NOTES & EXTENSIONS FOR A REMARKABLE CONTINUED FRACTION

PETER G. ANDERSON

Abstract. Let the Fibonacci words be $w_1 = 0, \quad w_2 = 1, \quad w_{n+1} = w_n w_{n-1}$ considered as integers expressed in binary. It is known that for $n \geq 2$ the numbers $0.\overline{w}_n = \frac{w_n}{2^{F_n}-1}$ have the continued fraction $[0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{F_{n-2}}]$. We provide a simple proof using Fibonacci-type recurrences of compositions of linear functions. We apply this to several related recurrences.

1. Introduction

The *Fibonacci words* are

$$\begin{aligned} w_1 = 0, \quad w_2 = 1, \quad w_{n+1} = w_n w_{n-1} \\ v_1 = 1, \quad v_2 = 0, \quad v_{n+1} = v_n v_{n-1} \end{aligned} \tag{1.1}$$

Consider these words as binary numbers, and let $0.\overline{w}_n$ and $0.\overline{v}_n$ be the infinite periodic fractions expressed in binary with period F_n. (We use the overline notation to denote the infinitely repeating pattern of binary digits.)

Each w_n is a prefix of w_{n+1} (and similarly v_n), so the infinite strings w_∞ and v_∞ are well defined: the kth letter of w_∞ is the kth letter of every w_n for which the length of w_n is at least k. Clearly $0.w_\infty$ and $0.v_\infty$ are the limits of $0.\overline{w}_n$ and $0.\overline{v}_n$.

Another point of view is that $0.w_\infty = \sum_{n\geq 1} 2^{-\lfloor n(1+\sqrt{5})/2 \rfloor}$.

Section 2 contains a proof of the following continued fractions for these numbers.

Theorem 1.1.

$$0.\overline{w}_n = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{F_{n-2}}] \tag{1.2}$$

$$0.\overline{v}_n = [0; 3, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{F_{n-2}}] \tag{1.3}$$

consequently

$$0.w_\infty = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{F_{n-2}}, \ldots] \tag{1.4}$$

$$0.v_\infty = [0; 3, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{F_{n-2}}, \ldots] \tag{1.5}$$

The mechanism of Section 2 is stronger than required—a much simpler proof will suffice—but that mechanism is interesting in its own right, and can be used for generalizations. Subsequent Sections generalize these ideas to word sequences of related recurrent sequences.

A much more general result encompassing Theorem 1.1 is given in [1], which includes an extensive bibliography of these types of results. The present presentation is far simpler than the earlier ones.

2. Fibonacci words and their continued fractions

The lengths of the Fibonacci words are $|w_n| = |v_n| = F_n$, the nth Fibonacci numbers. Treating these words as binary numbers, we have

$$w_{n+1} = 2^{F_{n-1}} w_n + w_{n-1}, \quad v_{n+1} = 2^{F_{n-1}} v_n + v_{n-1} \tag{2.1}$$

Define a sequence of functions $T_n : \mathbb{R}^2 \to \mathbb{R}^2, \quad i = 1, 2, \ldots$ by

$$T_n(\vec{x}) = \begin{cases} \frac{\vec{x}+(0,1)}{2} & \text{if } n = 1 \\ \frac{\vec{x}+(1,0)}{2} & \text{if } n = 2 \\ T_{n-1} \circ T_{n-2} & \text{if } n > 2 \end{cases} \tag{2.2}$$

