POLYNOMIAL EXTENSIONS OF A DIMINNIE DELIGHT REVISITED: PART I

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Abstract. Recently, we extended to Fibonacci polynomials a complex, but interesting, recurrence studied by C.R. Diminnie. We then studied the corresponding versions to Lucas, Pell, and Pell-Lucas polynomials, and extracted the respective number-theoretic counterparts. In this article, we explore extensions to Jacobsthal, Jacobsthal-Lucas, Vieta, and Chebyshev polynomials.

1. Introduction

The extended *gibonacci polynomials* $g_n(x)$ are defined by the second-order recurrence $g_{n+2}(x) = a(x)g_{n+1}(x) + b(x)g_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), g_0(x)$, and $g_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 7, 9, 15].

The *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The corresponding *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [6, 8, 9].

Suppose $a(x) = 1$ and $b(x) = x$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $g_0(x) = 2$ and $g_1(x) = 1$, $g_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 4, 9]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [5, 9, 12].

On the other hand, let $a(x) = 2x$ and $b(x) = -1$. When $g_0(x) = 1$ and $g_1(x) = x$, $g_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $g_0(x) = 1$ and $g_1(x) = 2x$, $g_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [5, 8, 9, 11].

1.1. Fibonacci Extension of the Diminnie Delight.

Recently, we investigated a Fibonacci polynomial extension of the recurrence [2, 14]

$$d_{n+1} = 5d_n(5d_n^4 - 5d_n^2 + 1),$$

where $d_0 = 1$ and $n \geq 0$. We found that the solution of the generalized recurrence

$$a_{n+1} = a_n(\Delta^4 a_n^4 - 5\Delta^2 a_n^2 + 5),$$

where $a_n = a_n(x), a_0 = 1$, and $\Delta = \sqrt{x^2+4}$ is $a_n(x) = f_{5^n}$ [10].

Polynomials $c_m(x)$, defined by the recurrence $c_{m+2}(x) = xc_{m+1}(x) - c_m(x)$, played a major part in the polynomial investigation. They are related to the polynomials $T_m(x)$ and $l_m(x)$:

$c_m(x) = 2T_m(x/2) = i^m l_m(-ix)$, where $i = \sqrt{-1}$. In addition, they satisfy a delightful property:

$$c_m\left(y + \frac{1}{y}\right) = y^m + \frac{1}{y^m},$$

where $y \neq 0$ and $m \geq 0$ [10, 14].

In the interest of brevity and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$.

Using the polynomials c_m, we studied an infinite class of recurrences, as in the following theorem [10].

Theorem 1.1. *The solution of the recurrence*

$$\Delta a_{n+1} = c_m(\Delta a_n), \tag{1.1}$$

is $a_n = f_{k\cdot m^n}$, where $a_n = a_n(x)$, $a_0 = f_k$, km is an odd positive integer, $k \neq m$, $m \geq 3$, and $n \geq 0$.

This theorem plays a pivotal role in our exploration of the extensions to Jacobsthal polynomials $J_n(x)$, Vieta polynomials $V_n(x)$, and Chebyshev polynomials $U_n(x)$. In the interest of brevity, we omit a lot of basic, but messy algebra.

2. Relationships Among the Gibonacci Subfamilies

Interestingly, Fibonacci, Pell, and Jacobsthal polynomials, and Chebyshev polynomials of the second kind are closely linked; and so are the Lucas, Pell-Lucas, and Jacobsthal-Lucas polynomials, and Chebyshev polynomials of the first kind [5, 9, 12]:

$$\begin{aligned} J_n(x) &= x^{(n-1)/2} f_n(1/\sqrt{x}) & j_n(x) &= x^{n/2} l_n(1/\sqrt{x}) \\ V_n(x) &= i^{n-1} f_n(-ix) & v_n(x) &= i^n l_n(-ix) \\ V_n(x) &= U_{n-1}(x/2) & v_n(x) &= 2T_n(x/2) \\ xV_n(x^2+2) &= f_{2n} & xv_n(x^2+2) &= l_{2n} \\ J_{2n}(x) &= x^{n-1} V_n\left(\frac{2x+1}{x}\right) & j_{2n}(x) &= x^n v_n\left(\frac{2x+1}{x}\right). \end{aligned}$$

With these tools at our finger tips, we are now ready for the explorations.

