w(m)$ modulo m. Let $h = h_w(m)$. It follows easily that

$$w_{n+hi} \equiv M^i w_n \pmod{m}. \tag{1.4}$$

Given the recurrence (w), we let the set $S_w(m)$ consist of all residues i such that $0 \le i \le m-1$ and $w_n \equiv i \pmod{m}$ for some n. By $A_w(d, m)$ we denote the number of times that the residue d appears in a full period of (w) modulo m and by $N_w(m)$ we denote the number of residues such that $0 \le d \le m-1$ and $A_w(d, m) \ge 1$. Clearly, m is nondefective with respect to (w) if $A_w(d, m) \ge 1$ for all residues d modulo m and $N_w(m) = m$. It is evident that m is defective with respect to (w) if $\lambda_w(m) < m$.

The following results will be helpful for our future work.

Theorem 1.1. *Consider the recurrence $w(a, b)$. Suppose that m is defective. Then every positive multiple of m is also defective.*

Proof. Let tm be a multiple of m, where $t \ge 1$, and suppose that m is defective. Then there exists an integer d such that $0 \le d \le m-1$ and $A_w(d, m) = 0$. Then $A_w(d, tm) = 0$ also and tm is defective. □

Theorem 1.1 was proved in [15] in the case of the Fibonacci sequence.

Theorem 1.2. *Consider the recurrence $w(a, b)$. Then m is nondefective with respect to $w(a, b)$ only if m is nondefective with respect to $u(a, b)$.*

Proof. Suppose that m is nondefective with respect to $w(a, b)$. Since $w(a, b)$ is purely periodic modulo m and $A_w(0, m) \ge 1$, we can assume without loss of generality that $w_0 \equiv 0 \pmod{m}$. If $d = \gcd(w_1, m) > 1$, then by the recursion relation (1.1) defining $w(a, b)$, $d \mid w_n$ for all $n \ge 0$ and $A_w(1, m) = 0$, contrary to assumption. Thus, $w_1 \equiv c \pmod{m}$, where c is invertible modulo m. Then by the recursion relation defining $u(a, b)$,

$$u_n(a, b) \equiv c^{-1} w_n(a, b) \pmod{m}$$

for all $n \ge 0$, and m is also nondefective with respect to $u(a, b)$. □

Remark 1.3. By virtue of Theorem 1.2, we need only consider the LSFK $u(a, b)$ when looking for all possible nondefective integers m with respect to the recurrence $w(a, b)$.

Theorem 1.4. *Consider the LSFK $u(a, b)$ with discriminant D, where $\gcd(a, b) = 1$. Let p be a fixed odd prime such that $p \nmid b$ and let m be a fixed positive integer such that $\gcd(m, b) = 1$. Let $h = h_u(p)$ and $\lambda = \lambda_u(p)$.*

(i) *If $r \mid s$, then $u_r \mid u_s$.*
(ii) *If $d = \gcd(r, s)$, then $\gcd(u_r, u_s) = |u_d|$.*
(iii) *$h > 1$ and $h \mid p - (D/p)$, where (D/p) denotes the Legendre symbol and $(D/p) = 0$ if $p \mid D$.*
(iv) *If $(D/p) = 0$, then $h = p$.*
(v) *If $p \nmid D$, then $h \mid (p - (D/p))/2$ if and only if $(-b/p) = 1$.*
(vi) *$u_n \equiv 0 \pmod{m}$ if and only if $h_u(m) \mid n$.*
(vii) *If $(D/p) = 1$, then $\lambda \mid p - 1$.*
(viii) *$h_u(mn) = \operatorname{lcm}(h_u(m), h_u(n))$ and $\lambda_u(mn) = \operatorname{lcm}(\lambda_u(m), \lambda_u(n))$.*

Proof. We note that $h > 1$, since $u_0 = 0$ and $u_1 = 1$. Part (i) follows from the Binet formula in (1.3). Part (ii) is proved in Theorem VI of [6]. Parts (iii) and (vii) are proved in [6, pp. 44–45] and [12, pp. 290, 296, 297]. Part (iv) is proved in [10, p. 424], while part (v) is proved in [10, p. 441]. Part (vi) follows from part (ii). Part (viii) follows from part (i), (ii), and (vi). □

Corollary 1.5. *Consider the LSFK $u(a, b)$ with discriminant D, where $\gcd(a, b) = 1$. Let p be a fixed odd prime such that $p \nmid b$. If $(D/p) = 1$, then p is defective.*

Proof. By part (vii) of Theorem 1.4, $\lambda_u(p) \le p - 1$. Then $N_u(p) \le \lambda_u(p) \le p - 1$, and p is defective. □

Corollary 1.6. *Consider the LSFK $u(a, b)$, where $\gcd(a, b) = 1$. Suppose that $m_1 \mid m_2$, where $m_2 > m_1$ and $h_u(m_1) = h_u(m_2)$. Then m_2 is defective. In particular, if a is even and $h_u(m_1)$ is even, then $h_u(m_1) = h_u(2m_1)$ and $2m_1$ is defective.*

Proof. First suppose that $m_1 \mid m_2$, where $m_2 > m_1$, and $h_u(m_1) = h_u(m_2)$. By Theorem 1.4 (vi), $u_n \equiv 0 \pmod{m_1}$ if and only if $u_n \equiv 0 \pmod{m_2}$. It thus follows that $m_1 \notin S_u(m_2)$, and thus m_2 is defective.

Now suppose that a is even and $h_u(m_1)$ is even. Then $u_2 = a \equiv 0 \pmod 2$ and $h_u(2) = 2$. Then by Theorem 1.4 (viii),

$$h_u(2m_1) = \operatorname{lcm}(h_u(2), h_u(m_1)) = \operatorname{lcm}(2, h_u(m_1)) = h_u(m_1).$$

Thus, by our argument above, $m_1 \notin S_u(2m_1)$, and $2m_1$ is defective. □

The recurrence $w(a, b)$ is said to be *uniformly distributed* (u.d.) modulo m if $A_w(d_1, m) = A_w(d_2, m)$ for all pairs (d_1, d_2) of residues modulo m such that $0 \le d_1 < d_2 \le m - 1$. It is clear that m is nondefective with respect to (w) if (w) is u.d. modulo m. Then m is said to be *purely nondefective* with respect to (w) if (w) is u.d. modulo m, while the nondefective integer m is said to be *impurely nondefective* with respect to (w) otherwise. In particular, 1 is always considered to be purely nondefective with respect to (w). Theorem 1.7 due to Bumby [3] and Webb and Long [25] completely determines all purely nondefective integers m with respect to the LSFK $u(a, b)$ with discriminant D. In particular, it is shown that m is purely nondefective with respect to $u(a, b)$ only if $p \mid D$ for each prime divisor p of m.

Theorem 1.7. *Consider the LSFK $u(a, b)$ with discriminant D. Then*

(i) *$u(a, b)$ is uniformly distributed modulo m if and only if it is u.d. modulo all prime power factors of m. In particular, $u(a, b)$ is u.d. modulo m if and only if $u(a, b)$ is u.d. modulo m_1 for each divisor m_1 of m.*

(ii) *$u(a, b)$ is u.d. modulo p if and only if $p \mid D$.*

(iii) *$u(a, b)$ is u.d. modulo p^e for $e \ge 2$ only if $u(a, b)$ is u.d. modulo p.*

(iv) *If $p \ge 5$, then $u(a, b)$ is u.d. modulo p^e for $e \ge 2$ if and only if $u(a, b)$ is u.d. modulo p.*

(v) *If $p = 2$, then $u(a, b)$ is u.d. modulo 2^e for $e \ge 2$ if and only if $a \equiv 2 \pmod 4$ and $b \equiv 3 \pmod 4$.*

(vi) *If $p = 3$, then $u(a, b)$ is u.d. modulo 3^e for $e \ge 2$ if and only if $a \equiv \pm 1 \pmod 3$ and $b \equiv -1 \pmod 3$, but $a^2 \not\equiv -b \pmod 9$.*

(vii) *If $u(a, b)$ is u.d. modulo p^e for some $e \ge 1$, then $h_u(p^e) = p^e$ and*

$$E_u(p^e) = A_u(d, p^e) = \begin{cases} 1, & \text{if } p = 2, \\ \operatorname{ord}_p a/2, & \text{if } p \ge 3; \end{cases}$$

where d is any residue such that $0 \le d \le p^e - 1$ and $\operatorname{ord}_{p^e} r$ denotes the multiplicative order of r modulo p^e.

(viii) *If $b = 1$, then $u(a,b)$ is u.d. modulo p only if $p = 2$ or $p \equiv 1 \pmod 4$.*

Proof. Parts (i)–(vii) follow from the results in [3] and [25]. Part (vii) is also proved in [23]. We now prove part (viii). Suppose that $u(a,1)$ is u.d. modulo p. By part (ii), $D = a^2 + 4 \equiv 0 \pmod p$, which implies that either $p = 2$ or $\left(\frac{-4}{p}\right) = \left(\frac{-1}{p}\right) = 1$ if p is odd. By the law of quadratic reciprocity, $p \equiv 1 \pmod 4$ if p is odd. □

Corollary 1.8. *Consider the LSFK $u(a,b)$.*

(i) *If $u(a,b)$ is u.d. modulo p, but $u(a,b)$ is not u.d. modulo p^2, then p^e is defective for all $e \geq 2$.*

(ii) *If $b = 1$ and p is an odd prime such that $u(a,b)$ is u.d. modulo p^e for some $e \geq 1$, then*

$$E_u(p^e) = A_u(d,p^e) = 4 \tag{1.5}$$

for all $d \in \{0,1,2,\ldots,p^e-1\}$.

