LUCAS SEQUENCES AND THE HOSOYA INDEX OF GRAPHS

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Abstract. In this note, we construct families of graphs whose Hosoya indices, i.e., the total numbers of matchings, can be described by Lucas sequences.

1. Introduction

Throughout this paper, a and b denote positive integers. Then, as usual (see, e.g., [4]), Lucas sequences $(u_n(a,-b))_{n=0}^{\infty}$ and $(v_n(a,-b))_{n=0}^{\infty}$ are defined by the initial values and binary linear recurrence relations

$$u_0(a,-b)=0,\ u_1(a,-b)=1,\ \ u_{n+2}(a,-b)=au_{n+1}(a,-b)+bu_n(a,-b)\ \ (n\geq 0);$$

and

$$v_0(a,-b)=2,\ v_1(a,-b)=a,\ \ v_{n+2}(a,-b)=av_{n+1}(a,-b)+bv_n(a,-b)\ \ (n\geq 0).$$

In particular, they include the sequences of Fibonacci and Lucas numbers for parameters $a=b=1$.

In their recent paper [1], J. Alexander and P. Hearding constructed families of graphs in which the total number of independent sets can be described by Lucas sequences with parameters satisfying $a\geq b$.

This result gave us the idea to find graphs with the elements of Lucas sequences as the total number of independent sets excluding the restrictive inequality $a\geq b$. We obtain such graphs by providing another graph theoretical interpretation of Lucas sequences as the Hosoya index of certain graphs.

2. Lucas Sequences and Matchings

To give our graph theoretical interpretation of Lucas sequences, we construct two families of graphs. Here, we mention that we use the term graph in the broader sense in which loops and multiple edges are allowed.

Let $n,a,b\geq 1$ be integers. The graph $P_{n,a,b}$ is built up as follows: Consider a graph that is the disjoint union of n copies of the star graph with a vertices, and denote their central points by $w_1,\dots,w_n$. Then join the vertices w_i and w_{i+1} by b parallel edges ($i=1,\dots,n-1$).

If we additionally connect vertices w_n and w_1 by b parallel edges, which edges will be referred to as additional edges, then this expanded graph will be denoted by $C_{n,a,b}$. For clarity, we should emphasize that in $C_{1,a,b}$ the additional edges are loops that join w_1 to itself, whereas in $C_{2,a,b}$ we have $2b$ parallel edges between w_1 and w_2 altogether.

We notice that these graphs are generalizations of path and cycle graphs, since for $a=b=1$ they simply coincide with the n-vertex path and cycle graphs, respectively. In the figures below, as further examples, we illustrate graphs $P_{6,4,2}$ and $C_{6,4,2}$.

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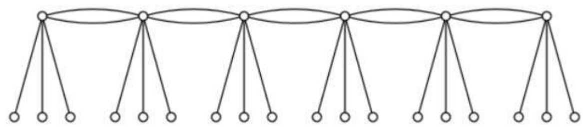

FIGURE 1. The graph $P_{6,4,2}$.

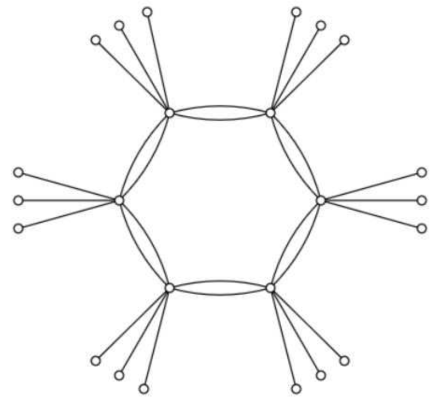

FIGURE 2. The graph $C_{6,4,2}$.

Before stating our theorems, we recall that the Hosoya index of a graph G is defined as the total number of matchings in G, including the empty matching. This notion is originated in chemical graph theory, see [2, 3].

Theorem 1. *For $n, a, b \geq 1$, the Hosoya index of the graph $P_{n,a,b}$ is $u_{n+1}(a, -b)$.*

Proof. We prove this theorem by induction on n. It is easy to check that the Hosoya indices of $P_{1,a,b}$ and $P_{2,a,b}$ are $a = u_2(a, -b)$ and $a^2 + b = u_3(a, -b)$, respectively.

Let $n \geq 3$ and suppose that the assertion is true for $n - 2$ and $n - 1$. If a matching of $P_{n,a,b}$ contains no edges between w_{n-1} and w_n, then it is a matching of $P_{n-1,a,b}$ together with at most one of the $a - 1$ edges of the star graph with central point w_n. On the other hand, if a matching contains one of the b edges between w_{n-1} and w_n, then it is a matching of $P_{n-2,a,b}$ together with this edge. Therefore, by the induction hypothesis, the total number of matchings in $P_{n,a,b}$ is

$$au_n(a, -b) + bu_{n-1}(a, -b) = u_{n+1}(a, -b).$$

□

Theorem 2. *For $n, a, b \geq 1$, the Hosoya index of the graph $C_{n,a,b}$ is $v_n(a, -b)$.*

Proof. It can be easily verified that the Hosoya indices of $C_{1,a,b}$ and $C_{2,a,b}$ are $a = v_1(a, -b)$ and $a^2 + 2b = v_2(a, -b)$, respectively. (In case of $C_{1,a,b}$ the additional edges, which are loops, cannot be contained in a matching, since a loop has a common vertex with itself.)

Let $n \geq 3$. If a matching of $C_{n,a,b}$ contains none of the additional edges, then it is simply a matching of $P_{n,a,b}$. On the other hand, if a matching contains one of the b additional edges, then it is a matching of $P_{n-2,a,b}$ together with this edge. Hence, it follows from Theorem 1 and a well-known identity for Lucas sequences that the total number of matchings in $C_{n,a,b}$ is

$$u_{n+1}(a, -b) + bu_{n-1}(a, -b) = v_n(a, -b).$$

□

3. Lucas Sequences and Independent Sets

Now, we are able to conclude this paper with the solution of the original motivating problem, which is to give families of graphs with total number of independent sets described by Lucas sequences.

Since a matching in a graph G becomes an independent set in its line graph $L(G)$, our theorems can be translated into this context.

Corollary. *For $n, a, b \geq 1$, the total numbers of independent sets in the graphs $L(P_{n,a,b})$ and $L(C_{n,a,b})$ are $u_{n+1}(a, -b)$ and $v_n(a, -b)$, respectively.*

We can give a description of the structures of these graphs directly:

The graph $L(P_{n,a,b})$ is obtained from the disjoint union of n copies $G_1, \ldots, G_n$ of the $(a-1)$-vertex complete graph, and $n-1$ copies $H_1, \ldots, H_{n-1}$ of the b-vertex complete graph by joining each vertex of H_i to all vertices of H_{i+1} $(i = 1, \ldots, n-2)$, and joining each vertex of H_j to all vertices of $G_j \cup G_{j+1}$ $(j = 1, \ldots, n-1)$.

Similarly, the graph $L(C_{n,a,b})$ is obtained from the disjoint union of n copies $G_1, \ldots, G_n$ of the $(a-1)$-vertex complete graph, and n copies $H_1, \ldots, H_n$ of the b-vertex complete graph by joining each vertex of H_i to all vertices of H_{i+1} $(i = 1, \ldots, n-1)$, joining each vertex of H_n to all vertices of H_1, joining each vertex of H_j to all vertices of $G_j \cup G_{j+1}$ $(j = 1, \ldots, n-1)$, finally joining each vertex of H_n to all vertices of $G_n \cup G_1$.

References

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