

SOME INFINITE PRODUCT IDENTITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. We derive interesting general infinite product identities involving Fibonacci and Lucas numbers. Our procedure consists of applying the classic telescoping summation formula and its variants to identities involving inverse hyperbolic tangent functions having inverse powers of the golden ratio as arguments, thereby obtaining infinite summation identities. While taking into consideration subtle properties of the Fibonacci and Lucas numbers, the infinite summation identities are then converted to infinite product identities.

1. INTRODUCTION

The following *striking* (in Frontczak's description [4]) infinite product identity involving Fibonacci numbers,

$$\prod_{k=1}^{\infty} \frac{F_{2k+2} + 1}{F_{2k+2} - 1} = 3,$$

originally obtained by Melham and Shannon [8], is an $n = 1 = q$ case of the following more general identity (to be derived in this present paper), valid for $q \in \mathbb{N}_0$, $n \in \mathbb{Z}^+$, \mathbb{N}_0 being the set of natural numbers including zero:

$$\prod_{k=1}^{\infty} \frac{F_{(2n-1)(2k+2q)} + F_{2q(2n-1)}}{F_{(2n-1)(2k+2q)} - F_{2q(2n-1)}} = \prod_{k=1}^q \frac{F_{(2n-1)(2k-1)} L_{(2n-1)2k}}{L_{(2n-1)(2k-1)} F_{(2n-1)2k}}.$$

Here the Fibonacci numbers, F_n , and Lucas numbers, L_n , are defined, for $n \in \mathbb{N}_0$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, with $L_0 = 2$, $L_1 = 1$.

Our goal in this paper is to derive several general infinite product identities, including the one given above. For this purpose, we require the following general telescoping summation property (equation (2.1) of [2], also (2.17) of [5]), which holds for $N, m \in \mathbb{Z}$:

$$\sum_{k=1}^N (f(k) - f(k+m)) = \sum_{k=1}^m f(k) - \sum_{k=1}^m f(k+N), \quad \text{for } N \geq m. \quad (1.1)$$

We also need the alternating version of the above identity, obtained by replacing $f(k)$ with $(-1)^{k-1} f(k)$, namely,

$$\begin{aligned} & \sum_{k=1}^N (-1)^{k-1} (f(k) + (-1)^{m-1} f(k+m)) \\ &= \sum_{k=1}^m (-1)^{k-1} f(k) + (-1)^{N-1} \sum_{k=1}^m (-1)^{k-1} f(k+N). \end{aligned} \quad (1.2)$$

Identities (1.1) and (1.2) will be applied to some inverse hyperbolic tangent identities with the inverse powers of the golden mean as arguments of the inverse hyperbolic tangent functions.

If $f(N) \rightarrow 0$ as $N \rightarrow \infty$, then we have, from (1.1) and (1.2), the useful identities:

$$\sum_{k=1}^{\infty} (f(k) - f(k+m)) = \sum_{k=1}^m f(k) \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} (f(k) + (-1)^{m-1} f(k+m)) = \sum_{k=1}^m (-1)^{k-1} f(k). \tag{1.4}$$

In handling infinite products, the following identity will often be useful:

$$\prod_{k=1}^m f(k) = \prod_{k=1}^{\lceil m/2 \rceil} f(2k-1) \prod_{k=1}^{\lfloor m/2 \rfloor} f(2k), \tag{1.5}$$

where $\lfloor u \rfloor$ is the greatest integer less than or equal to u and $\lceil u \rceil$ is the smallest integer greater than or equal to u .