Denote

$$T_n(\vec{v}) = \frac{\vec{v} + \vec{c}_n}{d_n} \tag{2.3}$$

with $\vec{c}_n \in \mathbb{R}^2$ and $d_n \in \mathbb{Z}$. We have the recurrences

$$\vec{c}_1 = (1, 0),\ \vec{c}_2 = (0, 1),\ \vec{c}_{n+1} = d_{n-1}\vec{c}_n + \vec{c}_{n-1} \tag{2.4}$$

and

$$d_1 = 2,\ d_2 = 2,\ d_{n+1} = d_n d_{n-1} \tag{2.5}$$

Clearly,

$$d_n = 2^{F_n} \tag{2.6}$$

We can now conclude from (Eqs. 2.2, 2.4, and 2.6),

$$\vec{c}_n = (w_n, v_n) \tag{2.7}$$

The fixed point, $\vec{f}_n = (f'_n, f''_n)$ of T_n is

$$\vec{f}_n = \frac{\vec{c}_n}{d_n - 1} = (0.\overline{w}_n, 0.\overline{v}_n) \tag{2.8}$$

Denote the denominator in (Eq. 2.8) by

$$D_n = d_n - 1 \tag{2.9}$$

Simple algebra gives us the recurrence

$$D_{n+1} = d_{n-1}D_n + D_{n-1} \tag{2.10}$$

Notice that the $D_n = 2^{F_n} - 1$, in binary, is a word of all ones of length F_n, so it obeys the same recurrence as w_n and v_n in (Eq. 1.1).

The recurrences of (Eqs. 2.4 and 2.10) give us

$$\vec{f}_{n+1} = (0.\overline{w}_{n+1}, 0.\overline{v}_{n+1}) = \frac{\vec{c}_{n+1}}{D_{n+1}} = \frac{d_{n-1}\vec{c}_n + \vec{c}_{n-1}}{d_{n-1}D_n + D_{n-1}} \tag{2.11}$$

The following table shows the continued fractions for the first few $0.\overline{w}_n$ and $0.\overline{v}_n$.

n	w_n	$0.\overline{w}_n$	v_n	$0.\overline{v}_n$
1	0	$0/1 = [0]$	1	$1/1 = [1]$
2	1	$1/1 = [1]$	0	$0/1 = [0]$
3	10	$2/3 = [0; 1, 2]$	01	$1/3 = [0; 3]$
4	101	$5/7 = [0; 1, 2, 2]$	010	$2/7 = [0; 3, 2]$

Combine the data of this table with the recurrence of (Eq. 2.11), to conclude the proof of Theorem 1.1.

Comment. We do not need the function sequence $\{T_n\}$ and its fixed points to prove Theorem 1.1. All we need to observe is that the numerators and denominators of the fractions $0.\overline{w}_n$ and $0.\overline{v}_n$ satisfy the recurrences (Eqs. 2.4 and 2.10), and that the first three fractions p_i/q_i satisfy $p_i q_{i+1} - p_{i+1} q_i = \pm 1,\ i = 1, 2$, which we verify with the above table. The function sequence adds an interesting dimension to the discussion, and in Section 6 it provides a generalization mechanism.

3. Words and their continued fractions for Lucas and similar sequences

Now consider second order linear recurrences, $\mathcal{U}_a$, of the form

$$u_1 = a, \quad u_2 = 1, \quad u_{n+1} = u_n + u_{n-1} \tag{3.1}$$

for $a \geq 2$ ($\mathcal{U}_2$ is the Lucas sequence, with a subscript offset). For these sequences, w_1 and v_1 will be words of length a, and w_2 and v_2 will be words of length 1. For $n \geq 2$, we use the same recurrence as the Fibonacci case, so $|w_n| = |v_n| = u_n$.

$$\begin{aligned} w_1 = 1^{a-1}0, \quad w_2 = 1, \quad w_{n+1} = w_n w_{n-1} \\ v_1 = 0^{a-1}1, \quad v_2 = 0, \quad v_{n+1} = v_n v_{n-1} \end{aligned} \tag{3.2}$$

The corresponding linear functions $\{T_n\}$ are

$$T_n(\vec{x}) = \begin{cases} \frac{\vec{x}+(2^a-2,1)}{2^a} & \text{if } n = 1 \\ \frac{\vec{x}+(1,0)}{2} & \text{if } n = 2 \\ T_{n-1} \circ T_{n-2} & \text{if } n > 2 \end{cases} \tag{3.3}$$

