A COMBINATORIAL PROOF FOR THE GENERATING FUNCTION OF POWERS OF THE FIBONACCI SEQUENCE

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Abstract. We derive a formula for the generating function of powers of the Fibonacci sequence. We illustrate the formula with an example, the generating function for the fourth powers of the Fibonacci sequence.

1. Introduction

Let $\mathcal{F}_k(x)$ be the generating function for kth power of Fibonacci numbers, defined by

$$\mathcal{F}_k(x) = \sum_{n=0}^{\infty} F_n^k x^n, \ \ k \in \mathbb{N}^+.$$

Example 1.1. $\mathcal{F}_1(x) = x/(1-x-x^2)$ *is known.*

The following recurrence relation is due to Riordan [6],

$$(1 - L_k x + (-1)^k x^2)\mathcal{F}_k(x) = 1 + kx \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{A_{kj}}{j} \mathcal{F}_{k-2j}((-1)^j x),$$

where L_k is the kth Lucas number and the doubly-indexed sequence A_{kj} has the generating function given by,

$$(1-x-x^2)^{-j} = \sum_{k=2j}^{\infty} A_{kj} x^{k-2j}, \ \ j \geq 0,$$

which was improved by Dujella [2].

Also, in [2], Dujella gave a bijective proof for Riordan's result by using Morse codes. Stănică [7] obtained a closed form for generating function of the powers of the nondegenerate second-order recurrence relation,

$$U_{n+2} = aU_{n+1} + bU_n, \ \text{with } a, b, U_0, U_1 \in \mathbb{Z},$$

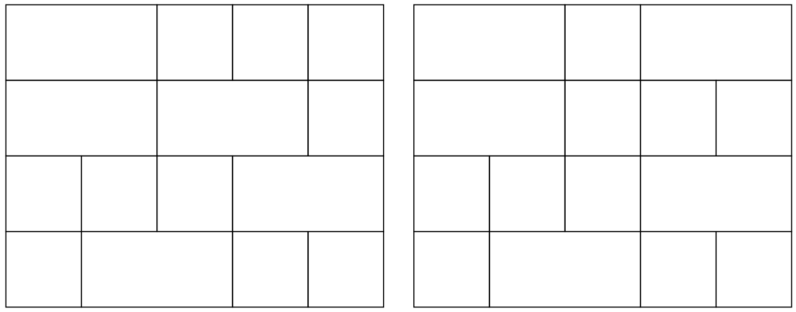
such that $\delta = a^2 + 4b \neq 0$.

Let $\alpha = \frac{1}{2}(a + \sqrt{a^2+4b})$, $\beta = \frac{1}{2}(a - \sqrt{a^2+4b})$ and $A = \frac{U_1 - U_0\beta}{\alpha-\beta}$, $B = \frac{U_1 - U_0\alpha}{\alpha-\beta}$, $V_n = \alpha^n + \beta^n$ with initial conditions $V_0 = 2$, $V_1 = a$. Stănică [7] showed that, if k is odd, then

$$U_k(x) = \sum_{j=0}^{\frac{k-1}{2}} (-AB)^j \binom{k}{j} \frac{A^{k-2j} - B^{k-2j} + (-b)^j((B\alpha)^{k-2j} - (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x - b^k x^2}.$$

He also showed that, if k is even, then

$$U_k(x) = \sum_{j=0}^{\frac{k}{2}-1} (-AB)^j \binom{k}{j} \frac{B^{k-2j} + A^{k-2j} - (-b)^j((B\alpha)^{k-2j} + (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x + b^k x^2} + \binom{k}{\frac{k}{2}} \frac{(-AB)^{\frac{k}{2}}}{1 - (-b)^{\frac{k}{2}} x}.$$



(a) This is an $M_{4\times 5}$. (b) This is not an $M_{4\times 5}$.

FIGURE 1. Two examples.

Mansour [4] obtained a formula for $\mathcal{F}_k(x)$ by using the determinant of certain matrices.

Horadam [3] defined the second-order linear recurrence sequence $W_n(a,b;p,q)$, or W_n, by $W_{n+2} = pW_{n+1} + qW_n$ with initial condition $W_0 = a$ and $W_1 = b$. The Fibonacci sequence $\{F_n\}$ in this notation is given by $W_n(0,1;1,1)$.

The Fibonacci number F_{n+1} gives the number of ways for 1×2 dominoes and squares to cover a $1 \times n$ checkerboard. In the next section, we begin with a series of definitions and lemmas, modifying this approach. We develop a counting method to find the generating function for powers of Fibonacci numbers which is the main result.

