THE NATURAL LOGARITHM OF THE GOLDEN SECTION

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Abstract. This paper looks at the ways in which the natural logarithm of the Golden Section may be expressed as summations and hyperbolic functions. It is a condensed version of the presentation kindly given on my behalf by Dr. Ron Knott at the recent Caen conference.

1. McLaurin's Series

From McLaurin's series, for $-1 \leq x < 1$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

and

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad .$$

If x $=\Phi = \frac{1}{2}(1+\sqrt{5})$, the golden section, we have:

$$\ln(1+\frac{1}{\Phi}) = \ln\Phi = \frac{1}{\Phi} - \frac{1}{2\Phi^2} + \frac{1}{3\Phi^3} - \frac{1}{4\Phi^4} + \cdots = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n\Phi^n} \tag{1.1}$$

and

$$-\ln(1-\frac{1}{\Phi}) = \ln\Phi^2 = 2\ln\Phi = \frac{1}{\Phi} + \frac{1}{2\Phi^2} + \frac{1}{3\Phi^3} + \frac{1}{4\Phi^4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\Phi^n} \quad . \tag{1.2}$$

Adding (1.1) to (1.2) gives:

$$3\ln\Phi = 2(\frac{1}{\Phi} + \frac{1}{3\Phi^3} + \frac{1}{5\Phi^5} + \dots) = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)\Phi^{2n-1}} = 2\,\mathrm{arctanh}\,\frac{1}{\Phi} \quad . \tag{1.3}$$

Subtracting (1.1) from (1.2) gives:

$$\ln\Phi = \frac{1}{\Phi^2} + \frac{1}{2\Phi^4} + \frac{1}{3\Phi^6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\Phi^{2n}} \quad . \tag{1.4}$$

2. Summations in Terms of Fibonacci Numbers, Lucas Numbers and Phi

from [1], p.52:

$$F_n\sqrt{5} = \Phi^n - \left(\frac{-1}{\Phi}\right)^n \tag{2.1}$$

$$L_n = \Phi^n + \left(\frac{-1}{\Phi}\right)^n \tag{2.2}$$

$$\Phi^n = F_n\Phi + F_{n-1} \tag{2.3}$$

Combining (2.1) and (2.2) gives:

$$\frac{F_n\sqrt{5} - L_n}{2} = -\left(\frac{-1}{\Phi}\right)^n \quad . \tag{2.4}$$

Substituting (2.4) into (1.1), we have:

$$\ln\Phi = \sum_{n=1}^{\infty} \frac{F_n\sqrt{5} - L_n}{2n} \quad . \tag{2.5}$$

Substituting (2.2) into (1.2), we have:

$$\ln\Phi = \sum_{n=1}^{\infty} \frac{\Phi^n - L_n}{n} \quad .$$

Using (2.3) gives:

$$\ln\Phi = \sum_{n=1}^{\infty} \frac{F_n\Phi + F_{n-1} - L_n}{n} \quad .$$

From [1], p.24:

$$L_n = F_{n-1} + F_{n+1}$$

So

$$\ln\Phi = \sum_{n=1}^{\infty} \frac{F_n\Phi + F_{n+1} - (F_{n-1} + F_{n+1})}{n}$$

or

$$\ln\Phi = \sum_{n=1}^{\infty} \frac{F_n\Phi - F_{n+1}}{n} \quad . \tag{2.6}$$

3. Hyperbolic Functions

These are closely related to Fibonacci series and the natural log of phi.

(i) Let $g = \ln\Phi$

From (1.3):

$$\tanh\left(\frac{3g}{2}\right) = \frac{1}{\Phi} \tag{3.1}$$

giving:

$$\cosh\left(\frac{3g}{2}\right) = \sqrt{\Phi} \tag{3.2}$$

and:

$$\sinh\left(\frac{3g}{2}\right) = \frac{1}{\sqrt{\Phi}} \quad . \tag{3.3}$$

(ii) Also, from [1], p.124:

$$\cosh g = \frac{\sqrt{5}}{2}$$
$$\sinh g = \frac{1}{2}$$

For any angle a, this produces:

$$\cosh(a+g) - \cosh(a-g) = \sinh(a) \tag{3.4}$$
$$\sinh(a+g) - \sinh(a-g) = \cosh(a) \tag{3.5}$$

(iii) From the identity

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

we have

$$\operatorname{arctanh}\left(\frac{a}{b}\right) = \frac{1}{2}\ln\left(\frac{b+a}{b-a}\right) \ .$$

Now b-a, a, b and a+b form a general Fibonacci series, where $G_0 = b-a$, $G_1 = a$, $G_2 = b$ and $G_3 = b+a$. So we have for $n \geq 2$:

$$G_n = G_{n-1} + G_{n-2}$$

and

$$\operatorname{arctanh}\left(\frac{G_{n+1}}{G_{n+2}}\right) = \frac{1}{2}\ln\left(\frac{G_{n+3}}{G_n}\right) \tag{3.6}$$

where $\mid G_{n+1} \mid < \mid G_{n+2} \mid$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Dover Publications, Inc., New York, 2008.

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