As in Section 2, the words w_n and v_n are complementary, and $0.\overline{w}_n + 0.\overline{v}_n = 1$

The following table shows the first few values of our continued fractions for the Lucas, $a = 2$ case.

n	w_n	$0.\overline{w}_n$	v_n	$0.\overline{v}_n$
1	10	$2/3 = [0; 1, 2]$	01	$1/3 = [0; 3]$
2	1	$1/1 = [1]$	0	$0/1 = [0]$
3	110	$6/7 = [0; 1, 6]$	001	$1/7 = [0; 7]$
4	1101	$13/15 = [0; 1, 6, 2]$	0010	$2/15 = [0; 7, 2]$
5	1101110	$110/127 = [0; 1, 6, 2, 8]$	0010001	$17/127 = [0; 7, 2, 8]$

So mimicking the program of Section 2, we conclude

$$0.\overline{w}_n = [0; 1, 6, 2^1, 2^3, \ldots, 2^{L_{n-2}}] \tag{3.4}$$

$$0.\overline{v}_n = [0; 7, 2^1, 2^3, \ldots, 2^{L_{n-2}}] \tag{3.5}$$

consequently

$$0.w_\infty = [0; 1, 6, 2^1, 2^3, \ldots, 2^{L_{n-2}}, \ldots] \tag{3.6}$$

$$0.v_\infty = [0; 7, 2^1, 2^3, \ldots, 2^{L_{n-2}}, \ldots] \tag{3.7}$$

For general a, we have:

n	w_n	$0.\overline{w}_n$
1	$1^{a-1}0$	$(2^a - 2)/(2^a - 1) = [0; 1, 2^a - 2]$
2	1	$1/1 = [1]$
3	1^a0	$(2^{a+1} - 2)/(2^{a+1} - 1) = [0; 1, 2^{a+1} - 2]$
4	1^a01	$(2^{a+2} - 3)/(2^{a+2} - 1) = [0; 1, 2^{a+1} - 2, 2]$
5	$1^a01^{a+1}0$	$(2^{2a+3} - 2^{a+2} - 2)/(2^{3a+3} - 1) = [0; 1, 2^{a+1} - 2, 2, a^{a+1}]$

n	v_n	$0.\overline{v}_n$
1	$0^{a-1}1$	$1/(2^a - 1) = [0; 2^a - 1]$
2	0	$0/1 = [0]$
3	0^a1	$1/(2^{a+1} - 1) = [0; 2^{a+1} - 1]$
4	0^a10	$2/(2^{a+2} - 1) = [0; 2^{a+1} - 1, 2]$
5	$0^a10^{a+1}1$	$(2^{a+2} + 1)/(2^{3a+1} - 1) = [0; 2^{a+1} - 1, 2, a^{a+1}]$

So,

$$0.\overline{w}_n = [0; 1, 2^{a+1} - 2, 2, a^{a+1}, \ldots, 2^{u_{n-2}}] \tag{3.8}$$

$$0.\overline{v}_n = [0; 2^{a+1} - 1, 2, a^{a+1}, \ldots, 2^{u_{n-2}}] \tag{3.9}$$

consequently

$$0.w_\infty = [0; 1, 2^{a+1} - 2, 2, a^{a+1}, \ldots, 2^{u_{n-2}}, \ldots] \tag{3.10}$$

$$0.v_\infty = [0; 2^{a+1} - 1, 2, a^{a+1}, \ldots, 2^{u_{n-2}}, \ldots] \tag{3.11}$$

4. Words and their continued fractions for Pell-type sequences

A *Pell-type* recurrence is $P_{n+1} = 2P_n + P_{n-1}$. The following sequence of words, w_n and v_n have lengths $1, 1, 3, 7, 17, 41, 99, 239, \ldots$ which satisfy a Pell-type recurrence.[1]

$$\begin{array}{lll} w_1 = 0, & w_2 = 1, & w_{n+1} = w_n w_n w_{n-1} \\ v_1 = 1, & v_2 = 0, & v_{n+1} = v_n v_n v_{n-1} \end{array} \tag{4.1}$$