2. New Closed Form of $\mathcal{F}_k(x)$

Definition 2.1. *Let $\mathcal{F}_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$ be the generating function of kth power of F_n where $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0$ and $F_1 = 1$.*

The following lemma is Combinatorial Theorem 1 from Benjamin and Quinn's book [1].

Lemma 2.2. *Let f_n count the ways to tile a length n board with squares and dominoes. Then f_n is a Fibonacci number, namely $f_n = F_{n+1}$ with $f_{-1} = 0$ and $f_1 = 1$.*

Definition 2.3. *For $k \geq 1$ and $n \geq 1$, the Fibonacci $k \times n$ checkerboard, $F_{k\times n}$ board in short, is a checkerboard with height k and length n covered by squares and dominoes such that the dominoes can only be placed horizontally. $|F_{k\times n}|$ is the number of different Fibonacci $k \times n$ checkerboards.*

The Fibonacci $k \times n$ minimal checkerboard, $M_{k\times n}$ board in short, is an $F_{k\times n}$ board that cannot be vertically divided into two Fibonacci checkerboards. Let $m_{k\times n}$ be the number of different Fibonacci $k \times n$ minimal checkerboards.

Lemma 2.4. $|F_{k\times n}| = f_n^k = F_{n+1}^k$ *for $k \geq 1$ and $n \geq 1$.*

Proof. From Lemma 2.2, $|F_{1\times n}| = f_n$ for $n \geq 1$. Since an $F_{k\times n}$ board has k layers, and each layer is an $F_{1\times n}$ board, there are f_n of them. Therefore, $|F_{k\times n}| = f_n^k = F_{n+1}^k$. □

In Figure 1, there are two examples of M boards. Figure 1(b) is not an $M_{4\times 5}$, since it can be divided into an $F_{4\times 3}$ and an $F_{4\times 2}$.

Table 1 contains values of $m_{4\times n}$ for small n. Small values were obtained by counting and large values were obtained from Lemma 2.23. These values are used to explain Examples 2.8 and 2.13.

Lemma 2.5. *Each $F_{k\times n}$ board, which is not an $F_{k\times n}$ minimal board, can be uniquely divided into at most n Fibonacci minimal boards.*

Proof. The proof is straightforward. □

Table 1. $m_{4\times n}$ for small n

n	1	2	3	4	5	6	7	8	9	10
$m_{4\times n}$	1	15	50	254	1202	5774	27650	132494	634802	3041534

Definition 2.6. *If an $F_{k\times n}$ board can be divided into $M_{k\times n_1}$, $M_{k\times n_2}, \ldots, M_{k\times n_j}$ boards, where order matters, $n_i \in \mathbb{N}^+$ for $i \in \{1, 2, \ldots, j\}$, then the $F_{k\times n}$ board can be written as an $M_{k\times(n_1,n_2,\ldots,n_j)}$ board. Let $m_{k\times(n_1,n_2,\ldots,n_j)}$ be the number of $M_{k\times(n_1,n_2,\ldots,n_j)}$ boards. With this notation, $m_{k\times(2,3)} \neq m_{k\times(2+3)}$.*

An $M_{4\times(3,2)}$ board is shown in Figure 1(b).

Lemma 2.7. *The number of $F_{k\times n}$ boards is,*

$$F^k_{n+1} = f^k_n = \sum_{j=1}^{n} \sum_{\substack{n_1+n_2+\cdots+n_j=n \\ n_i\in\mathbb{N}^+}} m_{k\times(n_1,n_2,\cdots,n_j)} = \sum_{j=1}^{n} \sum_{\substack{n_1+n_2+\cdots+n_j=n \\ n_i\in\mathbb{N}^+}} \prod_{i=1}^{j} m_{k\times n_i}.$$

Proof. From Lemma 2.5, each $F_{k\times n}$ board is either an $M_{k\times n}$ board or can be uniquely divided into j minimal boards, $2 \le j \le n$, and the number of $M_{k\times n_i}$ boards is $m_{k\times n_i}$, $1 \le i \le j$. □

Example 2.8. $F^4_4 = f^4_3 = m_{4\times 3} + m_{4\times(2,1)} + m_{4\times(1,2)} + m_{4\times(1,1,1)} = 50 + 15\cdot 1 + 1\cdot 15 + 1\cdot 1\cdot 1 = 81.$

From Mazur [5, p. 111], we have the following question: "In how many ways can we distribute n identical objects to 6 distinct recipients if each recipient receives at least one object?" Then the answer is the coefficient of x^n in the expansion of $(x + x^2 + x^3 + \cdots)^6 = (\frac{x}{1-x})^6$.