(iii) *Suppose that $b = -1$ and p is an odd prime such that $u(a,b)$ is u.d. modulo p. This occurs if and only if $a \equiv \pm 2 \pmod p$. Moreover, if $u(a,b)$ is u.d. modulo p^e for some $e \geq 1$, then*

$$E_u(p^e) = A_u(d,p^e) = \begin{cases} 1, & \text{if } a \equiv 2 \pmod p, \\ 2, & \text{if } a \equiv -2 \pmod p; \end{cases} \tag{1.6}$$

for all $d \in \{0,1,2\ldots,p^e-1\}$.

Proof. (i) By Theorem 1.7 (iv), we need to consider only the cases in which $p = 2$ or $p = 3$. By Theorem 1.7 (v) and (vi) and Corollary 1.6, it suffices to prove that if $p \in \{2,3\}$ and $u(a,b)$ is u.d. modulo p, but $u(a,b)$ is not u.d. modulo p^2, then p^2 is defective. First suppose that $p = 2$. Then by Theorem 1.7 (ii) and (v), $u(a,b)$ is u.d. modulo 2, but $u(a,b)$ is not u.d. modulo 2^e for some $e \geq 2$, if and only if b is odd and a is even, but it is not the case that $a \equiv 2 \pmod 4$ and $b \equiv 3 \pmod 4$. If $a \equiv 0 \pmod 4$ and $b \equiv 1 \pmod 2$, then $u_2 = a \equiv 0 \pmod 4$ and $h_u(2) = h_u(4) = 2$. It then follows from Corollary 1.6 that $2 \notin S_u(4)$ and thus 4 is defective. Now suppose that $a \equiv 2 \pmod 4$ and $b \equiv 1 \pmod 4$. By inspection, one sees that $3 \notin S_u(4)$ and 4 is again defective.

We now consider the case in which $u(a,b)$ is u.d. modulo 3, but $u(a,b)$ is not u.d. modulo 3^e for some $e \geq 2$. By Theorem 1.7 (ii) and (vi), this occurs if and only if $a \equiv \pm 1 \pmod 3$ and $b \equiv -a^2 \pmod 9$. Then $u_1 = 1$, $u_2 = a \equiv \pm 1 \pmod 3$, and $u_3 = a^2 + b \equiv 0 \pmod 9$. Therefore, $h_u(3) = h_u(9) = 3$ and by Corollary 1.6, $3 \notin S_u(9)$ and 9 is defective.

(ii) Suppose that $u(a,b)$ is u.d. modulo p^e for some $e \geq 1$. Then by Theorem 1.7 (iii), $u(a,b)$ is u.d. modulo p. Moreover, by Theorem 1.7 (ii), $p \mid D = a^2 + 4$. Hence,

$$\frac{a^2}{4} \equiv \left(\frac{a}{2}\right)^2 \equiv -1 \pmod p.$$

Thus, $\mathrm{ord}_p a/2 = 4$ and (1.5) holds by Theorem 1.7 (vii).

(iii) Suppose that $b = -1$ and $u(a,-1)$ is u.d. modulo p. Then $D = a^2 - 4 \equiv 0 \pmod p$ and $a \equiv \pm 2 \pmod p$. Then (1.6) holds by Theorem 1.7 (vii). □

Theorem 1.9. *Consider the LSFK $u(a,b)$ with discriminant D.*

(i) *m is purely nondefective only if $p \mid D$ for each prime divisor p of m.*

(ii) *2 is impurely nondefective if and only if $a \equiv b \equiv 1 \pmod 2$, in which case $h_u(2) = \lambda_u(2) = 3$.*

(iii) *m is impurely nondefective only if $(D/p) = -1$ for some odd prime divisor p of m or $2 \mid m$ and 2 is impurely nondefective.*

Proof. Part (i) follows from Theorem 1.7 (i)–(iii). Part (ii) follows by inspection. Part (iii) follows from parts (i) and (ii), Theorem 1.1, Corollary 1.5, and Corollary 1.8 (i). □

2. Previous Results and the Main Theorems

Shah [15] proved that the prime p is defective with respect to the Fibonacci sequence $\{F_n\} = u(1,1)$ if $p \equiv \pm 1 \pmod{10}$ or $p \equiv 13$ or $17 \pmod{20}$, while 2, 3, 5, and 7 are nondefective with respect to the Fibonacci sequence. Bruckner [2] proved the remaining cases that p is defective with respect to $\{F_n\}$ if $p > 7$ with $p \equiv 3$ or $7 \pmod{20}$. Somer [17] partially generalized the results of Shah and Bruckner by showing that p is defective with respect to $u(a,1)$ if $p > 7$, $p \nmid D = a^2+4$, and $p \not\equiv 1$ or $9 \pmod{20}$. Schinzel [14], completely generalized the results of Shah and Bruckner by proving Theorem 2.1 below.

Theorem 2.1. *Consider the LSFK $u(a,1)$. Then p is defective if $p > 7$ and $p \nmid D = a^2+4$.*

Li [11] also proved Theorem 2.1 by extending the methods of Somer [17].

Somer [19] also obtained similar results in Theorem 2.2 to those in Theorem 2.1 by considering the LSFK $u(a,-1)$.

Theorem 2.2. *Consider the LSFK $u(a,-1)$. Then p is defective if $p \geq 5$ and $p \nmid D = a^2-4$.*

Remark 2.3. By Theorems 1.1 and 2.1, one sees that m can be nondefective with respect to the LSFK $u(a,1)$ only if $p \leq 7$ or $p \mid a^2+4$ for each prime divisor p of m. It similarly follows from Theorem 2.2 that p can be nondefective with respect to $u(a,-1)$ only if $p = 2$ or 3 or $p \mid a^2-4$ for each prime factor p of m.

Burr [4] generalized the results of Shah and Bruckner by completely determining all nondefective integers m with respect to the Fibonacci sequence.

Theorem 2.4. *Consider the Fibonacci sequence $\{F_n\}$. Then m is nondefective if and only if m has one of the following forms:*

$$5^k,\ 2\cdot 5^k,\ 4\cdot 5^k,\ 3^j\cdot 5^k,\ 6\cdot 5^k,\ 7\cdot 5^k,\ 14\cdot 5^k, \tag{2.1}$$

where $k \geq 0$ and $j \geq 1$. Moreover, m is purely nondefective if and only if m is of the form 5^k.

Theorem 2.5 due to Avila and Chen [1] complements Theorem 2.4 by finding all nondefective integers m for the Lucas sequence $\{L_n\} = v(1,1)$.

Theorem 2.5. *Consider the Lucas sequence $\{L_n\}$. Then m is nondefective if and only if m is equal to one of the following numbers:*

$$2,\ 4,\ 6,\ 7,\ 14,\ 3^k,$$

where $k \geq 0$.

Theorems 2.6 and 2.9 below generalize Burr's result for the LSFK's $u(a,1)$ and $u(a,-1)$.

Theorem 2.6. *Consider the LSFK $u(a,1)$ with discriminant $D = a^2+4$. Let L be the set of integers $\ell \geq 1$ such that each prime divisor of ℓ also divides a^2+4 and $4 \nmid \ell$. Then m is nondefective only if m is of the form*

$$\ell,\ 2\ell\ (\ell \text{ odd}),\ 3^k\ell\ (\ell \text{ odd}),\ 4\ell\ (\ell \text{ odd}),\ 5^k\ell\ (\ell \text{ odd}),\ 6\ell\ (\ell \text{ odd}),\ 7\ell\ (\ell \text{ odd}),\ \text{or } 14\ell\ (\ell \text{ odd}), \tag{2.2}$$

where $k \geq 0$, $\ell \in L$, and either $m/\ell = 1$ or $m/\ell \notin L$. Moreover the following hold:

(i) *$\ell \in L$ only if $\ell = 1$ or each odd prime divisor of ℓ is congruent to 1 modulo 4.*
(ii) *m is purely nondefective if and only if $m = \ell$ for some $\ell \in L$ and a is even if $2 \mid m$.*

(iii) 2ℓ (ℓ *odd*) *is impurely nondefective if and only if* a *is odd. In particular,* 2ℓ (ℓ *odd*) *is nondefective no matter what value* a *has.*
(iv) 3ℓ *is impurely nondefective if and only if* ℓ *is odd and* $a \equiv \pm 1 \pmod 3$.
(v) $3^k\ell$ *is impurely nondefective for* $k \geq 2$ *if and only if* ℓ *is odd, and* $a \equiv \pm 1$ *or* $\pm 2 \pmod 9$.
(vi) 4ℓ *is impurely nondefective if and only if* ℓ *is odd and* $a \equiv \pm 1 \pmod 4$.
(vii) *If* $5 \nmid a^2 + 4$, *then* 5ℓ *is impurely nondefective if and only if* ℓ *is odd and* $a \equiv \pm 2 \pmod 5$.
(viii) *If* $5 \nmid a^2 + 4$, *then* $5^k\ell$ *is impurely nondefective for* $k \geq 2$ *if and only if* ℓ *is odd,* $a \equiv \pm 2 \pmod 5$, *but* $a \not\equiv \pm 7 \pmod{25}$.
(ix) 6ℓ *is impurely nondefective if and only if* ℓ *is odd and* $a \equiv \pm 1 \pmod 6$.
(x) 7ℓ *is impurely nondefective if and only if* ℓ *is odd, and* $a \equiv \pm 1$ *or* $\pm 3 \pmod 7$.
(xi) 14ℓ *is impurely nondefective if and only if* ℓ *is odd, and* $a \equiv \pm 1$ *or* $\pm 3 \pmod{14}$.