3. Jacobsthal Extensions

Let $b_n = b_n(x) = x^{(k\cdot m^n-1)/2} a_n(1/\sqrt{x}) = x^{(k\cdot m^n-1)/2} f_{k\cdot m^n}(1/\sqrt{x})$. It follows from recurrence (1.1) that

$$\sqrt{x^2+4}\, f_{k\cdot m^{n+1}} = c_m\left[\sqrt{x^2+4}\, f_{k\cdot m^n}\right].$$

Replacing x with $1/\sqrt{x}$, and then multiplying both sides of the resulting equation by $x^{(k\cdot m^{n+1}-1)/2}$ yields

$$\sqrt{\frac{4x+1}{x}}\, b_{n+1} = x^{(k\cdot m^{n+1}-1)/2} c_m\left[\sqrt{\frac{4x+1}{x}} \cdot \frac{1}{x^{(k\cdot m^n-1)/2}}\, b_n\right]$$

$$\sqrt{4x+1}\, b_{n+1} = x^{(k\cdot m^{n+1})/2} c_m\left[\sqrt{\frac{4x+1}{x^{(k\cdot m^n)/2}}}\, b_n\right], \tag{3.1}$$

where $b_0 = x^{(k-1)/2} f_k(1/\sqrt{x}) = J_k(x)$.

The solution of recurrence (3.1) is $b_n = x^{(k\cdot m^n-1)/2} f_{k\cdot m^n}(1/\sqrt{x}) = J_{k\cdot m^n}(x)$.

In particular, let $x = 2$ and $B_n = b_n(2)$. It then follows from (3.1) that

$$3B_{n+1} = 2^{(k\cdot m^{n+1})/2} c_m \left[\frac{3}{2^{(k\cdot m^n)/2}}\, B_n\right], \tag{3.2}$$

where $B_0 = b_0(2) = J_k$. The solution of the recurrence is $B_n = J_{k\cdot m^n}$.

Suppose we let $m = 3$. It then follows from recurrence (3.2) that

$$\begin{aligned} 3B_{n+1} &= 2^{(k\cdot 3^{n+1})/2} c_3(u) \\ &= 2^{(k\cdot 3^{n+1})/2}(u^3 - 3u) \\ B_{n+1} &= 9B_n^3 - 3\cdot 2^{k\cdot 3^n} B_n, \end{aligned} \tag{3.3}$$

where $u = \dfrac{3B_n}{2^{(k\cdot 3^n)/2}}$; $B_0 = J_k$; and hence $B_n = J_{k\cdot 3^n}$, where $n \geq 0$.

Letting $m = 5$ in (3.2), we similarly get

$$B_{n+1} = 81B_n^5 - 45\cdot 2^{k\cdot 5^n} B_n^3 + 5\cdot 4^{k\cdot 5^n} B_n, \tag{3.4}$$

where $B_0 = J_k$; and the solution of the recurrence is $B_n = J_{k\cdot 5^n}$, where $n \geq 0$.

Similarly, by letting $m = 7$, we get the recurrence

$$B_{n+1} = 729B_n^7 - 567\cdot 2^{k\cdot 7^n} B_n^5 + 126\cdot 4^{k\cdot 7^n} B_n^3 - 7\cdot 8^{k\cdot 7^n} B_n, \tag{3.5}$$

where $B_0 = J_k$. The solution is $B_n = J_{k\cdot 7^n}$, where $n \geq 0$.

In particular, let $k = 1$. Since $B_1 = J_7 = 43, B_2 = 729\cdot 43^7 - 567\cdot 2^7\cdot 43^5 + 126\cdot 4^7\cdot 43^3 - 7\cdot 8^7\cdot 43 = 187,649,984,473,771 = J_{7^2}$.