This is equivalent to: "If there is exactly one word for each length, in how many ways can we write an n-letter sentence with 6 words?"

Question 2.9. *In how many ways can we write an n-letter sentence?*

The following examples answer Question 2.9 with different conditions.

Example 2.10. *If there is exactly one word for each length and we are not sure how many words are there, then the answer to Question 2.9 is the coefficient of x^n in*

$$\sum_{j=1}^{\infty} (x + x^2 + x^3 + \cdots)^j = \frac{1}{1 - (x + x^2 + \cdots)} - 1 = \frac{1}{1 - \frac{x}{1-x}} - 1 = \frac{x}{1 - 2x}.$$

Example 2.11. *Suppose there is one 1-letter word and one 2-letter word, then the number of ways of writing an n-letter sentence is the coefficient of x^n in*

$$\sum_{j=1}^{\infty} (x + x^2)^j = \frac{1}{1 - (x + x^2)} - 1 = \frac{x(1 + x)}{1 - x - x^2}.$$

Example 2.12. *Now, if there are $m_{k\times i}$ different i-letter words, then f_n^k is the number of ways to write an n-letter sentence.*

Example 2.13. *Suppose there are one 1-letter word, 15 2-letter words, 50 3-letter words, ..., and $m_{4\times n}$ n-letter words, $n \geq 1$. Then the answer to Question 2.9 is the coefficient of x^n in*

$$\sum_{j=1}^{\infty}(x+15x^2+50x^3+254x^4+\cdots)^j = \frac{1}{1-(x+15x^2+50x^3+254x^4+\cdots)} - 1.$$

We have the following lemma.

Lemma 2.14. *Let $e_k = \sum_{n=1}^{\infty} m_{k\times n}x^n$. Then*

$$\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i}\right)x^n\right) = \frac{1}{1-e_k} - 1.$$

Moreover, $\mathcal{F}_k(x) = \frac{x}{1-e_k}$.

Proof. The first part of the lemma follows from Lemma 2.7 and as in Example 2.13. From Lemma 2.7 we obtain,

$$\begin{aligned}\mathcal{F}_k(x) &= \sum_{n=0}^{\infty} F_n^k x^n = F_0^k + F_1^k x + \sum_{n=2}^{\infty} f_{n-1}^k x^n \\ &= x + x\sum_{n=1}^{\infty} f_n^k x^n = x + x\left(\frac{1}{1-e_k} - 1\right) = \frac{x}{1-e_k}.\end{aligned}$$

□

Example 2.15. $e_4 = \sum_{n=1}^{\infty} m_{4\times n}x^n = x + 15x^2 + 50x^3 + \cdots, \mathcal{F}_4(x) = \frac{x}{1-e_4}$.

In order to find the closed form of e_4, or the closed form for e_k, we will utilize the following lemma.

Lemma 2.16. *Let $S_{1,n}$, $S_{2,n}$, ..., $S_{m,n}$ be sequences (not necessarily distinct) with*
$S_{1,n+1} = a_{1,1}S_{1,n} + a_{1,2}S_{2,n} + \cdots + a_{1,m}S_{m,n}$,
$S_{2,n+1} = a_{2,1}S_{1,n} + a_{2,2}S_{2,n} + \cdots + a_{2,m}S_{m,n}$,
...,
$S_{m,n+1} = a_{m,1}S_{1,n} + a_{m,2}S_{2,n} + \cdots + a_{m,m}S_{m,n}$.

If $[a_{i,j}]_{m\times m} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m}\end{bmatrix}_{m\times m}$, $[S_{j,n}]_{m\times 1} = \begin{bmatrix} S_{1,n} \\ S_{2,n} \\ \vdots \\ S_{m,n}\end{bmatrix}_{m\times 1}$, $B_{m+1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1\end{bmatrix}^T_{1\times m}$,
then $\sum_{j=1}^{m} S_{j,n} = B_{m+1}[a_{i,j}]^{n-l}_{m\times m}[S_{j,l}]_{m\times 1} \in \mathbb{R}$.