Specifically, we have:

$$\prod_{k=1}^{2q} f(k) = \prod_{k=1}^q f(2k-1) f(2k) \tag{1.6}$$

and

$$\prod_{k=1}^{2q-1} f(k) = \prod_{k=1}^q f(2k-1) \prod_{k=1}^{q-1} f(2k). \tag{1.7}$$

We shall adopt the following empty product convention:

$$\prod_{k=1}^0 f(k) = 1.$$

The golden ratio, having the numerical value of $(\sqrt{5} + 1)/2$, denoted throughout this paper by ϕ , will appear frequently. We shall often make use of the following algebraic properties of ϕ , valid for $n \in \mathbb{Z}$:

$$\phi^2 = 1 + \phi, \tag{1.8a}$$

$$\sqrt{5} = 2\phi - 1, \tag{1.8b}$$

$$\phi - 1 = 1/\phi, \tag{1.8c}$$

$$\phi^n = \phi F_n + F_{n-1}, \tag{1.8d}$$

$$\phi^{-n} = (-1)^n (-\phi F_n + F_{n+1}), \quad \text{and} \tag{1.8e}$$

$$\phi^n = \phi^{n-1} + \phi^{n-2}. \tag{1.8f}$$

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Using Binet's formula, $F_n\sqrt{5} = \phi^n - (-\phi)^{-n}$, $n \in \mathbb{Z}$ and identities (1.8d) and (1.8e), it is straightforward to establish the following useful identities:

$$\phi^n - \phi^{-n} = F_n\sqrt{5}, \quad n \text{ even} \tag{1.9a}$$

$$\phi^n + \phi^{-n} = L_n, \quad n \text{ even} \tag{1.9b}$$

$$\phi^n + \phi^{-n} = F_n\sqrt{5}, \quad n \text{ odd} \tag{1.9c}$$

$$\phi^n - \phi^{-n} = L_n, \quad n \text{ odd}. \tag{1.9d}$$

From (1.9) we also have the following useful results:

$$\frac{\phi^n + 1}{\phi^n - 1} = \frac{F_n\sqrt{5} + 2}{L_n}, \quad n \text{ odd, and} \tag{1.10a}$$

$$\frac{\phi^n + 1}{\phi^n - 1} = \frac{F_n\sqrt{5}}{L_n - 2}, \quad n \text{ even}. \tag{1.10b}$$

We shall make repeated use of the following identities connecting Fibonacci and Lucas numbers:

$$F_{2n} = F_n L_n, \tag{1.11a}$$

$$L_{2n} - 2(-1)^n = 5F_n^2, \tag{1.11b}$$

$$5F_n^2 - L_n^2 = 4(-1)^{(n+1)}, \quad \text{and} \tag{1.11c}$$

$$L_{2n} + 2(-1)^n = L_n^2. \tag{1.11d}$$

Identities (1.11) or their variations can be found in [1, 3, 6].

Finally, the relevant inverse hyperbolic function identities that we shall require are the following, valid for $x, y \in \mathbb{R}$ (see [8]):

$$\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1} \left(\frac{x + y}{1 + xy} \right), \quad xy < 1, \tag{1.12a}$$

$$\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1} \left(\frac{x - y}{1 - xy} \right), \quad xy > -1, \quad \text{and} \tag{1.12b}$$

$$\tanh^{-1} \left(\frac{x}{y} \right) = \frac{1}{2} \log \left(\frac{y + x}{y - x} \right), \quad |x| < |y|. \tag{1.12c}$$

Infinite product identities involving Fibonacci numbers and Lucas numbers were also derived in [4, 8, 7] and references therein.

2. INFINITE PRODUCT IDENTITIES

2.1. Identities resulting from taking $f(k) = \tanh^{-1}(\frac{1}{\phi^{2pk}})$ in (1.3).

Theorem 2.1. *For $q, n \in \mathbb{Z}^+$ the following infinite product identities hold:*

$$\prod_{k=1}^{\infty} \frac{L_{(2n-1)(2k+2q-1)} + L_{(2n-1)(2q-1)}}{L_{(2n-1)(2k+2q-1)} - L_{(2n-1)(2q-1)}} = \sqrt{5} \prod_{k=1}^q \frac{F_{(2k-1)(2n-1)}}{L_{(2k-1)(2n-1)}} \prod_{k=1}^{q-1} \frac{L_{2k(2n-1)}}{F_{2k(2n-1)}}, \tag{2.1}$$