The corresponding linear functions are

$$T_n(\vec{x}) = \begin{cases} \frac{\vec{x}+(0,1)}{2} & \text{if } n = 1 \\ \frac{\vec{x}+(1,0)}{2} & \text{if } n = 2 \\ T_{n-1} \circ T_{n-1} \circ T_{n-2} & \text{if } n > 2 \end{cases} \tag{4.2}$$

Follow the same program used in Sections 2 and 3, and conclude

$$0.w_\infty = [0; 1, 6, 18, 1032, \ldots, 2^{P_{n-1}}(2^{P_n} + 1), \ldots] \tag{4.3}$$

$$0.v_\infty = [0; 7, 18, 1032, \ldots, 2^{P_{n-1}}(2^{P_n} + 1), \ldots] \tag{4.4}$$

5. Variation one: Binary words from different types of recurrences

These final two sections report some computational experiments attempting to discover a wider variety of remarkable continued fractions.

This section explores three recurrences of the form

$$u_1 = 1, \quad u_2 = 2, \quad u_{n+1} = u_n + u_{n-1} + M_n \tag{5.1}$$

for which $M_n = 1$ or 2, for all n, and $M_n = n - 3$.

The first rule gives us the sequence

$$1, \quad 2, \quad 4, \quad 7, \quad 12, \quad 20, \quad 33, \quad 54, \quad 88, \quad 143, \quad 232, \quad 376, \quad 609, \quad 986, \ldots$$

Each term is one less than the corresponding Fibonacci number (A000071 in [2]). Build two binary sequences N_n and D_n (for numerator and denominator) as follows.

$$\begin{array}{lll} N_1 = 1, & N_2 = 10, & N_{n+1} = N_n 0 N_{n-1} \\ D_1 = 1, & D_2 = 11, & D_{n+1} = D_n 0 D_{n-1} \end{array} \tag{5.2}$$

This gives

$$\frac{N_8}{D_8} = \frac{100101001001010010100100101001001010010100100101001010}{110101101101011010110110101101101011010110110101101011} \tag{5.3}$$

[1]This is sequence A001333, numerators of continued fraction convergents to $\sqrt{2}$ [2].

which has the continued fraction

$$\begin{aligned}[0;1,2,4,8,32,256,8192,2097152] &= [0;2^0,2^1,2^2,2^3,2^5,2^8,2^{13},2^{21}] \\ &= [0;2^{F_0},2^{F_1},2^{F_2},2^{F_3},2^{F_4},2^{F_5},2^{F_6},2^{F_7}]\end{aligned} \tag{5.4}$$

and generally

$$\frac{N_n}{D_n} = [0;2^{F_0},2^{F_1},2^{F_2},2^{F_3},\ldots,2^{F_{n-1}}] \tag{5.5}$$

Complementing the numerator's seeds, as

$$N_0 = 1, \quad N_2 = 01, \quad N_{n+1} = N_n 0 N_{n-1} \tag{5.6}$$

yields the continued fraction

$$\frac{N_8}{D_8} = [0;3,4,8,32,256,8192,2097152] = [0;3^1,2^2,2^3,2^5,2^8,2^{13},2^{21}] \tag{5.7}$$

To explain the similarity of this continued fraction to Theorem 1.1, note that the *difference* sequence in the present case is $u_{n+1} - u_n = F_{n+1}$.