Theorem 2.6 will be proved in Section 4. Corollary 2.7 below will determine all those integers a for which m is nondefective with respect to $u(a, 1)$ if and only if m is purely nondefective. Corollary 2.8 will find all integers a for which there are nondefective integers with respect to $u(a, 1)$ satisfying each of the possible forms given in Theorem 2.6.

Corollary 2.7. *Consider the LSFK* $u(a, 1)$. *Then the integer* m *is nondefective if and only if* m *is purely nondefective exactly when*

$$a \equiv 0,\ 30,\ 54,\ 84,\ 96,\ 114,\ 126,\ 156,\ \text{or } 180 \pmod{210}.$$

Proof. This follows from Theorem 2.6 and the Chinese Remainder Theorem. □

Corollary 2.8. *Consider the LSFK* $u(a, 1)$. *Then there exists a nondefective integer* m *satisfying each of the possible cases in Theorem 2.6 if and only if* $a \equiv 1 \pmod 2$, $a \equiv \pm 1, \pm 2 \pmod 9$, $a \equiv \pm 2, \pm 3, \pm 8, \pm 12 \pmod{25}$, *and* $a \equiv \pm 1, \pm 3 \pmod 7$. *The least such integer in absolute value is* ± 17.

Proof. This follows from Theorem 2.6 and the use of the Chinese Remainder Theorem. □

Theorem 2.9. *Consider the LSFK* $u(a, -1)$ *with discriminant* $D = a^2 - 4$. *Let* L' *be the set of odd integers* $\ell' \geq 1$ *such that each prime divisor of* ℓ' *divides* $a^2 - 4$ *and is greater than or equal to* 5. *In particular,* $p \mid \ell'$ *if and only if* $p \geq 5$ *and* $a \equiv \pm 2 \pmod p$. *Then* m *is nondefective only if* m *is of the form*

$$2^i 3^j \ell', \tag{2.3}$$

where $i \geq 0$, $j \geq 0$, *and* $\ell' \in L'$. *Moreover, the following hold:*

(i) $m = 2^i 3^j \ell'$ *is purely nondefective if and only if* 2^i *and* 3^j *are both purely nondefective. Moreover,* 2 *is purely nondefective if and only if* a *is even; and* 3 *is purely nondefective if and only if* $a \equiv \pm 1 \pmod 3$. *Further,* 2^i *is purely nondefective for* $i \geq 2$ *if and only if* $a \equiv 2 \pmod 4$, *while* 3^j *is purely nondefective for* $j \geq 2$ *if and only if* $a \equiv \pm 2$ *or* $\pm 4 \pmod 9$.
(ii) *If* $a = \pm 2$ *then* $D = 0$ *and all positive integers* m *are purely nondefective. Furthermore,* $u_n = n$ *for all* $n \geq 0$ *if* $a = 2$, *while* $u_n = (-1)^{n+1} n$ *for* $n \geq 0$ *if* $a = -2$.
(iii) m *is impurely nondefective if and only if* $m = 2\ell', 3\ell'$, *or* $6\ell'$.
(iv) $2\ell'$ *is impurely nondefective if and only if* a *is odd. In particular,* $2\ell'$ *is nondefective for any integer* a.
(v) $3\ell'$ *is impurely nondefective if and only if* $a \equiv 0 \pmod 3$. *In particular,* $3\ell'$ *is nondefective for any integer* a.

(vi) $6\ell'$ *is impurely nondefective if and only if* $a \equiv 3 \pmod 6$.

Theorem 2.9 will be proved in Section 4.

Corollary 2.10. *Consider the LSFK* $u(a,-1)$. *Then the integer* m *is nondefective if and only if* m *is purely nondefective exactly when* $a \equiv 2$ *or* $4 \pmod 6$.

Proof. This follows from Theorem 2.9 and application of the Chinese Remainder Theorem. □

Corollary 2.11. *Consider the LSFK* $u(a,-1)$. *Then there exists a nondefective integer* m *satisfying each of the possible cases in Theorem 2.9 if and only if* $a \equiv 3 \pmod 6$.

Proof. This follows from Theorem 2.9 and the Chinese Remainder Theorem. □

3. Further Results and Definitions

The following definitions and results will be needed for the proofs of our main theorems, Theorem 2.6 and Theorem 2.9.

Theorem 3.1. *Let* $u(a,1)$ *be a LSFK with discriminant* D. *Let* p *be an odd prime. Then*

(i) $E_u(p) = 1, 2,$ *or* 4.
(ii) $E_u(p) = 1$ *if and only if* $h_u(p) \equiv 2 \pmod 4$. *Moreover, if* $E_u(p) = 1$, *then* $(D/p) = 1$.
(iii) $E_u(p) = 2$ *if and only if* $h_u(p) \equiv 0 \pmod 4$. *Moreover, if* $E_u(p) = 2$, *then* $(D/p) = (-1/p)$.
(iv) $E_u(p) = 4$ *if and only if* $h_u(p)$ *is odd. Moreover, if* $E_u(p) = 4$, *then* $p \equiv 1 \pmod 4$.
(v) *If* $p \equiv 3 \pmod 4$ *and* $(D/p) = 1$, *then* $h_u(p) \equiv 2 \pmod 4$, $E_u(p) = 1$, *and* $M_u(p) \equiv 1 \pmod p$.
(vi) *If* $p \equiv 3 \pmod 4$ *and* $(D/p) = -1$, *then* $h_u(p) \equiv 0 \pmod 4$, $E_u(p) = 2$, *and* $M_u(p) \equiv -1 \pmod p$.
(vii) *If* $p \equiv 1 \pmod 4$ *and* $(D/p) = -1$, *then* $h_u(p)$ *is odd and* $E_u(p) = 4$.

This follows from Lemma 3 and Theorem 13 of [16].

Theorem 3.2. *Let* $u(a,-1)$ *be a LSFK with discriminant* D. *Let* p *be an odd prime. Then*

(i) $E_u(p) = 1$ *or* 2.
(ii) *If* $\lambda_u(p)$ *is odd, then* $h_u(p)$ *is odd,* $E_u(p) = 1$, *and* $M_u(p) \equiv 1 \pmod p$.
(iii) *If* $\lambda_u(p) \equiv 2 \pmod 4$, *then* $h_u(p)$ *is odd,* $E_u(p) = 2$, *and* $M_u(p) \equiv -1 \pmod p$.
(iv) *If* $\lambda_u(p) \equiv 0 \pmod 4$, *then* $h_u(p)$ *is even,* $E_u(p) = 2$, *and* $M_u(p) \equiv -1 \pmod p$.

This follows Theorem 16 of [16].

Lemma 3.3. *Consider the LSFK* $u(a,1)$ *with discriminant* D. *Let* p *be an odd prime.*

(i) *If* $p \equiv 3 \pmod 4$ *and* p *is nondefective, then* $h_u(p) = p + 1$.
(ii) *If* $p \equiv 1 \pmod 4$ *and* p *is impurely nondefective, then* $h_u(p) = (p+1)/2$.

Proof. (i) Suppose that $p \equiv 3 \pmod 4$ and p is nondefective. By Theorem 1.7 (viii), p is impurely nondefective. Then by Theorem 1.9 (iii), $(D/p) = -1$. Thus, $h_u(p) \mid p+1$ by Theorem 1.4 (iii). Since $p \equiv 3 \pmod 4$, it follows by the law of quadratic reciprocity that $(-1/p) = -1$. Thus by Theorem 1.4 (v), $h_u(p) \nmid (p+1)/2$. Therefore, if $h_u(p) \neq p+1$, then $h_u(p) \leq (p+1)/3$. However, by Theorem 3.1 (vi), $E_u(p) = 2$. Thus,

$$\lambda_u(p) = E_u(p) \cdot h_u(p) \leq 2\frac{p+1}{3} < p.$$

Hence, $N_u(p) \leq \lambda_u(p) < p$, and p is defective in this case. Consequently $h_u(p) = p + 1$.

(ii) Suppose that $p \equiv 1 \pmod 4$ and p is impurely nondefective. Then by Theorem 1.9 (iii), $(D/p) = -1$. Since $p \equiv 1 \pmod 4$, we see by the law of quadratic reciprocity that $(-1/p) = 1$. Then by Theorem 1.4 (iii) and (v), $h_u(p) \mid (p+1)/2$. Suppose that $h_u(p) \neq (p+1)/2$. Since $p \equiv 1 \pmod 4$, it follows that $p + 1 \equiv 2 \pmod 4$. Thus, $h_u(p) \leq (p+1)/6$. By Theorem 3.1 (i), $E_u(p) \leq 4$. Hence,

$$N_u(p) \leq \lambda_u(p) = E_u(p) \cdot h_u(p) \leq 4\frac{p+1}{6} < p.$$

Therefore, $h_u(p) = (p+1)/2$. □

Recall that a *Mersenne prime* is a prime of the form $2^q - 1$, where q is a prime.