Proof. Since

$$\begin{aligned}[a_{i,j}]^{n-l}_{m\times m}[S_{j,l}]_{m\times 1} &= [a_{i,j}]^{n-l-1}_{m\times m}[a_{i,j}]_{m\times m}[S_{j,l}]_{m\times 1} \\ &= [a_{i,j}]^{n-l-1}_{m\times m}[S_{j,l+1}]_{m\times 1} = \cdots = [S_{j,n}]_{m\times 1},\end{aligned}$$

then

$$\sum_{j=1}^{m} S_{j,n} = B_{m+1}[S_{j,n}]_{m\times 1} = B_{m+1}[a_{i,j}]_{m\times m}^{n-l}[S_{j,l}]_{m\times 1}.$$

□

Example 2.17. *Since* $F_{n-1} = F_{n-2} + F_{n-3}$ *and* $F_{n-2} = F_{n-2} + 0 \cdot F_{n-3}$, *then*

$$F_n = F_{n-1} + F_{n-2} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, n \geq 2.$$

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *is called the Fibonacci Q-matrix.*

Definition 2.18. *Let* $S_{j,k\times n}$ *board* ($j \in \{1, 2, 3, \ldots, k\}, k \geq 1, n \geq 2$) *be an* $M_{k\times n}$ *board with* j *dominoes in the last two columns. Let* $s_{j,k\times n}$ *be the number of different* $S_{j,k\times n}$ *boards and* $s_{j,k\times n} = 0$ *if there does not exist any* $S_{j,k\times n}$ *board.*

Example 2.19. *Figure 1(a) is an* $S_{1,4\times 5}$ *board and Figure 1(b) is a combination of an* $S_{1,4\times 3}$ *board and* $S_{2,4\times 2}$ *board.*

Note 2.20. *There are some special cases:*

(1) *If* $k = 1$ *and* $n = 2$, *then* $j = 1$, $S_{1,1\times 2}$ *is a* 1×2 *domino and* $s_{1,1\times 2} = 1$.
(2) *If* $k = 1$ *and* $n \geq 3$, *then* $j = 1$, $S_{1,1\times n}$ *does not exist and* $s_{1,1\times n} = 0$.
(3) *If* $n = 2$, $k \geq 2$ *and* $1 \leq j \leq k$, *then* $s_{j,k\times 2} = \binom{k}{j}$.
(4) *If* $n \geq 3$, $k \geq 2$ *and* $j = k$, *then* $s_{k,k\times n} = 0$.
(5) *If* $n = 1$, *then* $m_{k\times 1} = 1$.
(6) *If* $n = 2$, *then* $m_{k\times 2} = 2^k - 1 = \sum_{j=1}^{k} s_{j,k\times 2} = \sum_{j=1}^{k-1} s_{j,k\times 2} + 1$.
(7) *If* $n \geq 3$, *then* $m_{k\times n} = \sum_{j=1}^{k} s_{j,k\times n} = \sum_{j=1}^{k-1} s_{j,k\times n}$.

Lemma 2.21. $s_{j,k\times(n+1)} = \sum_{i=1}^{k-j} \binom{k-i}{j} s_{i,k\times n}$ *for* $1 \leq j \leq k-1, k \geq 2, n \geq 2$.

Proof. Each $S_{j,k\times(n+1)}$ board, $1 \leq j \leq k-1$, can be obtained from any $S_{i,k\times n}$ board with $1 \leq i \leq k-j$. An $S_{i,k\times n}$ board with $1 \leq i \leq k-j$, has $k-i$ squares in the last column; choose j squares from them, replace the chosen squares by dominoes, then fill the $S_{j,k\times(n+1)}$ board with squares. Figure 2 illustrates this with an example. Hence, $s_{j,k\times(n+1)} = \sum_{i=1}^{k-j} \binom{k-i}{j} s_{i,k\times n}$ for $1 \leq j \leq k-1$. □

Definition 2.22. *For* $k \geq 2$, *define matrices*

$$A_k = \begin{bmatrix} \binom{k-1}{1} & \binom{k-2}{1} & \cdots & \binom{1}{1} \\ \binom{k-1}{2} & \binom{k-2}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k-1}{k-1} & 0 & \cdots & 0 \end{bmatrix}_{(k-1)\times(k-1)}, C_k = \begin{bmatrix} \binom{k}{1} \\ \binom{k}{2} \\ \vdots \\ \binom{k}{k-1} \end{bmatrix}_{(k-1)\times 1}, B_k = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{1\times(k-1)}^{T}$$

and I_k *is the* $(k-1) \times (k-1)$ *identity matrix. If* $k = 1$, *let* $A_1 = B_1 = C_1 = 0$, $I_1 = 1$. *Note that the matrix* A_k *contains portions of Pascal's triangle.*

Lemma 2.23. *For* $k \geq 2$, $m_{k\times 2} = 1 + B_k A_k^0 C_k$, $m_{k\times n} = B_k A_k^{n-2} C_k$ *for* $n \geq 3$.