$$\prod_{k=1}^{\infty} \frac{F_{2n(2k+2q-1)} + F_{2n(2q-1)}}{F_{2n(2k+2q-1)} - F_{2n(2q-1)}} = \frac{1}{(\sqrt{5})^{2q-1}} \prod_{k=1}^{2q-1} \frac{L_{2nk}}{F_{2nk}}, \tag{2.2}$$

$$\prod_{k=1}^{\infty} \frac{F_{4n(k+q)} + F_{4nq}}{F_{4n(k+q)} - F_{4nq}} = \frac{1}{5^q} \prod_{k=1}^{2q} \frac{L_{2nk}}{F_{2nk}} \tag{2.3}$$

and

$$\prod_{k=1}^{\infty} \frac{F_{(2n-1)(2k+2q)} + F_{2q(2n-1)}}{F_{(2n-1)(2k+2q)} - F_{2q(2n-1)}} = \prod_{k=1}^q \frac{F_{(2n-1)(2k-1)} L_{(2n-1)2k}}{L_{(2n-1)(2k-1)} F_{(2n-1)2k}}. \tag{2.4}$$

Proof. Choosing $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right)$ in (1.3) and using (1.12b), we have

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{\phi^{2pk+2pm} - \phi^{2pk}}{\phi^{2pk}\phi^{2pk+2pm} - 1}\right) = \sum_{k=1}^m \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right). \tag{2.5}$$

Dividing through the argument of the inverse hyperbolic tangent function in the summation on the left hand side of identity (2.5) by ϕ^{2pk+pm} gives

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{\phi^{pm} - \phi^{-pm}}{\phi^{2pk+pm} - \phi^{-2pk-pm}}\right) = \sum_{k=1}^m \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right). \tag{2.6}$$

It then follows from the identity (2.6), using identity (1.9d) and identity (1.9a), respectively, that:

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{L_{pm}}{L_{p(2k+m)}}\right) = \sum_{k=1}^m \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right), \quad pm \text{ odd}, \tag{2.7}$$

and

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{F_{pm}}{F_{p(2k+m)}}\right) = \sum_{k=1}^m \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right), \quad pm \text{ even}. \tag{2.8}$$

Now by identity (1.12c), we can express the left hand side of identity (2.7) as:

$$\begin{aligned} \sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{L_{pm}}{L_{p(2k+m)}}\right) &= \frac{1}{2} \sum_{k=1}^{\infty} \log\left(\frac{L_{p(2k+m)} + L_{pm}}{L_{p(2k+m)} - L_{pm}}\right) \\ &= \frac{1}{2} \log\left(\prod_{k=1}^{\infty} \frac{L_{p(2k+m)} + L_{pm}}{L_{p(2k+m)} - L_{pm}}\right). \end{aligned} \tag{2.9}$$

Similarly, the right hand side of identity (2.7) can be expressed as:

$$\begin{aligned} \sum_{k=1}^m \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right) &= \frac{1}{2} \sum_{k=1}^m \log\left(\frac{\phi^{2pk} + 1}{\phi^{2pk} - 1}\right) \\ &= \frac{1}{2} \log\left(\prod_{k=1}^m \frac{\phi^{2pk} + 1}{\phi^{2pk} - 1}\right) \\ &= \frac{1}{2} \log\left(\prod_{k=1}^m \frac{F_{2pk}\sqrt{5}}{L_{2pk} - 2}\right), \quad \text{by identity (1.10b)}. \end{aligned} \tag{2.10}$$

Using (2.9) and (2.10) in the identity (2.7) while setting $p = 2n - 1$ and $m = 2q - 1$ (since pm is required to be odd) we have:

$$\begin{aligned}
 & \prod_{k=1}^{\infty} \frac{L_{(2n-1)(2k+2q-1)} + L_{(2n-1)(2q-1)}}{L_{(2n-1)(2k+2q-1)} - L_{(2n-1)(2q-1)}} \\
 &= \prod_{k=1}^{2q-1} \frac{F_{2k(2n-1)}\sqrt{5}}{L_{2k(2n-1)} - 2} \\
 &= (\sqrt{5})^{2q-1} \prod_{k=1}^{2q-1} \frac{F_{2k(2n-1)}}{L_{2k(2n-1)} - 2} \\
 &= (\sqrt{5})^{2q-1} \prod_{k=1}^q \frac{F_{2(2k-1)(2n-1)}}{L_{2(2k-1)(2n-1)} - 2} \prod_{k=1}^{q-1} \frac{F_{4k(2n-1)}}{L_{4k(2n-1)} - 2}, \quad \text{by identity (1.7)}.
 \end{aligned} \tag{2.11}$$