Lemma 3.4. *Consider the LSFK $u(a,1)$ with discriminant D and let p be a Mersenne prime. If $(D/p) = -1$, then $h_u(p) = p+1$.*

Proof. If follows from the fact that p is a Mersenne prime that $p \equiv 3 \pmod 4$. Thus, $(-1/p) = -1$. It now follows from Theorem 1.4 (iii) and (v) that $h_u(p) \mid p+1$ and $h_u(p) \nmid (p+1)/2$. Since $p + 1 = 2^q$ for some prime q, we see that $h_u(p) = p + 1$. □

Lemma 3.5. *Consider the LSFK $u(a,1)$ with discriminant D. Suppose that $p \in \{3,7\}$ and that $(D/p) = -1$. Then p is impurely nondefective.*

Proof. By Lemma 3.4, $h_u(p) = p+1$. By Theorem 8 (v) of [18], it follows that p is nondefective. We now see by Theorem 1.7 (ii) that p is in fact impurely nondefective. □

Theorem 3.6. *Let $e > 1$. Consider the LSFK $u(a,b)$. Let p be an odd prime. Suppose that $h_u(p^2) \neq h_u(p)$. Then the following hold:*

(i) $\lambda_u(p^2) \neq \lambda_u(p)$.
(ii) $h_u(p^e) = p^{e-1}h_u(p)$.
(iii) $\lambda_u(p^e) = p^{e-1}\lambda_u(p)$.
(iv) $E_u(p^e) = E_u(p)$.

Proof. Part (i) follows from the discussion in [5, p. 697]. Part (ii) is proved in [6, p. 42] and part (iii) is proved in [24, pp. 619–620]. Part (iv) follows from parts (ii) and (iii). □

Given the recurrence $w(a,b)$, we define the function

$$\Delta^{(n)}w = w_{n+1}^2 - aw_nw_{n+1} - bw_n^2. \tag{3.1}$$

The recurrence $w(a,b)$ is said to be *p-regular* if $\Delta^0(w) \not\equiv 0 \pmod p$; otherwise, the recurrence is called *p-irregular*. We note that $w(a,b)$ is p-irregular if and only if $w(a,b)$ also satisfies a recursion relation modulo p of order less than two. It is well-known and proved in Lemma 2.1 of [22] that if $w(a,b)$ is p-regular, then

$$\Delta^{(n)}(w) = (-b)^n\Delta^{(0)}(w), \tag{3.2}$$

(see also [9, p. 723]).

Theorem 3.7. *Suppose that the recurrence $w(a,b)$ and $w'(a,b)$ are both p-regular. Then*

$$\lambda_w(p) = \lambda_{w'}(p), \quad h_w(p) = h_{w'}(p), \quad E_w(p) = E_{w'}(p),$$

and

$$M_w(p) \equiv M_{w'}(p) \pmod p.$$

This is proved in [5, p. 695].

Theorem 3.8. *Let p be a fixed prime. Consider the LSFK $u(a,b)$ and the LSSK $v(a,b)$ with discriminant D. Then*

(i) *$u(a,b)$ is p-regular. In particular, if $b=1$, then*

$$\Delta^{(n)}(u) = (-1)^n. \tag{3.3}$$

(ii) *$v(a,b)$ is p-regular if and only if $p \nmid D$.*

Proof. (i) We note that

$$\Delta^{(0)}(u) = u_1^2 - au_0u_1 - bu_0^2 = 1^2 - a \cdot 0 \cdot 1 - b \cdot 0^2 \equiv 1 \pmod{p}. \tag{3.4}$$

Thus, $u(a,b)$ is p-regular. Moreover, if $b=1$, then equation (3.3) follows from (3.4) and (3.2).

(ii) We observe that

$$\Delta^{(0)}(v) = v_1^2 - av_0v_1 - bv_0^2 = a^2 - a \cdot 2 \cdot a - b \cdot 2^2 = -a^2 - 4b = -D.$$

Thus $v(a,b)$ is p-regular if and only if $p \nmid D$. □

Definition 3.9. *The recurrences $w(a,b)$ and $w'(a,b)$ are called p^e-equivalent for $e \geq 1$ if there exists an integer c and a fixed t such that $\gcd(c,p^e)=1$ and*

$$w'_n \equiv cw_{n+t} \pmod{p^e} \tag{3.5}$$

for all $n \geq 0$.

Definition 3.10. *The recurrences $w(a,b)$ and $w'(a,b)$ are said to be cyclically p^e-equivalent for $e \geq 1$ if there exists a fixed integer t such that*

$$w'_n \equiv w_{n+t} \pmod{p^e} \tag{3.6}$$

for all $n \geq 0$.

Remark 3.11. It is clear that if two recurrences are cyclically p^e-equivalent, then they are p^e-equivalent. It is also evident that both p^e-equivalence and cyclic p^e-equivalence are indeed equivalence relations on the set of recurrences $w(a,b)$ modulo p^e. It is easily seen that if $w(a,b)$ and $w'(a,b)$ are cyclically p^e-equivalent, then

$$A_{w'}(d,p^e) = A_w(d,p^e) \tag{3.7}$$

for all $d \in \{0,1,2,\ldots,p^e-1\}$.

Lemma 3.12. *Suppose that $w(a,b)$ and $w'(a,b)$ are p^e-equivalent recurrences. Then $w(a,b)$ and $w'(a,b)$ are either both p-regular or both p-irregular.*

This is proved in [5, p. 694]

Lemma 3.13. *Suppose that $w'(a,b)$ and $w(a,b)$ are both p-regular recurrences and that $w'(a,b)$ is p^e-equivalent to $w(a,b)$. Suppose further that*

$$w'_n \equiv cw_{n+t} \pmod{p^e} \quad \text{for all } n \geq 0,$$

where $\gcd(c,p^e)=1$ and t is a fixed integer. Let $M = M_w(p^e)$. Then $w'(a,b)$ is also cyclically p^e-equivalent to $w(a,b)$ if and only if

$$c \equiv M^i \pmod{p^e}$$

for some $i \in \{0,1,\ldots,E_w(p^e)-1\}$.

Proof. This follows from the definition of $M_w(p)$ and from (1.4). □

Lemma 3.14. *Let $\mathcal{F}(a,b)$ denote the set of all p-regular recurrences $w(a,b)$. Let $D = a^2 + 4b$. In particular, $u(a,b) \in \mathcal{F}(a,b)$. Suppose that $h_u(p^2) \neq h_u(p)$. Then the number of p^e-equivalence classes $T(a,b,p^e)$ of $\mathcal{F}(a,b)$ for any $e \geq 1$ is given by*

$$T(a,b,p^e) = \frac{p - (D/p)}{h_u(p)}. \tag{3.8}$$

Proof. By Theorem 3.8 (i), $u(a,b) \in \mathcal{F}(a,b)$, and by Theorem 1.4 (iii), $h_u(p) \mid (p - D/p)$. Equation (3.8) follows from Theorem 2.14 and Corollary 2.15 in [5]. □

Lemma 3.15. *Let p be an odd prime and let $e \geq 1$ be a fixed integer. Consider the LSFK $u(a,1)$ with discriminant D. Suppose that $(D/p) = -1$, $h_u(p) = p+1$, and $h_u(p^2) \neq h_u(p)$. Then $p \equiv 3 \pmod 4$. Suppose that $w(a,1)$ is a p-regular recurrence. Then $w(a,1)$ is cyclically p^e-equivalent to $u(a,1)$ if and only if*

$$\Delta^{(0)}(w) \equiv \pm 1 \pmod{p^e}. \tag{3.9}$$

Proof. The fact that $p \equiv 3 \pmod 4$ follows from Theorem 1.4 (iii) and (v) and the observation that $(-1/p) = -1$ if and only if $p \equiv 3 \pmod 4$. First suppose that congruence (3.9) holds. By equation (3.8) in the statement of Lemma 3.14, there is only one p^e-equivalence class of p-regular recurrences in $\mathcal{F}(a,b)$. Since $u(a,1)$ is p-regular by Theorem 3.8 (i), $w(a,1)$ is p^e-equivalent to $u(a,1)$. Thus,

$$w_n(a,1) \equiv c u_{n+t}(a,1) \pmod{p^e} \tag{3.10}$$

for some fixed integer t, some integer c such that $\gcd(c,p^e) = 1$, and all $n \geq 0$. Then by (3.1), (3.3), (3.9), and (3.10),

$$\begin{aligned}\Delta^{(0)}(w) &= w_1^2 - a w_0 w_1 - b w_0^2 \equiv (c u_{t+1})^2 - a c u_t c u_{t+1} - b(c u_t)^2 \\ &= c^2 \Delta^{(t)}(u) = (-1)^t c^2 \equiv \pm 1 \pmod{p^e}. \end{aligned} \tag{3.11}$$

Let $M = M_u(p^e)$. It follows from Theorem 3.1 (vi) and Theorem 3.6 (iv) that

$$E_u(p^e) = E_u(p) = 2. \tag{3.12}$$

Hence,

$$M \equiv -1 \pmod{p^e}. \tag{3.13}$$

Then

$$M^2 \equiv 1 \pmod{p^e}. \tag{3.14}$$

Since $p \equiv 3 \pmod 4$, there exists no solution to the congruence

$$x^2 \equiv -1 \pmod{p^e}. \tag{3.15}$$

Thus, by (3.11) and (3.13)–(3.15), $c \equiv M^i \pmod{p^e}$ for some $i \in \{0,1\}$. It now follows from Lemma 3.13 that $w(a,1)$ is cyclically p^e-equivalent to $u(a,1)$.