The second experiment, with $M_n = 2$, produces a similar result. The sequence of word lengths for numerators and denominators is

$$1, \quad 2, \quad 5, \quad 9, \quad 16, \quad 27, \quad 45, \quad 74, \quad 121, \quad 197, \quad 320, \quad 519, \quad 841, \quad 1362, \ldots$$

For this sequence, $u_n = L_{n+1} - 2$ (A014739 in [2]). Define the sequence of numerator and denominator words as

$$\begin{array}{lll} N_1 = 1, & N_2 = 10, & N_{n+1} = N_n 00 N_{n-1} \\ D_1 = 1, & D_2 = 11, & D_{n+1} = D_n 00 D_{n-1} \end{array} \tag{5.8}$$

This gives

$$\frac{N_7}{D_7} = [0;1,2,8,16,128,2048,262144,536870912] = [0;1,2^1,2^3,2^4,2^7,2^{11},2^{18},2^{29}] \tag{5.9}$$

The exponents are the Lucas numbers. Generally,

$$\frac{N_n}{D_n} = [0;1,2^{L_1},2^{L_2},2^{L_3},\ldots,2^{L_n}] \tag{5.10}$$

Now let $M_n = n - 3$, producing the sequence

$$1, \quad 2, \quad 3, \quad 6, \quad 11, \quad 20, \quad 35, \quad 60, \quad 101, \quad 168, \quad 277, \quad 454, \quad 741, \quad 1206, \ldots$$

(A131269 in [2]). The difference sequence, $a_n = u_n - u_{n-1}$ for this sequence satisfies $a_{n+1} = a_n + a_{n-1} + 1$, which occurs as the exponents in the continued fraction below (Eq. 5.12). Define the sequence of numerator and denominator words as

$$\begin{array}{lll} N_1 = 1, & N_2 = 10, & N_{n+1} = N_n 0^{n-3} N_{n-1} \\ D_1 = 1, & D_2 = 11, & D_{n+1} = D_n 0^{n-3} D_{n-1} \end{array} \tag{5.11}$$

$$\frac{N_8}{D_8} = [0;1,2,2,8,32,512,32768,33554432] = [0;1,2^1,2^1,2^3,2^5,2^9,2^{15},2^{25}] \tag{5.12}$$

The sequence of exponents appears to be A001595 in [2]: "$a(n) = a(n-1) + a(n-2) + 1$, with $a(0) = a(1) = 1$."

6. Variation two: Richer linear functions

In the Section 5, variations of the Fibonacci words treated as binary numbers yielded some remarkable continued fractions. But it is difficult to see what sequence of linear functions would correspond to those continued fractions. The present section, on the other hand, presents a sequence of linear functions that does not readily correspond to a sequence of binary words (although a sequence of words using a mixed radix may apply).

The functions T_n are

$$T_1(\vec{x}) = \frac{\vec{x} + (1,0)}{2} \tag{6.1}$$

$$T_2(\vec{x}) = \frac{\vec{x} + (1,1)}{3} \tag{6.2}$$

$$T_n(\vec{x}) = T_{n-1}(T_{n-2}(\vec{x})), \quad \text{if } n > 2 \tag{6.3}$$

The fixed points of these functions $\vec{f} = (f', f'')$ have the form

$$f' = [0; 1, 1, 2, 3, 6, 18, 108, 1944, 209952, 408146688, \ldots] \tag{6.4}$$

$$f'' = [0; 2, 2, 3, 6, 18, 108, 1944, 209952, 408146688, \ldots] \tag{6.5}$$

The remarkable thing about these continued fractions is that starting with the element 6, every element is the product of the two preceding elements (essentially sequence A000304 in [2]). For now, we present this as an experimental result, and reserve its proof and generalization for a sequel.

References

[1] P. G. Anderson, T. C. Brown, and P. J.-S. Shiue, *A Simple Proof of a Remarkable Continued Fraction Identity*, Proceedings of the American Mathematical Society, Volume 123, Number 7, July 1995.

[2] OEIS Foundation Inc. (2016), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

MSC2010: 30B70

Computer Science Department, Rochester Institute of Technology, Rochester, NY 14531
E-mail address: `pga@cs.rit.edu`