Next we explore Vieta extensions.

4. Vieta Extensions

This time, we let $b_n = b_n(x) = i^{k\cdot m^n-1} a_n(-ix) = i^{k\cdot m^n-1} f_{k\cdot m^n}(-ix)$. Replacing x with $-ix$ in recurrence (1.1), and then multiplying both sides of the resulting equation by $x^{k\cdot m^{n+1}-1}$, we get

$$\sqrt{4-x^2}\, b_{n+1} = i^{k\cdot m^{n+1}-1} c_m \left[\frac{\sqrt{4-x^2}}{i^{k\cdot m^n-1}}\, b_n\right], \tag{4.1}$$

where $b_0 = i^{k-1} f_k(-ix) = V_k(x)$. Since $b_n = i^{k\cdot m^n-1} f_{k\cdot m^n}(-ix)$, it follows that $b_n = V_{k\cdot m^n}(x)$, where $n \geq 0$.

Suppose $m = 3$. Then recurrence (4.1) yields

$$\begin{aligned} \sqrt{4-x^2}\, b_{n+1} &= i^{k\cdot 3^{n+1}-1} c_3(u) \\ &= i^{k\cdot 3^{n+1}-1}(u^3 - 3u) \\ b_{n+1} &= (x^2-4)b_n^3 + 3b_n, \end{aligned} \tag{4.2}$$

where $u = \dfrac{\sqrt{4-x^2}}{i^{k\cdot m^n-1}}\, b_n$ and $b_0 = V_k(x)$. The solution of this recurrence is $b_n = V_{k\cdot 3^n}(x)$.

For example, when $k = 1, b_1 = x^2 - 1 = V_3(x)$ and $b_2 = (x^2-4)(x^2-1)^3 + 3(x^2-1) = x^8 - 7x^6 + 15x^4 - 10x^2 + 1 = V_{3^2}(x)$.

The cases $m = 5$ and $m = 7$ can be studied similarly.

Next we present three charming byproducts of recurrence (4.1).

4.1. **A Fibonacci Byproduct.** Let $d_n = d_n(x) = xb_n(x^2+2)$. Replacing x with x^2+2 in (4.1), we get

$$\Delta\, i\, d_{n+1} = i^{k\cdot m^{n+1}-1} c_m \left(\frac{\Delta\, i}{i^{k\cdot m^n -1}}\, d_n \right)$$

$$\Delta\, d_{n+1} = i^{k\cdot m^{n+1}-2} c_m \left(\frac{\Delta}{i^{k\cdot m^n -2}}\, d_n \right), \tag{4.3}$$

where $d_0 = xb_0(x^2+2) = xV_k(x^2+2) = f_{2k}$. The solution of this recurrence is $d_n = xb_n(x^2+2) = xV_{k\cdot m^n}(x^2+2) = f_{2k\cdot m^n}$, where $n \geq 0$.

In particular, let $d_n(1) = D_n$. Then equation (4.3) yields the recurrence

$$\sqrt{5} D_{n+1} = i^{k\cdot m^{n+1}-2} c_m \left(\frac{\sqrt{5}}{i^{k\cdot m^n -2}}\, D_n \right), \tag{4.4}$$

where $D_0 = F_{2k}$. Clearly, $D_n = F_{2k\cdot m^n}$, where $n \geq 0$.

Letting $k = 5$ and $m = 3$, it follows from equation (4.4) that

$$D_{n+1} = 5D_n^3 + 3D_n, \tag{4.5}$$

where $D_0 = F_{10} = 55$. Then $D_n = F_{10\cdot 3^n}$, where $n \geq 0$.

For example, $D_1 = 5D_0^3 + 3D_0 = 832,040 = F_{10\cdot 3}$ and hence, $D_2 = 5\cdot 832040^3 + 3\cdot 832040 = 2,880,067,194,370,816,120 = F_{10\cdot 3^2}$.