Now assume that $w(a,1)$ is cyclically p^e-equivalent to $u(a,1)$. Then by Lemma 3.13 and (3.13),

$$w_n(a,1) \equiv M^i u_{n+t}(a,1) \equiv (-1)^i u_{n+t}(a,1) \pmod{p^e} \tag{3.16}$$

for some fixed integer t and some $i \in \{0,1\}$. Thus by (3.11) and (3.16),

$$\Delta^{(0)}(w) \equiv (M^i)^2 \Delta^{(t)}(u) \equiv (-1)^{2i}(-1)^t \equiv \pm 1 \pmod{p^e}.$$

□

By Remark 2.3, one sees that m can be nondefective with respect to the LSFK $u(a,1)$ only if m is of the form $m = m_1\ell$, where $p \leq 7$ and $p \nmid a^2+4$ for each prime divisor p of m_1, whereas $q \mid a^2+4$ for each prime factor q of ℓ. Similarly, by Remark 2.3, m is nondefective with respect to the LSFK $u(a,-1)$ only if m is of the form $m = m_2\ell'$, where $p \leq 3$ for each prime divisor p of m_2, while $q \geq 5$ and $q \mid a^2-4$ for each prime q dividing ℓ'. The remaining results of this section will mostly examine exactly which integers having the forms given above are purely nondefective or impurely nondefective with respect to $u(a,1)$ or $u(a,-1)$.

Theorem 3.16. *Consider the LSFK $u(a,b)$ with discriminant D. Suppose that p^e is purely nondefective for $e \geq 1$. Then $p \mid D$ and*

$$\lambda(p^e) = p^e E, \tag{3.17}$$

where $E = E_u(p)$. Let m be an integer and let $\lambda = \lambda_u(m)$. Suppose that $p \nmid \lambda$. Then

$$A(d, m\cdot p^e) = \frac{E}{\gcd(\lambda, E)} \cdot A_u(d,m) \tag{3.18}$$

for all $d \in \{0,1,\ldots,mp^e-1\}$.

Proof. The fact that $p \mid D$ follows from Theorem 1.7 (ii) and (iii), while equation (3.17) is proved in Theorem 1.7 (vii). Equation (3.18) is proved in [8]. □

Corollary 3.17. *Consider the LSFK $u(a,b)$ with discriminant $D = a^2+4b$, where $b = \pm 1$. Suppose that ℓ is purely nondefective and that $p \geq 5$ for each prime p dividing ℓ. Then $p \mid D$ for each prime divisor p of ℓ. Suppose also that m is impurely nondefective and let $\lambda = \lambda_u(m)$. Suppose further that* $\gcd(m,D) = 1$. *Then*

$$\gcd(\lambda, \ell) = 1,$$

and $m\ell$ is also impurely nondefective.

Proof. The result is obvious if $\ell = 1$, so we now suppose that $\ell > 1$. Let $\ell = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, where $r \geq 1$ and $p_1 < p_2 < \cdots < p_r$. First suppose that $b = 1$. By Theorem 1.7 and Corollary 1.8 (ii),

$$p_i \mid D, \quad p_i \equiv 1 \pmod 4, \quad \text{and } E_u(p_i^{e_i}) = 4 \tag{3.19}$$

for $1 \leq i \leq r$. By Remark 2.3, $p \mid m$ only if $p \leq 7$. It now follows from Theorem 1.4, Theorem 1.1, Corollary 1.6, Theorem 1.9 (ii) and (iii), and Theorem 3.6 that $5 \nmid \ell$ if $5 \mid m$ and $p \mid \lambda(m)$ only if $p \leq 7$. We now see from (3.19) that

$$\gcd(\lambda, \ell) = 1. \tag{3.20}$$

If $r = 1$, then by Theorem 3.16, (3.19), (3.20), and the fact that m is impurely nondefective, we have that

$$A_u(d, mp_1^{e_1}) = \frac{E_u(p_1^{e_1})}{\gcd(\lambda, E_u(p_1^{e_1}))} \cdot A_u(d,m) = \frac{4}{\gcd(\lambda,4)} \cdot A_u(d,m) \geq g_1 \tag{3.21}$$

for all $d \in \{0,1,\ldots,mp_1^{e_1}-1\}$ and some positive integer g_1 such that $g_1 \mid 4$. Hence, $mp_1^{e_1}$ is impurely nondefective by Theorem 1.7 (i)–(iii).

Now suppose that $r = 2$. Then by Theorem 1.4 (viii), Theorem 1.7 (vii), and Corollary 1.8 (ii),

$$\lambda(mp_1^{e_1}) = \operatorname{lcm}(\lambda(m), \lambda(p_1^{e_1})) = \operatorname{lcm}(\lambda(m), 4p_1^{e_1}) \mid 4\lambda(m)p_1^{e_1}. \tag{3.22}$$

Since $\gcd(\lambda(m), \ell) = 1$ by (3.20) and $2 < p_1 < p_2$, we see from (3.22) that

$$4 \mid \lambda(mp_1^{e_1}) \quad \text{and} \quad \gcd(\lambda(mp_1^{e_1}), p_2) = 1. \tag{3.23}$$

Hence, by Theorem 3.16, (3.19), (3.21), and (3.23),

$$A(d, mp_1^{e_1}p_2^{e_2}) = \frac{E(p_2^{e_2})}{\gcd(\lambda(mp_1^{e_1}), E(p_2))} A(d, mp_1^{e_1}) = \frac{4}{4} g_1 A(d, m) \geq g_1 \tag{3.24}$$

for all $d \in \{0, 1, \ldots, mp_1^{e_1}p_2^{e_2} - 1\}$ and some integer g_1 such that $g_1 \mid 4$. Therefore, $mp_1^{e_1}p_2^{e_2}$ is impurely nondefective. Iterating the above argument r times when $r > 2$, we see that

$$A(d, m\ell) = g_1 A(d, m) \geq g_1 \tag{3.25}$$

for all $d \in \{0, 1, \ldots, ml - 1\}$ and some integer g_1 such that $g_1 \mid 4$. Consequently, ml is impurely nondefective as desired.

Now suppose that $b = -1$. Then by Remark 2.3, $p \mid m$ only if $p = 2$ or 3. Moreover, by Theorem 1.7 and Corollary 1.8 (iii),

$$p_i \mid D \quad \text{and} \quad E_u(p_i^{e_1}) = 1 \text{ or } 2 \tag{3.26}$$

for $1 \leq i \leq r$. It now follows from Theorems 1.4 and 1.1, Corollary 1.6, Theorem 1.9 (ii) and (iii), and Theorem 3.6 that $p \mid \lambda(m)$ only if $p = 2$ or 3. We now see from (3.26) that

$$\gcd(\lambda, l) = 1. \tag{3.27}$$

It now follows from (3.26), (3.27), and a completely similar argument to that given above for the case in which $b = 1$ that

$$A_u(d, m\ell) = g_2 A_u(d, m) \geq g_2 \tag{3.28}$$

for all $d \in \{0, 1, \ldots, ml - 1\}$ and some integer g_2 for which $g_2 \mid 2$. Thus ml is impurely nondefective as desired. □

In contrast to Corollary 3.17, we see in Theorem 3.18 that $m\ell$ is defective with respect to the LSFK $u(a, 1)$ with discriminant D if m is impurely nondefective, ℓ is purely nondefective, $\gcd(m, D) = 1$, and $2 \mid \ell$.

Theorem 3.18. *Consider the LSFK $u(a, 1)$ with discriminant D. Suppose that ℓ is purely nondefective and that $2 \mid \ell$. Moreover, let m be impurely nondefective and* $\gcd(m, D) = 1$. *Then $m\ell$ is defective.*

Proof. Since ℓ is purely nondefective and $2 \mid \ell$, it follows from Theorem 1.7 (i)–(iii) that $2 \mid D$, which implies that $h_u(2) = 2$ and a is even. Since $\gcd(m, D) = 1$, we see that m is odd. By Remark 2.3, the only primes which can divide m are 3 or 5 or 7. To show that $m\ell$ is defective, it suffices by Theorem 1.1 to show that $2p$ is defective, when $p = 3$ or 5 or 7, and $p \mid m$.

Suppose that $p \in \{3, 7\}$ and $p \mid m$. Then p is impurely nondefective by Theorem 1.1 and Theorem 1.7 (ii). It then follows by Lemma 3.3 (i) that $h_u(p) = p + 1$, and thus $h_u(p)$ is even. It now follows from Corollary 1.6 that $2p$ is defective.

We now suppose that $p = 5$ and $p \mid m$. Then p is impurely nondefective and we see by Lemma 3.3 (ii) that $h_u(p) = (p + 1)/2 = 3$. Then $E_u(5) = 4$ and $\lambda_u(5) = 12$ by Theorem 3.1 (iv). We will show that 3 and $7 \notin S_u(10)$, and thus 10 is defective. We first observe that by Theorem 1.4 (viii),

$$h_u(10) = \operatorname{lcm}(h_u(2), h_u(5)) = \operatorname{lcm}(2, 3) = 6 \tag{3.29}$$

and

$$\lambda_u(10) = \operatorname{lcm}(\lambda_u(2), \lambda_u(5)) = \operatorname{lcm}(2, 12) = 12. \tag{3.30}$$

Thus,

$$E_u(10) = \frac{\lambda_u(10)}{h_u(10)} = \frac{12}{6} = 2, \tag{3.31}$$

and consequently,

$$M_u(10) \equiv 9 \pmod{10}. \tag{3.32}$$

Since $h_u(2) = 2$ and $h_u(5) = 3$, we observe that $2 \mid u_n$ if $2 \mid n$ and $5 \mid u_n$ if $3 \mid n$. Moreover, $u_1 = 1$ and

$$u_{11} = u_{13} - au_{12} \equiv 1 - a \cdot 0 \equiv 1 \pmod{10}.$$

Thus, u_n can be congruent to 3 or 7 modulo 10 for $0 \le n \le \lambda_u(10) - 1 = 11$ only if $n = 5$ or 7. However, by (3.29) and (3.32), $h_u(10) = 6$ and $M_u(10) \equiv 9 \pmod{10}$. Hence, by (1.4),

$$u_7 \equiv 9u_1 \equiv 9 \pmod{10} \quad \text{and} \quad u_5 \equiv u_7 - au_6 \equiv 9 - a \cdot 0 \equiv 9 \pmod{10}. \tag{3.33}$$

Therefore, by (3.33), we find that 3 and $7 \notin S_u(10)$, and 10 is defective. □

Let $P(m)$ denote the largest prime factor of m. By virtue of Remark 2.3, Corollary 3.17, and Theorem 3.18, given the LSFK $u(a, b)$ with discriminant D, where $b = \pm 1$, in searching for impurely nondefective integers m with respect $u(a, b)$, we need only examine those m for which $\gcd(m, D) = 1$ and additionally $P(m) \le 7$ when $b = 1$ and $P(m) \le 3$ when $b = -1$.