Now:

$$L_{2(2k-1)(2n-1)} - 2 = L_{(2k-1)(2n-1)}^2, \quad \text{by (1.11d)}, \tag{2.12}$$

and

$$L_{4k(2n-1)} - 2 = 5F_{2k(2n-1)}^2, \quad \text{by (1.11b)}. \tag{2.13}$$

Also by (1.11a) we have:

$$F_{2(2k-1)(2n-1)} = F_{(2k-1)(2n-1)}L_{(2k-1)(2n-1)} \tag{2.14}$$

and

$$F_{4k(2n-1)} = F_{2k(2n-1)}L_{2k(2n-1)}. \tag{2.15}$$

Using (2.12), (2.13), (2.14), and (2.15) in identity (2.11), we finally have:

$$\begin{aligned}
 & \prod_{k=1}^{\infty} \frac{L_{(2n-1)(2k+2q-1)} + L_{(2n-1)(2q-1)}}{L_{(2n-1)(2k+2q-1)} - L_{(2n-1)(2q-1)}} \\
 &= (\sqrt{5})^{2q-1} \prod_{k=1}^q \frac{F_{(2k-1)(2n-1)}L_{(2k-1)(2n-1)}}{L_{(2k-1)(2n-1)}^2} \prod_{k=1}^{q-1} \frac{F_{2k(2n-1)}L_{2k(2n-1)}}{5F_{2k(2n-1)}^2} \\
 &= \sqrt{5} \prod_{k=1}^q \frac{F_{(2k-1)(2n-1)}}{L_{(2k-1)(2n-1)}} \prod_{k=1}^{q-1} \frac{L_{2k(2n-1)}}{F_{2k(2n-1)}},
 \end{aligned}$$

which proves (2.1).

The possibilities for the choice of m and p in (2.8) are (i) p even, m odd, (ii) p even, m even and (iii) p odd, m even. Converting the infinite sums in (2.8) to infinite products and setting $p = 2n$ and $m = 2q - 1$ proves identity (2.2); while the identity (2.3) is proved by setting $p = 2n$ and $m = 2q$. Finally, identity (2.4) is proved by setting $p = 2n - 1$ and $m = 2q$ in the infinite product identity resulting from (2.8). □

2.2. Identities resulting from taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{p(2k-1)}}\right)$ in (1.3).

Theorem 2.2. For $q, n \in \mathbb{Z}^+$ the following infinite product identities hold:

$$\prod_{k=1}^{\infty} \frac{F_{4n(2k+q-1)} + F_{4nq}}{F_{4n(2k+q-1)} - F_{4nq}} = \frac{1}{\sqrt{5}^q} \prod_{k=1}^q \frac{L_{2n(2k-1)}}{F_{2n(2k-1)}}, \tag{2.16}$$

$$\prod_{k=1}^{\infty} \frac{F_{(4n-2)(2k+q-1)} + F_{2q(2n-1)}}{F_{(4n-2)(2k+q-1)} - F_{2q(2n-1)}} = \sqrt{5}^q \prod_{k=1}^q \frac{F_{(2n-1)(2k-1)}}{L_{(2n-1)(2k-1)}}, \tag{2.17}$$

$$\prod_{k=1}^{\infty} \frac{L_{(2k-1+2q)(2n-1)} + \sqrt{5}F_{2q(2n-1)}}{L_{(2k-1+2q)(2n-1)} - \sqrt{5}F_{2q(2n-1)}} = \prod_{k=1}^{2q} \frac{F_{(2k-1)(2n-1)}\sqrt{5} + 2}{L_{(2k-1)(2n-1)}} \tag{2.18}$$

and

$$\prod_{k=1}^{\infty} \frac{\sqrt{5}F_{2(2n-1)(k+q-1)} + L_{(2n-1)(2q-1)}}{\sqrt{5}F_{2(2n-1)(k+q-1)} - L_{(2n-1)(2q-1)}} = \prod_{k=1}^{2q-1} \frac{F_{(2k-1)(2n-1)}\sqrt{5} + 2}{L_{(2k-1)(2n-1)}}. \tag{2.19}$$