4.2. **A Jacobsthal Byproduct.** Let $e_n = e_n(x) = x^{k\cdot m^n -1} b_n(u)$, where $u = \dfrac{2x+1}{x}$. Replacing x with u in equation (4.1), and then multiplying the resulting equation by $x^{k\cdot m^{n+1}-1}$, we get the recurrence

$$\sqrt{4x+1}\, e_{n+1} = i^{k\cdot m^{n+1}-2} \cdot x^{k\cdot m^{n+1}} c_m \left[\frac{\sqrt{4x+1}}{i^{k\cdot m^n -2}\, x^{k\cdot m^n}}\, e_n \right], \tag{4.6}$$

where $e_0 = J_{2k}(x)$. Its solution is $e_n = x^{k\cdot m^n -1} b_n(u) = x^{k\cdot m^n -1} V_{k\cdot m^n}(x) = J_{2k\cdot m^n}(x)$, where $n \geq 0$.

In particular, let $e_n(2) = E_n$. Then equation (4.6) yields the recurrence

$$3E_{n+1} = -(2i)^{k\cdot m^{n+1}} c_m \left[\frac{-3}{(2i)^{k\cdot m^n}}\, E_n \right], \tag{4.7}$$

where $E_0 = J_{2k}$. Its solution is $E_n = J_{2k\cdot m^n}$, where $n \geq 0$.

When $k = 5$ and $m = 3$, equation (4.7) yields

$$E_{n+1} = 9E_n^3 + 3\cdot 1024^{3^n} E_n, \tag{4.8}$$

where $E_0 = J_{10} = 341$. The solution of this recurrence is $E_n = J_{10\cdot 3^n}$, where $n \geq 0$.

For example, $E_1 = 9E_0^3 + 3\cdot 1024\, E_0 = 357,913,941$; and hence, $E_2 = 9(357913941)^3 + 3\cdot 1024^3 \cdot 357913941 = 412,646,679,761,793,424,966,374,741 = J_{10\cdot 3^2}$.

Next we present a byproduct to Chebyshev polynomials.

4.3. **A Chebyshev Byproduct.** Let $h_n = h_n(x) = b_n(2x)$. Replacing x with $2x$ in equation (4.1), we get the recurrence

$$2\sqrt{1-x^2}\, h_{n+1} = i^{k\cdot m^{n+1}-1} c_m \left(\frac{2\sqrt{1-x^2}}{i^{k\cdot m^n -1}}\, h_n \right), \tag{4.9}$$

where $h_0 = U_{k-1}(x)$. The solution of this recurrence is $h_n = b_n(2x) = V_{k\cdot m^n}(2x) = U_{k\cdot m^n -1}(x)$.

In particular, when $m = 3$, equation (4.10) yields the recurrence

$$h_{n+1} = 4(x^2 - 1)h_n^3 + 3h_n,$$

where $h_0 = U_{k-1}(x)$. Its solution is $h_n = U_{k\cdot 3^n - 1}$, where $n \geq 0$.

For example, let $k = 5$. Then $h_0 = U_4(x) = 16x^4 - 12x^2 + 1$. Consequently,

$$\begin{aligned} h_1 &= 4(x^2 - 1)(16x^4 - 12x^2 + 1)^3 + 3(16x^4 - 12x^2 + 1) \\ &= 16384x^{14} - 53248x^{12} + 67584x^{10} - 42240x^8 + 13440x^6 - 2016x^4 + 112x^2 - 1 \\ &= U_{5\cdot 3 - 1}. \end{aligned}$$

Next we focus on recurrence (1.1), where m is an even positive integer. Theorem 1.1 has a parallel result, as the following theorem shows.