Theorem 3.19. *Consider the LSFK $u(a, 1)$ with discriminant $D = a^2 + 4$. Then 3 is impurely nondefective if and only if $a \equiv \pm 1 \pmod 3$, while 3^e is impurely nondefective for $e \ge 2$ if and only if $a \equiv \pm 1$ or $\pm 2 \pmod 9$.*

Proof. We observe by Theorem 1.9 (iii) and Lemma 3.5 that 3 is impurely nondefective if and only if $(D/3) = ((a^2 + 4)/3) = -1$. By inspection, we find that $((a^2 + 4)/3) = -1$ if and only if $a \equiv \pm 1 \pmod 3$.

We now assume that $e \ge 2$. By Theorems 1.1 and 1.7, 3^e is impurely nondefective only if a is impurely nondefective. By Theorem 1.1, Corollary 1.6, Lemma 3.3 (i), and our above discussion, 9 is impurely nondefective only if $a \equiv \pm 1 \pmod 3$, $h_u(3) = 3 + 1 = 4$, and $h_u(3) \ne h_u(9)$. However, if $a \equiv 4 \pmod 9$, then

$$u_4 = a^3 + 2a = a(a^2 + 2) \equiv a(7 + 2) \equiv 0 \pmod 9,$$

and $h_u(3) = h_u(9) = 4$. Hence, we see that $a \equiv \pm 1 \pmod 3$ and $h_u(3) \ne h_u(9)$ if and only if $a \equiv \pm 1$ or $\pm 2 \pmod 9$.

Suppose that, in fact, $a \equiv \pm 1$ or $\pm 2 \pmod 9$. We will prove that 3^e is impurely nondefective for $e \ge 2$. It suffices to show that for any integer r such that $0 \le r \le 3^e - 1$, there exists a p-regular recurrence $w(a, 1)$ such that $w_0 \equiv r \pmod{3^e}$ and $w(a, 1)$ is cyclically 3^e-equivalent to $u(a, 1)$. Then by (3.7),

$$A_u(r, p^e) = A_w(r, p^e) \ge 1$$

and 3^e is impurely nondefective with respect to $u(a, 1)$. Since $((a^2 + 4)/3) = -1$, $h_u(3) = 4$, and $h_u(3^2) \ne h_u(3)$, we see by Lemma 3.15 that $w(a, 1)$ is cyclically 3^e-equivalent to $u(a, 1)$ if and only if

$$\Delta^{(0)}(w) \equiv \pm 1 \pmod{3^e}. \tag{3.34}$$

We thus seek to find an integer s such that for the recurrence $w(a, 1)$ with initial terms $w_0 = r$ and $w_1 = s$,

$$\Delta^{(0)}(w) = s^2 - ars - r^2 \equiv \pm 1 \pmod{3^e}. \tag{3.35}$$

First suppose that $r \equiv 1 \pmod 3$. We will determine an integer s such that

$$\Delta^{(0)}(w) = s^2 - ars - r^2 \equiv -1 \pmod{3^e}. \tag{3.36}$$

Then by the quadratic formula,

$$s \equiv \frac{ar \pm \sqrt{a^2r^2 + 4r^2 - 4}}{2} \pmod{3^e}. \tag{3.37}$$

Hence, congruence (3.37) has a solution s if and only if there exists an integer y such that

$$y^2 \equiv a^2r^2 + 4r^2 - 4 \pmod{3^e}. \tag{3.38}$$

Note that

$$a^2r^2 + 4r^2 - 4 \equiv (\pm 1)^2(\pm 1)^2 + 1(\pm 1)^2 - 1 \equiv 1 \pmod{3}. \tag{3.39}$$

Consider the function $g(x) = x^2 - 1$. Then $g(1) \equiv 0 \pmod{3}$ and $g'(1) = 2 \not\equiv 0 \pmod{3}$. Therefore, by Hensel's lemma (see Theorem 2.23 in [13, p. 87]), there exists an integer y satisfying (3.38). Thus, congruence (3.37) has a solution.

Now suppose that $r \equiv 0 \pmod{3}$. We now search for an integer s such that

$$s^2 - ars - r^2 \equiv 1 \pmod{3^e}. \tag{3.40}$$

Then

$$s \equiv \frac{ar \pm \sqrt{a^2r^2 + 4r^2 + 4}}{2} \pmod{3^e}. \tag{3.41}$$

Observe that

$$a^2r^2 + 4r^2 + 4 \equiv a^2(0)^2 + 1(0)^2 + 1 \equiv 1 \pmod{3}.$$

By the same argument as given above, congruence (3.41) has a solution s, and the proof is complete. □

The proof of Theorem 3.19 generalizes and slightly corrects the proof of Lemma 2 in [4], showing that 3^e is nondefective for $e \geq 1$ with respect to the Fibonacci sequence.

Theorem 3.20. *Let p be a fixed odd prime. Consider the recurrences $u(a,b)$ and $v(a,b)$. Let $h = h_u(p)$ and $\lambda = \lambda_u(p)$. Then $v(a,b)$ is p-equivalent to $u(a,b)$ if and only if h is even.*

This is proved in Lemma 2 (i) of [20].

Theorem 3.21. *Let $e > 1$, $\varepsilon \in \{-1, 1\}$, and p be an odd prime. Consider the p-regular recurrence $w(a,\varepsilon)$ with discriminant D. Suppose that $p \nmid D$ and $h_u(p^2) \neq h_u(p)$. Suppose further that $w(a,\varepsilon)$ is not p-equivalent to $v(a,\varepsilon)$. Then*

$$A_w(d, p^e) = A_w(d, p) \quad \text{for all } d \in \{0, 1, \ldots, p^e - 1\}.$$

This follows from Theorem 3.6 (i) of this paper and from Theorem 6.5, 6.8, and 6.9 of [5].

Theorem 3.22. *Consider the LSFK $u(a,1)$ with discriminant $D = a^2+4$. Then 5 is impurely nondefective if and only if $a \equiv \pm 2 \pmod{5}$, while 5^e is impurely nondefective for $e \geq 2$ if and only if $a \equiv \pm 2 \pmod{5}$ and $a \not\equiv \pm 7 \pmod{25}$.*

Proof. It follows from Theorem 1.9 (iii) and Lemma 3.3 (ii) that 5 is impurely nondefective only if $(D/5) = -1$ and $h_u(5) = (5+1)/2 = 3$. If $h_u(5) = 3$, it follows from Theorem 3.1 (iv) that if $M = M_u(5)$, then

$$E_u(5) = \operatorname{ord}_5(M) = 4.$$

Thus, by (1.4),

$$u_0 = 0 \quad \text{and} \quad u_{1+3i} \equiv M^i u_1 \equiv M^i \cdot 1 \pmod{5}$$

for $0 \leq i \leq 3$. Hence, $N_u(5) = 5$ and 5 is impurely nondefective. By inspection,

$$\left(\frac{D}{5}\right) = \left(\frac{a^2+4}{5}\right) = -1$$

if and only if $a \equiv \pm 2 \pmod{5}$. Thus, 5 is impurely nondefective if and only if $a \equiv \pm 2 \pmod{5}$.

Now it follows from Theorems 3.20 and 3.21 and our above argument that if $e \geq 2$, $h_u(5) = 3$, $h_u(25) \neq h_u(5)$, and $d \in \{0, 1, \ldots, 5^e - 1\}$, then

$$A_u(d, 5^e) = A_u(d, 5) \geq 1. \tag{3.42}$$

We note that $u_3 = a^2 + 1$. By inspection, we see that $h_u(5) = 3$ and $h_u(25) \neq h_u(5)$ if and only if $a \equiv \pm 2 \pmod 5$ and $a \not\equiv \pm 7 \pmod{25}$. Hence, by (3.42), 5^e is impurely nondefective for $e \geq 2$ exactly when $a \equiv \pm 2 \pmod 5$ and $a \not\equiv \pm 7 \pmod{25}$. □

Lemma 3.23. *Consider the LSFK $u(a, 1)$ with discriminant $D = a^2 + 4$.*

(i) 4 *is impurely nondefective if and only if* $a \equiv \pm 1 \pmod 4$.
(ii) 6 *is impurely nondefective if and only if* $a \equiv \pm 1 \pmod 6$.
(iii) 7 *is impurely nondefective if and only if* $a \equiv \pm 1$ *or* $\pm 3 \pmod 7$.
(iv) 14 *is impurely nondefective if and only if* $a \equiv \pm 1$ *or* $\pm 3 \pmod{14}$.

Proof. (i) By Theorem 1.7 (ii) and Corollary 1.8 (i), 4 is impurely nondefective only if $2 \nmid D$, which occurs if and only if $a \equiv \pm 1 \pmod 4$. By inspection 4 is indeed impurely nondefective if $a \equiv \pm 1 \pmod 4$.