Proof. Choosing $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{p(2k-1)}}\right)$ in (1.3) and proceeding as in the previous section, we obtain the following infinite sum identities:

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{F_{mp}}{F_{2kp+mp-p}}\right) = \sum_{k=1}^m \tanh^{-1}\frac{1}{\phi^{p(2k-1)}}, \quad p \text{ even},$$

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{F_{mp}\sqrt{5}}{L_{2kp+mp-p}}\right) = \sum_{k=1}^m \tanh^{-1}\frac{1}{\phi^{p(2k-1)}}, \quad p \text{ odd, } m \text{ even},$$

and

$$\sum_{k=1}^{\infty} \tanh^{-1}\left(\frac{L_{mp}}{\sqrt{5}F_{2kp+mp-p}}\right) = \sum_{k=1}^m \tanh^{-1}\frac{1}{\phi^{p(2k-1)}}, \quad p, m \text{ odd}.$$

Identities (2.16) — (2.19) are established by converting each infinite sum identity to an infinite product identity with the appropriate choice of m and p in each case. □

In closing this section, we observe that $q = 1 = n$ in (2.18) and (2.19) yield the following special evaluations:

$$\prod_{k=1}^{\infty} \frac{L_{2k+1} + \sqrt{5}}{L_{2k+1} - \sqrt{5}} = \phi^4, \quad \prod_{k=1}^{\infty} \frac{\sqrt{5}F_{2k} + 1}{\sqrt{5}F_{2k} - 1} = \phi^3.$$

2.3. Identities resulting from taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right)$ in (1.4).

Theorem 2.3. The following infinite product identities hold for $q, n \in \mathbb{Z}^+$:

$$\prod_{k=1}^{\infty} \frac{F_{4nq+4nk} + (-1)^{k-1}F_{4nq}}{F_{4nq+4nk} + (-1)^k F_{4nq}} = \prod_{k=1}^q \frac{L_{2n(2k-1)}F_{4nk}}{F_{2n(2k-1)}L_{4nk}}, \tag{2.20}$$

$$\prod_{k=1}^{\infty} \frac{F_{(2n-1)(2q+2k)} + (-1)^{k-1}F_{(2n-1)2q}}{F_{(2n-1)(2q+2k)} + (-1)^k F_{(2n-1)2q}} = 5^q \prod_{k=1}^q \frac{F_{(2n-1)(2k-1)}F_{(2n-1)2k}}{L_{(2n-1)(2k-1)}L_{(2n-1)2k}}, \tag{2.21}$$

$$\prod_{k=1}^{\infty} \frac{L_{4nk+4nq-2n} + (-1)^{k-1} L_{4nq-2n}}{L_{4nk+4nq-2n} + (-1)^k L_{4nq-2n}} = \frac{1}{\sqrt{5}} \prod_{k=1}^q \frac{L_{2n(2k-1)}}{F_{2n(2k-1)}} \prod_{k=1}^{q-1} \frac{F_{4nk}}{L_{4nk}} \quad (2.22)$$

and

$$\begin{aligned} & \prod_{k=1}^{\infty} \frac{F_{(2n-1)(2q+2k-1)} + (-1)^{k-1} F_{(2n-1)(2q-1)}}{F_{(2n-1)(2q+2k-1)} + (-1)^k F_{(2n-1)(2q-1)}} \\ &= (\sqrt{5})^{2q-1} \prod_{k=1}^q \frac{F_{(2n-1)(2k-1)}}{L_{(2n-1)(2k-1)}} \prod_{k=1}^{q-1} \frac{F_{(2n-1)2k}}{L_{(2n-1)2k}}. \end{aligned} \quad (2.23)$$