Theorem 4.1. *Let m be an even positive integer and k a positive integer such that $m \nmid k$. Then the solution of the recurrence*

$$a_{n+1} = c_m(a_n) \tag{4.10}$$

is $a_n = l_{k\cdot m^n}$, where $a_n = a_n(x), a_1 = l_{km}$, and $n \geq 1$. □

The proof follows by the Binet-like formula for l_n [10], the property that

$$c_m\left(y + \frac{1}{y}\right) = y^m + \frac{1}{y^m},$$

and induction on n, where $y \neq 0$ and m is an even positive integer. In the interest of brevity, we omit the proof.

Theorem 4.1 also has interesting implications to Jacobsthal, Vieta, and Chebyshev families.

5. Jacobsthal-Lucas Extensions

Let $b_n = x^{(k\cdot m^n)/2} a_n(1/\sqrt{x})$. Now replace x with $1/\sqrt{x}$ in (4.10) and then multiply the resulting equation by $x^{(k\cdot m^{n+1})/2}$. We then get the recurrence

$$b_{n+1} = x^{(k\cdot m^{n+1})/2}\, c_m\left[\frac{1}{x^{(k\cdot m^n)/2}}\, b_n\right], \tag{5.1}$$

where $b_1 = x^{(km)/2} l_{km}(1/\sqrt{x}) = j_{km}(x)$. The solution of this recurrence is

$$b_n = x^{(k\cdot m^n)/2} l_{k\cdot m^n}(1/\sqrt{x}) = j_{k\cdot m^n}(x).$$

Suppose we let $x = 2$ and $B_n = b_n(2)$. Then equation (5.1) yields the recurrence

$$B_{n+1} = 2^{(k\cdot m^{n+1})/2}\, c_m\left[\frac{B_n}{2^{(k\cdot m^n)/2}}\right], \tag{5.2}$$

where $B_1 = j_{km}$. The corresponding solution is $B_n = j_{k\cdot m^n}$, where $n \geq 1$.

When $m = 2$, we get

$$B_{n+1} = B_n^2 - 2^{k\cdot 2^n + 1}, \tag{5.3}$$

where $B_1 = j_{2k}$.

Similarly, the cases $m = 4$ and $m = 6$ yield the recurrences

$$B_{n+1} = B_n^4 - 4\cdot 2^{k\cdot 4^n} B_n^2 + 4^{k\cdot 4^n} \tag{5.4}$$

$$B_{n+1} = B_n^6 - 6\cdot 2^{k\cdot 6^n} B_n^4 + 9\cdot 4^{k\cdot 6^n} B_n^2 - 2\cdot 8^{k\cdot 6^n}, \tag{5.5}$$

where $B_1 = j_{4k}$ and $B_1 = j_{6k}$, respectively.

For example, let $k = 1$ in recurrence (5.5). Then $B_1 = j_6 = 65$, and hence, $B_2 = 65^6 - 6 \cdot 2^6 \cdot 65^4 + 9 \cdot 4^6 \cdot 65^2 - 2 \cdot 8^6 = 68,719,476,737 = j_{6^2}$.

Next we pursue Vieta-Lucas polynomial extensions.

6. Vieta-Lucas Extensions

Since $v_n(x) = i^n l_n(-ix)$, we let $b_n = b_n(x) = i^{k \cdot m^n} a_n(-ix)$. Replacing x with $-ix$ in (4.10) and multiplying the resulting equation by $i^{k \cdot m^{n+1}}$ yields the recurrence

$$b_{n+1} = c_m \left(\frac{1}{i^{k \cdot m^n}} \, b_n \right), \tag{6.1}$$

where $b_1 = i^{k \cdot m} l_{km}(-ix) = v_{km}(x)$. Its solution is $b_n = i^{k \cdot m^n} l_{k \cdot m^n}(-ix) = v_{k \cdot m^n}(x)$.

Next we study three interesting implications of recurrence (6.1).