(ii) By Theorem 1.1, 6 is impurely nondefective only if 3 is nondefective. By Theorem 1.7 (viii) and Theorem 3.19, 3 is nondefective if and only if $a \equiv \pm 1 \pmod 3$, in which case, 3 is impurely nondefective. Moreover, by Lemmas 3.3 and 3.5, 3 is impurely nondefective if and only if $h_u(3) = 4$. It now follows from Corollary 1.6 that 6 is defective if a is even. Thus, 6 is impurely nondefective only if $a \equiv \pm 1 \pmod 6$. By inspection, 6 is in fact impurely nondefective when $a \equiv \pm 1 \pmod 6$.

(iii) By Theorem 1.9 (iii) and Lemma 3.5, 7 is impurely nondefective if and only if $(D/7) = ((a^2 + 4)/7) = -1$. By inspection, we find that $((a^2 + 4)/7) = -1$ if and only if $a \equiv \pm 1$ or $\pm 3 \pmod 7$.

(iv) By Theorem 1.1, 14 is nondefective only if 7 is nondefective. By Theorem 1.7 (ii) and part (iii) of this lemma, 7 is nondefective if and only if 7 is also impurely nondefective, which occurs if and only if $a \equiv \pm 1$ or $\pm 3 \pmod 7$. Moreover, if 7 is impurely nondefective, then by Lemma 3.3 (i), $h_u(7) = 8$. Hence, by Corollary 1.6, we see that 14 is defective if a is even. Therefore, 14 is impurely nondefective only if $a \equiv \pm 1$ or $\pm 3 \pmod{14}$. By inspection, we observe that 14 is indeed impurely nondefective when $a \equiv \pm 1$ of $\pm 3 \pmod{14}$. □

Lemma 3.24. *Consider the LSFK $u(a, -1)$ with discriminant $D = a^2 - 4$.*

(i) 3 *is impurely nondefective if and only if* $a \equiv 0 \pmod 3$.
(ii) 6 *is impurely nondefective if and only if* $a \equiv 3 \pmod 6$.

Proof. (i) By Theorem 1.9 (iii), 3 is impurely nondefective only if $(D/3) = -1$, which occurs only if $a \equiv 0 \pmod 3$. By inspection, 3 is indeed impurely nondefective when $a \equiv 0 \pmod 3$.

(ii) If $a \equiv \pm 2 \pmod 6$, then both 2 and 3 divide D, which implies by Theorem 1.7 (i) that 6 is purely nondefective. If $a \equiv 0$ or $\pm 1 \pmod 6$, then we see by inspection that 2, 3, and $4 \notin S_u(6)$ and 6 is defective. By examination, we find that 6 is indeed impurely nondefective when $a \equiv 3 \pmod 6$. □

Lemma 3.25. *Consider the LSFK $u(a, 1)$ with discriminant $D = a^2 + 4$. Then*

(i) 8 *is defective,*
(ii) 21 *is defective,*
(iii) 12 *is defective,*
(iv) 28 *is defective,*
(v) *If* $5 \nmid D$ *then* 10 *is defective,*

(vi) 18 *is defective.*

Proof. (i) First suppose that a is even. Then 2 is purely nondefective. It now follows from Theorem 1.7 (v) and Corollary 1.8 (ii) that 2^e is defective for $e \geq 2$.

Now suppose that is odd. Then $a^2 \equiv 1 \pmod 8$. We note by the Binet formulas (1.3) that $u_3 = a^2 + 1 \equiv 2 \pmod 8$ and $u_6 = u_3 v_3 = (a^2+1)a(a^2+3) \equiv 2 \cdot a \cdot 4 \equiv 0 \pmod 8$. Thus, $h_u(4) = h_u(8) = 6$. It now follows from Corollary 1.6 that 8 is defective in this case, too.

(ii) By Theorem 1.1, 21 is nondefective only if both 3 and 7 are nondefective. By Lemma 3.3 (i), 3 is nondefective only if $h_u(3) = 4$, while 7 is nondefective only if $h_u(7) = 8$. By Theorem 3.1 (iii), it follows that $\lambda_u(3) = 8$ and $\lambda_u(7) = 16$. Then by Theorem 1.4 (viii),

$$\lambda_u(21) = \operatorname{lcm}(\lambda_u(3), \lambda_u(7)) = \operatorname{lcm}(8, 16) = 16.$$

Thus,

$$N_u(21) \leq \lambda_u(21) = 16 < 21,$$

and 21 is defective.

(iii) and (iv) We prove the more general result that if $p \equiv 3 \pmod 4$, then $4p$ is defective. By Theorem 1.1, $4p$ is nondefective only if 4 and p are both nondefective. By Lemma 3.3 (i), if p is nondefective, then $h_u(p) = p + 1 \equiv 0 \pmod 4$. If follows from the proof of part (i) of this lemma that 4 is nondefective only if a is odd, in which case, $h_u(2) = 3$ and $h_u(4) = 6$. Then by Theorem 1.4 (viii),

$$h_u(2p) = \operatorname{lcm}(h_u(2), h_u(p)) = \operatorname{lcm}(3, h_u(p)) = h_u(4p) = \operatorname{lcm}(h_u(4), h_u(p)) = \operatorname{lcm}(6, h_u(p)).$$

We now see by Corollary 1.6 that $4p$ is defective.

(v) Since $5 \nmid D$, we see by Theorem 1.1 and Theorem 1.7 (ii) that 10 is nondefective only if 2 is nondefective and 5 is impurely nondefective. It follows from Lemma 3.3 (ii) that $h_u(5) = (5+1)/2 = 3$. If a is even, it was shown in the proof of Theorem 3.18 that 10 is defective. Now suppose that a is odd. Then by Theorem 1.9 (ii), $h_u(2) = 3$. We now see that

$$h_u(10) = \operatorname{lcm}(h_u(2), h_u(5)) = \operatorname{lcm}(3, 3) = 3 = h_u(5).$$

It now follows from Corollary 1.6 that 10 is defective in this case also.

(vi) It follows from Theorem 1.1 and Theorem 1.7 (i) and (viii) that 18 is nondefective only if both 6 and 9 are impurely nondefective. By Theorem 3.19 and Lemma 3.23 (ii) 9 is impurely nondefective if and only if $a \equiv \pm 1$ or $\pm 2 \pmod 9$, while 6 is impurely nondefective if and only if $a \equiv \pm 1 \pmod 6$. If $a \equiv \pm 1$ or $\pm 2 \pmod 9$, we find that $h_u(3) = 4$ and $h_u(9) = 12$, whereas if $a \equiv \pm 1 \pmod 6$, we see that $h_u(2) = 3$. Then

$$h_u(18) = \operatorname{lcm}(h_u(2), h_u(9)) = \operatorname{lcm}(3, 12) = 12 = h_u(9).$$

It now follows from Corollary 1.6 that 18 is defective. □

Lemma 3.26. *Consider the LSFK $u(a,1)$ with discriminant $D = a^2 + 4$. Suppose that $5 \nmid D$. Then*

(i) 15 *is defective,*
(ii) 35 *is defective.*

Proof. (i) Suppose that 15 is nondefective and $5 \nmid D$. Then by Theorem 1.1 and Theorem 1.7 (ii) and (viii), both 3 and 5 are impurely nondefective. By Lemma 3.3, $h_u(3) = 4$ and $h_u(5) = 3$. It now follows from Theorem 3.1 (iii) and (iv) that $\lambda_u(3) = 8$ and $\lambda_u(5) = 12$. Hence, by Theorem 1.4 (viii),

$$h_u(15) = \operatorname{lcm}(h_u(3), h_u(5)) = \operatorname{lcm}(4, 3) = 12 \tag{3.43}$$

and

$$\lambda_u(15) = \text{lcm}(\lambda_u(3), \lambda_u(5)) = \text{lcm}(8, 12) = 24. \tag{3.44}$$

We note by the above observations that

$$u_{13} = u_{1+4h_u(5)} = u_{1+3h_u(3)} = u_{1+h_u(15)}. \tag{3.45}$$

Noting that $u_1 = 1$ and applying (1.4), we now find that

$$u_{13} \equiv [M_u(5)]^4 u_1 \equiv 1 \pmod 5, \quad u_{13} \equiv [M_u(3)]^3 u_1 \equiv (-1)^3 \equiv -1 \pmod 3,$$

and

$$u_{13} \equiv M_u(15) u_1 \equiv M_u(15) \pmod{15}.$$

Hence, $M_u(15) \equiv 1 \pmod 5$ and $M_u(15) \equiv -1 \pmod 3$. By the Chinese Remainder Theorem, it follows that $M_u(15) \equiv 11 \pmod{15}$.

We now show that there exists a residue $3s$ modulo 15 such that $0 < 3s < 15$ and $3s \notin S_u(15)$. Since $h_u(3) = 4$ and $h_u(15) = 12$, we see that $u_n \equiv 3s \pmod{15}$ for $0 < 3s < 15$ and $0 \le n < \lambda_u(15) = 24$ only if $n \in \{4, 8, 16, 20\}$. Then $u_4 \equiv 3s_1$ and $u_8 \equiv 3s_2 \pmod{15}$. But then

$$u_{16} = u_{4+h_u(15)} \equiv M(15) u_4 \equiv 11(3s_1) \equiv 3s_1 \equiv u_4 \pmod{15}.$$

Similarly,

$$u_{20} \equiv 3s_2 \equiv u_8 \pmod{15}.$$

Thus, there exists an integer $3s_3$ such that $1 \le s_3 \le 4$ and $3s_3 \notin S_u(15)$. Hence, 15 is defective.