Proof. Taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{2pk}}\right)$ in (1.4), setting $m = 2q$, and making use of the identities (1.12b) and (1.9a), we have:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{F_{2qp}}{F_{2kp+2qp}}\right) = \sum_{k=1}^{2q} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk}},$$

from which follows

$$\begin{aligned} \prod_{k=1}^{\infty} \frac{F_{2pq+2pk} + (-1)^{k-1} F_{2pq}}{F_{2pq+2pk} + (-1)^k F_{2pq}} &= \prod_{k=1}^{2q} \frac{\phi^{2pk} + (-1)^{k-1}}{\phi^{2pk} + (-1)^k} \\ &= \prod_{k=1}^q \frac{\phi^{2p(2k-1)} + 1}{\phi^{2p(2k-1)} - 1} \prod_{k=1}^q \frac{\phi^{4pk} - 1}{\phi^{4pk} + 1}, \quad \text{by (1.5),} \\ &= \prod_{k=1}^q \frac{F_{2p(2k-1)} \sqrt{5}}{L_{2p(2k-1)} - 2} \frac{L_{4pk} - 2}{F_{4pk} \sqrt{5}}, \quad \text{by (1.10b),} \\ &= 5^q \prod_{k=1}^q \frac{F_{2p(2k-1)}}{L_{2p(2k-1)} - 2} \frac{F_{2pk}}{L_{2pk}}. \end{aligned} \quad (2.24)$$

The final evaluation of the denominator of the product term in the last line of identity (2.24) depends on the parity of p . Setting $p = 2n$ in (2.24), making use of (1.11b) proves (2.20) while setting $p = 2n - 1$ in (2.24), utilizing (1.11d) proves (2.21).

Taking $f(k) = \tanh^{-1}(\phi^{-2pk})$ in (1.4), setting $m = 2q - 1$, and making use of the identities (1.12a) and (1.9), we have:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{L_{2qp-p}}{L_{2kp+2qp-p}}\right) = \sum_{k=1}^{2q-1} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk}}, \quad p \text{ even,}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{F_{2qp-p}}{F_{2kp+2qp-p}}\right) = \sum_{k=1}^{2q-1} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk}}, \quad p \text{ odd,}$$

from which identities (2.22) and (2.23) follow immediately. □

2.4. Identities resulting from taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{p(2k-1)}}\right)$ in (1.4).

Theorem 2.4. *The following infinite product identities hold for $q, n \in \mathbb{Z}^+$:*

$$\prod_{k=1}^{\infty} \frac{F_{8nk+8nq-4n} + (-1)^{k-1}F_{8nq}}{F_{8nk+8nq-4n} + (-1)^k F_{8nq}} = \prod_{k=1}^q \frac{L_{2n(4k-3)}F_{2n(4k-1)}}{F_{2n(4k-3)}L_{2n(4k-1)}}, \tag{2.25}$$

$$\prod_{k=1}^{\infty} \frac{F_{(4n-2)(2q+2k-1)} + (-1)^{k-1}F_{(2n-1)4q}}{F_{(4n-2)(2q+2k-1)} + (-1)^k F_{(2n-1)4q}} = \prod_{k=1}^q \frac{F_{(2n-1)(4k-3)}L_{(2n-1)(4k-1)}}{L_{(2n-1)(4k-3)}F_{(2n-1)(4k-1)}}, \tag{2.26}$$

$$\begin{aligned} \prod_{k=1}^{\infty} \frac{L_{(2n-1)(2q+2k-1)} + (-1)^{k-1}\sqrt{5}F_{(2n-1)2q}}{L_{(2n-1)(2q+2k-1)} + (-1)^k\sqrt{5}F_{(2n-1)2q}} \\ = \prod_{k=1}^q \frac{(F_{(2n-1)(4k-3)}\sqrt{5} + 2)L_{(2n-1)(4k-1)}}{(F_{(2n-1)(4k-1)}\sqrt{5} + 2)L_{(2n-1)(4k-3)}}, \end{aligned} \tag{2.27}$$