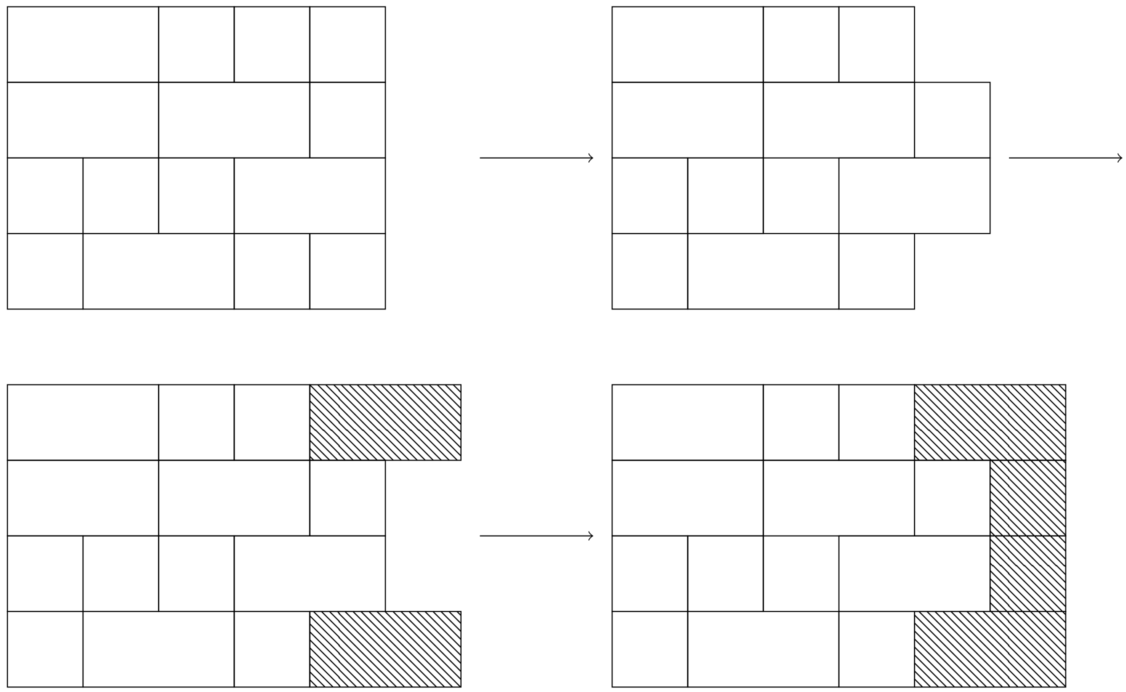


FIGURE 2. Transforming an $S_{1,4\times 5}$ board to an $S_{2,4\times 6}$ board.

Proof. From Lemmas 2.16, 2.21, and Note 2.20, for $k \geq 2$ and $n \geq 3$, we have

$$m_{k\times n} = \sum_{j=1}^{k-1} s_{j,k\times n} = B_k A_k^{n-2} \begin{bmatrix} s_{1,k\times 2} \\ s_{2,k\times 2} \\ \vdots \\ s_{k-1,k\times 2} \end{bmatrix}_{(k-1)\times 1} = B_k A_k^{n-2} C_k.$$

Further, $m_{k\times 2} = 1 + B_k A_k^0 C_k$ follows from Note 2.20. □

Example 2.24. *In Table 2, there are small values of $s_{j,4\times n}$, $j = 1, 2, 3$. We find that,*

$s_{1,4\times(n+1)} = 3 \cdot s_{1,4\times n} + 2 \cdot s_{2,4\times n} + 1 \cdot s_{3,4\times n}$,
$s_{2,4\times(n+1)} = 3 \cdot s_{1,4\times n} + 1 \cdot s_{2,4\times n} + 0 \cdot s_{3,4\times n}$,
$s_{3,4\times(n+1)} = 1 \cdot s_{1,4\times n} + 0 \cdot s_{2,4\times n} + 0 \cdot s_{3,4\times n}$.