(ii) Suppose that $5 \nmid D$ and 35 is nondefective. Then by Theorem 1.1 and Theorem 1.7 (ii) and (viii), both 5 and 7 are impurely nondefective. By Lemma 3.3 and Theorem 3.1 (iii) and (iv), we have $h_u(5) = 3$, $\lambda_u(5) = 12$, and $h_u(7) = 8$, $\lambda_u(7) = 16$. It now follows by Theorem 1.4 (viii) that

$$h_u(35) = \text{lcm}(h_u(5), h_u(7)) = \text{lcm}(3, 8) = 24$$
$$\lambda_u(35) = \text{lcm}(\lambda_u(5), \lambda_u(7)) = \text{lcm}(12, 16) = 48.$$

It now follows from a completely similar argument to that given in part (i) that there exists a residue $7t$ such that $0 < 7t < 35$ and $7t \notin S_u(35)$. Thus, 35 is defective. □

Lemma 3.27. *Consider the LSFK $u(a, 1)$. Suppose that $p \equiv 7 \pmod 8$ and $h = h_u(p) = p+1$. Then*

$$A(u_{h/2}, p) = 1.$$

Proof. This follows from Theorem 7 (vii) and Lemma 12 (ii) of [18]. □

Lemma 3.28. *Consider the LSFK $u(a, 1)$. Let p be an odd prime such that $h_u(p)$ is even and $h_u(p^2) \ne h_u(p)$. Let $h = h_u(p)$ and $\lambda = \lambda_u(p)$. Let n be a fixed integer such that $0 \le n \le \lambda - 1$. Then there exists a fixed integer M such that $M \equiv 1 \pmod p$, $1 \le M \le p^2 - 1$, and*

$$u_{n+\lambda i} \equiv M^i u_n \pmod{p^2}$$

for all $i \ge 0$. Moreover,

$$u_{h/2+\lambda i} \equiv u_{h/2} \pmod{p^2}$$

for all $i \ge 0$.

Proof. This follows from Theorem 3.5 of [5], Theorem 4.1 of [21], and the fact that $h \mid \lambda$. □

Lemma 3.29. *Consider the LSFK $u(a, 1)$. Then 49 is defective.*

Proof. Suppose that 49 is nondefective. Then by Theorem 1.1, Corollary 1.6, and Lemma 3.3 (i), 7 is nondefective, $h_u(49) \neq h_u(7)$, and $h_u(7) = 8$. Then by Theorem 3.1 (iii), $\lambda_u(7) = 16$. Since $h_u(49) \neq h_u(7)$, we see by Theorem 3.6 that

$$h_u(49) = 7h_u(7) = 56 \quad \text{and} \quad \lambda_u(49) = 7\lambda_u(7) = 112. \tag{3.46}$$

Let $h = h_u(7) = 8$ and $\lambda = \lambda_u(7) = 16$. Then by Lemmas 3.27 and 3.28,

$$A_u(u_{h/2}, 7) = A_u(u_4, 7) = 1 \tag{3.47}$$

and

$$u_{h/2+\lambda i} = u_{4+16i} \equiv u_4 \pmod{49} \tag{3.48}$$

for $0 \leq i \leq 6$. Let $d \equiv u_4 \pmod 7$, but $d \not\equiv u_4 \pmod{49}$, where $d \in \{0, 1, \ldots, 48\}$. Since $\lambda_u(7) = 16$ and $\lambda_u(49) = 112$, we see by (3.47) and (3.48) that $u_n \equiv d \pmod{49}$ for some n such that $0 \leq n < \lambda(49) = 112$ only if $n = 4 + 16i$ for some $i \in \{0, 1, \ldots, 6\}$. However by (3.48),

$$u_{4+16i} \equiv u_4 \not\equiv d \pmod{49}$$

for $0 \leq i \leq 6$. Thus, $d \notin S_u(49)$ and 49 is defective. □

Example 3.30. Consider the Fibonacci sequence $\{F_n\} = u(1, 1)$. We show that if $d \equiv F_4 \equiv 3 \pmod 7$, but $d \not\equiv 3 \pmod{49}$, then $d \notin S_F(49)$. The first 18 terms of $\{F_n\}$ modulo 7 are

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 1,\ 6,\ 0,\ 6,\ 6,\ 5,\ 4,\ 2,\ 6,\ 1,\ 0,\ 1.$$

Thus, $h_F(7) = 8$ and $\lambda_F(7) = 16$. Moreover, $F_8 = 21$, and thus, $h_F(7) \neq h_F(49)$. Further, by inspection, $A(F_4, 7) = A(3, 7) = 1$. We additionally observe that F_{4+16i} is indeed congruent to $3 \pmod{49}$ for $i \in \{0, 1, \ldots, 6\}$. In particular,

$$F_4 = 3, \quad F_{20} = 6765, \quad F_{36} = 14930352,$$
$$F_{52} = 32951280099, \quad F_{68} = 72723460248141,$$
$$F_{84} = 160500643816367088, \quad F_{100} = 354224848179261915075,$$

which are all in fact congruent to 3 modulo 49.

Lemma 3.31. *Consider the LSFK $u(a, -1)$ with discriminant $D = a^2 - 4$. Then*

(i) 4 *is defective if* $2 \nmid D$,
(ii) 12 *is defective if* $2 \nmid D$ *or* $3 \nmid D$,
(iii) 9 *is defective if* $3 \nmid D$,
(iv) 18 *is defective if* $2 \nmid D$ *or* $3 \nmid D$.

Proof. (i) We note that $2 \nmid D$ if and only if $a \equiv \pm 1 \pmod 4$. By inspection, we see that if $a \equiv \pm 1 \pmod 4$, then $h_u(2) = h_u(4) = 3$, and 4 is defective by Corollary 1.6.

(ii) If $2 \nmid D$, then 4 is defective by part (i). It now follows from Theorem 1.1 that 12 is defective. Now suppose that $3 \nmid D$. Then $a \equiv 0 \pmod 3$. Suppose that 12 is nondefective. Then by Theorem 1.1 and Theorem 1.7 (i) and (ii), 6 is impurely nondefective. By Lemma 3.24 (ii), this occurs if and only if $a \equiv 3 \pmod 6$. Thus, $a \equiv \pm 1 \pmod 4$, and by the proof of part (i), we see that 4 is defective. Hence, 12 is also defective by Theorem 1.1.

(iii) Suppose that $3 \nmid D$. Then $a \equiv 0 \pmod 3$. By inspection, we see that 2, 4, 5, and $7 \notin S_u(9)$, and 9 is defective.

(iv) If $3 \nmid D$, then by part (iii), 9 is defective and thus 18 is defective. We now suppose that $2 \nmid D$, $3 \mid D$, and 18 is nondefective. Then 6 is nondefective by Theorem 1.1. Moreover, a is odd and $a \equiv \pm 1 \pmod 3$, which implies that $a \equiv \pm 1 \pmod 6$. Since $2 \nmid D$, it follows from Theorem 1.7(i) and (ii) that 6 is impurely nondefective. However, by Lemma 3.24 (ii), 6 is impurely nondefective if and only if $a \equiv 3 \pmod 6$. Hence, 18 is defective by Theorem 1.1. □

4. Proofs of the Main Theorems

Proof of Theorem 2.6. It follows from Theorems 1.1, 2.1, and 3.18, and from Lemmas 3.25, 3.26, and 3.29 that m is nondefective only if m satisfies one of the forms given in (2.2). Part (i) follows from Theorem 1.7 (viii) while part (ii) follows from part (i) and Theorem 1.7 (i)–(v). Part (iii) follows from Corollary 1.8 (i), Theorem 1.9 (ii), and Corollary 3.17 and Lemma 3.23 (i). Parts (iv) and (v) follow from Corollary 3.17 and Theorem 3.19. Part (vi) follows from Corollary 3.17 and Lemma 3.23 (i). Parts (vii) and (viii) follow from Corollary 3.17 and Theorem 3.22. Part (ix) follows from Corollary 3.17 and Lemma 3.23 (ii). Parts (x) and (xi) follow from Corollary 3.17 and Lemma 3.23 (iii) and (iv). □

Proof of Theorem 2.9. It follows from Theorems 1.1, 2.2, and 3.18, and Lemma 3.31 that m is nondefective only if m has one of the forms given in (2.3). We note by Theorem 1.7 (ii) that 3 is purely nondefective if and only if $3 \mid D = a^2 - 4$, which occurs if and only if $a \equiv \pm 1 \pmod 3$. Part (i) now follows from Theorem 1.7 (i)–(vi).

Suppose that $a = \pm 2$. Then $D = 0$ and it follows by induction that $u_n = n$ for $n \geq 0$ if $a = 2$, while $u_n = (-1)^{n+1}n$ for $n \geq 0$ if $a = -2$. Part (ii) now follows from Theorem 1.7 (i)-(vi). Part (iii) follows from Theorem 1.7, Corollary 3.17, and Lemma 3.31. Part (iv) follows from Theorem 1.9 (ii) and Corollary 3.17. Part (v) follows from Lemma 3.24 (i) and Corollary 3.17, while part (vi) follows from Lemma 3.24 (ii) and Corollary 3.17. □

5. Concluding Remarks

Except for this paper and the paper [1], which appeared in 2013, all the papers cited in this article concerning second-order recurrences having a complete system of residues appeared before 2000. We encourage interested readers to investigate this problem for more general recurrences than those treated in this paper.

Acknowledgement

This paper was supported by RVO 67985840 of the Czech Republic.

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MSC2010: 11B39, 11A07

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