$$\prod_{k=1}^{\infty} \frac{L_{8n(k+q-1)} + (-1)^{k-1}L_{4n(2q-1)}}{L_{8n(k+q-1)} + (-1)^k L_{4n(2q-1)}} = \frac{1}{\sqrt{5}} \prod_{k=1}^q \frac{L_{2n(4k-3)}}{F_{2n(4k-3)}} \prod_{k=1}^{q-1} \frac{F_{2n(4k-1)}}{L_{2n(4k-1)}}, \tag{2.28}$$

$$\begin{aligned} \prod_{k=1}^{\infty} \frac{L_{(8n-4)(k+q-1)} + (-1)^{k-1}L_{(4n-2)(2q-1)}}{L_{(8n-4)(k+q-1)} + (-1)^k L_{(4n-2)(2q-1)}} \\ = \sqrt{5} \prod_{k=1}^q \frac{F_{(2n-1)(4k-3)}}{L_{(2n-1)(4k-3)}} \prod_{k=1}^{q-1} \frac{L_{(2n-1)(4k-1)}}{F_{(2n-1)(4k-1)}}, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} \prod_{k=1}^{\infty} \frac{L_{(4n-2)(q+k-1)} + (-1)^{k-1}\sqrt{5}F_{(2n-1)(2q-1)}}{L_{(4n-2)(q+k-1)} + (-1)^k\sqrt{5}F_{(2n-1)(2q-1)}} \\ = \prod_{k=1}^q \frac{(F_{(2n-1)(4k-3)}\sqrt{5} + 2)}{L_{(2n-1)(4k-3)}} \prod_{k=1}^{q-1} \frac{L_{(2n-1)(4k-1)}}{(F_{(2n-1)(4k-1)}\sqrt{5} + 2)}. \end{aligned} \tag{2.30}$$

Proof. Taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{p(2k-1)}}\right)$ in (1.4) and setting $m = 2q$, we obtain the following summation identities:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{F_{2qp}}{F_{2kp+2qp-p}}\right) = \sum_{k=1}^{2q} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk-p}}, \quad p \text{ even}, \tag{2.31}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{F_{2qp}\sqrt{5}}{L_{2kp+2qp-p}}\right) = \sum_{k=1}^{2q} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk-p}}, \quad p \text{ odd}. \tag{2.32}$$

Identities (2.25) and (2.26) follow from (2.31), while the identity (2.27) follows directly from (2.32).

Taking $f(k) = \tanh^{-1}\left(\frac{1}{\phi^{p(2k-1)}}\right)$ in (1.4) and setting $m = 2q - 1$, we obtain the following summation identities:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1}\left(\frac{L_{p(2q-1)}}{L_{2p(k+q-1)}}\right) = \sum_{k=1}^{2q-1} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk-p}}, \quad p \text{ even} \tag{2.33}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} \tanh^{-1} \left(\frac{F_{p(2q-1)} \sqrt{5}}{L_{2p(k+q-1)}} \right) = \sum_{k=1}^{2q-1} (-1)^{k-1} \tanh^{-1} \frac{1}{\phi^{2pk-p}}, \quad p \text{ odd}. \quad (2.34)$$

Identities (2.28) and (2.29) follow from evaluating the summation identity (2.33) at $p = 4n$ and $p = 4n-2$, respectively. Identity (2.30) is obtained by converting the summation identity (2.34) to an infinite product identity at $p = 2n - 1$. □

From (2.27) and (2.30) at $q = 1 = n$ come the special evaluations:

$$\prod_{k=1}^{\infty} \frac{L_{2k+1} + (-1)^{k-1} \sqrt{5}}{L_{2k+1} + (-1)^k \sqrt{5}} = \phi^2, \quad \prod_{k=1}^{\infty} \frac{L_{2k} + (-1)^{k-1} \sqrt{5}}{L_{2k} + (-1)^k \sqrt{5}} = \phi^3.$$

3. ACKNOWLEDGEMENT

The author thanks the anonymous referee whose comments helped to improve this paper.

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MSC2010: 11B39, 11Y60

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