6.1. A Lucas Byproduct.

Let $d_n = d_n(x) = x b_n(x^2+2)$. Replacing x with x^2+2 in equation (6.1) and then multiplying the resulting equation by x, we get

$$d_{n+1} = x c_m \left(\frac{1}{x i^{k \cdot m^n}} \, d_n \right), \tag{6.2}$$

where $d_1 = x v_{km}(x^2+2) = l_{2km}$. Its solution is $d_n = x v_{k \cdot m^n}(x^2+2) = l_{2k \cdot m^n}$.

In particular, let $d_n(1) = D_n$. Then equation (6.2) yields

$$D_{n+1} = c_m \left(\frac{D_n}{i^{k \cdot m^n}} \right), \tag{6.3}$$

where $D_1 = L_{2km}$. Clearly, $D_n = L_{2k \cdot m^n}$.

For example, let $k = 5$ and $m = 4$. Then $D_{n+1} = C_4(D_n)$, where $D_1 = L_{40} = 228,826,127$. Then $D_2 = 228826127^4 - 4 \cdot 228826127^2 + 2 = 2,741,715,832,729,650,778,856,894,742,296,127 = L_{160}$.

Next we present an implication to Jacobsthal-Lucas polynomials.

6.2. A Jacobsthal-Lucas Byproduct.

Let $e_n = e_n(x) = x^{k \cdot m^n} b_n(u)$, where $u = \dfrac{2x+1}{x}$. It then follows from recurrence (6.1) that

$$e_{n+1} = x^{k \cdot m^{n+1}} c_m \left[\frac{1}{(ix)^{k \cdot m^n}} \, e_n \right], \tag{6.4}$$

where $e_1 = x^{km} v_{km}(u) = j_{2km}(x)$. The solution of this recurrence is $e_n = x^{k \cdot m^n} b_n(u) = x^{k \cdot m^n} v_{k \cdot m^n}(u) = j_{2k \cdot m^n}(x)$, where $n \geq 1$.

Letting $E_n = e_n(2)$, it follows that

$$E_{n+1} = 2^{k \cdot m^{n+1}} c_m \left[\frac{1}{(2i)^{k \cdot m^n}} \, E_n \right], \tag{6.5}$$

where $E_1 = j_{2km}$. The solution of this recurrence is $E_n = j_{2k \cdot m^n}$, where $n \geq 1$.

Suppose we let $k = 3$ and $m = 4$. Then recurrence (6.5) yields

$$E_{n+1} = E_n^4 - 4 \cdot 64^{4^n} E_n^2 + 2 \cdot 4096^{4^n},$$

where $E_1 = j_{24} = 16,777,217$. Consequently, $E_2 = E_1^4 - 4 \cdot 64^4 E_1^2 + 2 \cdot 4096^4 = 79,228,162,514,264,337,593,543,950,337 = j_{6 \cdot 4^2}$.

Finally, we present an interesting consequence to Chebyshev polynomials $T_n(x)$.

6.3. **A Chebyshev Byproduct.** Letting $h_n = h_n(x) = \frac{1}{2}b_n(2x)$, equation (6.1) yields the recurrence

$$2h_{n+1} = c_m\left(\frac{2}{i^{k\cdot m^n}}\,h_n\right),$$

where $h_1 = \frac{1}{2}b_1(2x) = \frac{1}{2}v_{km}(2x) = T_{km}(x)$. Its solution is $h_n = \frac{1}{2}b_n(2x) = \frac{1}{2}v_{k\cdot m^n}(2x) = T_{k\cdot m^n}(x)$, where $n \ge 1$.

When $m = 2$, it follows that $2h_{n+1} = c_2\left(\frac{2}{i^{k\cdot 2^n}}\,h_n\right)$; so $h_{n+1} = 2h_n^2 - 1$, where $h_1 = T_{2k}(x)$. Likewise, when $m = 4$ and $m = 6$, we get the recurrences

$$h_{n+1} = 8h_n^4 - 8h_n^2 + 2$$
$$h_{n+1} = 32h_n^6 - 48h_n^4 + 36h_n^2 - 2,$$

where $h_1 = T_{4k}(x)$ and $h_1 = T_{6k}(x)$, respectively.

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