From Lemma 2.23, for $n \geq 3$,

$$m_{4\times n} = s_{1,4\times n} + s_{2,4\times n} + s_{3,4\times n} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}.$$

Lemma 2.25. *For $k \geq 2$, there exists an open interval J such that $\sum_{n=0}^{\infty} (xA_k)^n = (I_k - xA_k)^{-1}$ for all $x \in J$.*

Proof. For $k \geq 2$, we have $\det(A_k) = -1$ or 1; more precisely, $\det(A_k) = (-1)^{\lfloor \frac{k+3}{2} \rfloor}$. We can conclude that A_k is not a singular matrix and 0 is not an eigenvalue of A_k.

TABLE 2. $s_{j,4\times n}$, $j = 1, 2, 3$, for small n

n	1	2	3	4	5	6	7	8	9	10
$m_{4\times n}$	1	15	50	254	1202	5774	27650	132494	634802	3041534
$s_{1,4\times n}$		4	28	124	604	2884	13828	66244	317404	1520764
$s_{2,4\times n}$		6	18	102	474	2286	10938	52422	251154	1203366
$s_{3,4\times n}$		4	4	28	124	604	2884	13828	66244	317404

Suppose all $k-1$ eigenvalues of A_k are λ_1, λ_2, ..., λ_{k-1}. Let $|x| < \min\left\{|\lambda_1|^{-1}, |\lambda_2|^{-1}, \ldots, |\lambda_{k-1}|^{-1}\right\}$. It follows that, $|x\lambda_i| < 1$ for $1 \leq i \leq k-1$. Since the $k-1$ eigenvalues of xA_k can be written as $x\lambda_i$ for $1 \leq i \leq k-1$, it follows $|xA_k| < 1$.

Thus, $\sum_{n=0}^{\infty}(xA_k)^n = (I_k - xA_k)^{-1}$. □

Theorem 2.26. *For $k \in \mathbb{N}$,*

$$\mathcal{F}_k(x) = \frac{x}{1 - x - x^2 - x^2 B_k (I_k - xA_k)^{-1} C_k}.$$

Proof. For $k \geq 2$, according to Lemmas 2.14, 2.23, and 2.25

$$\begin{aligned} e_k &= \sum_{n=1}^{\infty} m_{k\times n} x^n \\ &= x + x^2 + \sum_{n=2}^{\infty} B_k A_k^{n-2} C_k x^n \\ &= x + x^2 + x^2 \sum_{n=0}^{\infty} B_k A_k^n C_k x^n \\ &= x + x^2 + x^2 B_k \sum_{n=0}^{\infty} (xA_k)^n C_k \\ &= x + x^2 + x^2 B_k (I_k - xA_k)^{-1} C_k. \end{aligned}$$

Then

$$\mathcal{F}_k(x) = \frac{x}{1 - e_k} = \frac{x}{1 - x - x^2 - x^2 B_k (I_k - xA_k)^{-1} C_k}.$$

Since $B_1(I_1 - xA_1)^{-1}C_1 = 0$, the theorem is also true for $k = 1$. □

Example 2.27. *If $k = 4$ we obtain,*

$$(I_4 - xA_4)^{-1} = \frac{1}{1 - 4x - 4x^2 + x^3} \begin{bmatrix} 1-x & 2x & x(1-x) \\ 3x & 1 - 3x - x^2 & 3x^2 \\ x(1-x) & 2x^2 & 1 - 4x - 3x^2 \end{bmatrix}.$$

Also, $B_4(I_4 - xA_4)^{-1}C_4 = \frac{-2x^2 - 6x + 14}{1 - 4x - 4x^2 + x^3}$. *Hence,*

$$\mathcal{F}_4(x) = \frac{x}{1 - x - x^2 - x^2 \frac{-2x^2 - 6x + 14}{1 - 4x - 4x^2 + x^3}} = \frac{x(1+x)(1 - 5x + x^2)}{(1-x)(1 - 7x + x^2)(1 + 3x + x^2)}.$$

Corollary 2.28. *If* $\det(I_k - xA_k) = a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-1}x^0$, *then*

$$a_{k-1}A_k^{k-1} + a_{k-2}A_k^{k-2} + \cdots + a_0A_k^0 = 0.$$

Proof. Since $\det(I_k - xA_k) = x^{k-1}\det(x^{-1}I_k - A_k)$, then $\det(x^{-1}I_k - A_k) = a_0x^0 + a_1x^{-1} + \cdots + a_{k-1}x^{1-k}$. Employing the Cayley-Hamilton Theorem, we obtain

$$a_{k-1}A_k^{k-1} + a_{k-2}A_k^{k-2} + \cdots + a_0A_k^0 = 0.$$

□

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