ON CANDIDO LIKE IDENTITIES

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Abstract. We give some identities for Horadam and (general) Lucas numbers analogous to the famous Candido identity for the Fibonacci numbers.

1. Introduction

Let $Q = (a + b + c)\,(b + c - a)\,(c + a - b)\,(a + b - c)$. We first recall the following two well-known alternate representations of Q.

$$\left(a^2 + b^2 + c^2\right)^2 - 2\left(a^4 + b^4 + c^4\right) = Q, \tag{1}$$

$$4\left[(b\,c)^2 + (c\,a)^2 + (a\,b)^2\right] - \left(a^2 + b^2 + c^2\right)^2 = Q. \tag{2}$$

Of course, they are related to the Heron's formula $16\,P^2 = Q$ for the area P of a triangle in terms of its sides a, b and c (see [11]).

An immediate consequence of (1) is that when real numbers a, b and c are three consecutive members of a binary recurrence sequence $\{s_n\}$ satisfying the recurrence $s_{n+2} = s_n + s_{n+1}$ then $Q = 0$. Since every Fibonacci number F_n is the sum of two previous Fibonacci numbers F_{n-1} and F_{n-2}, in particular, we conclude from (1) that

$$\left(F_n^2 + F_{n+1}^2 + F_{n+2}^2\right)^2 = 2\left(F_n^4 + F_{n+1}^4 + F_{n+2}^4\right). \tag{3}$$

This is one of the most beautiful among myriads of identities for Fibonacci numbers known as the Candido identity (quoted in [1] and first appeared in [2]). Let $S_3 = F_n^2 + F_{n+1}^2 + F_{n+2}^2$. In the same way, from (2), we also have the following (less known) identity.

$$S_3^2 = 4\left[(F_{n+1}\,F_{n+2})^2 + (F_{n+2}\,F_n)^2 + (F_n\,F_{n+1})^2\right]. \tag{4}$$

Both (3) and (4) hold also for Lucas numbers since they also satisfy the above recurrence.

The identity (3) was again considered by Melham [8] in 2004 when he decided to increase the numbers of terms. For example, he discovered the following identity.

$$3\left(F_n^4 + 4\,F_{n+1}^4 + 4\,F_{n+2}^4 + F_{n+3}^4\right) = 2\left(F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2\right)^2.$$

Let $S_4 = F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2$ and for integers i and j, let $P_{i,j}$ denote the product $F_{n+i}\,F_{n+j}$. Here we only mention the following three analogues of the identity (4) for S_4.

$$\tfrac{4}{9}\,S_4^2 = P_{0,1}^2 + 8\,P_{1,2}^2 + P_{2,3}^2 + 2\,P_{3,0}^2 + P_{0,2}^2 + P_{1,3}^2.$$

$$S_4^2 = 4\,P_{0,1}^2 + 4\,P_{1,2}^2 + 4\,P_{2,3}^2 + P_{3,0}^2 + 4\,P_{0,2}^2 + 4\,P_{1,3}^2.$$

$$\tfrac{16}{9}\,S_4^2 = 7\,P_{0,1}^2 + 8\,P_{1,2}^2 + 7\,P_{2,3}^2 + 2\,P_{3,0}^2 + 7\,P_{0,2}^2 + 7\,P_{1,3}^2.$$

In the rest of this paper we shall stick to the cases of at most three terms except in the section 10 when we return to four terms. Our main goal is to generalize the identities (3) and (4) in two directions.

Firstly, we replace Fibonacci numbers with more general Horadam numbers w_n. Moreover, we show that for (general) Lucas numbers x_n analogous identities are also true.

Secondly, instead of the indices n, $n+1$ and $n+2$ in (3) and (4), we consider the indices n, $n+m$ and $n+2\,m$ for any integers m and n. In this way, it follows that (3) and (4) are just single identities from a whole series of identities.

2. Horadam and (general) Lucas numbers

The sequence $F_0, F_1, F_2, F_3, \dots$ of Fibonacci numbers is defined recursively so that $F_0 = 0$ and $F_1 = 1$ and we require $F_{k+2} = F_{k+1} + F_k$ for every $k \geq 0$. Hence, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, etc.

Quite similar is also the sequence $L_0, L_1, L_2, L_3, \dots$ of Lucas numbers when we take $L_0 = 2$ and $L_1 = 1$ while the recursion $L_{k+2} = L_{k+1} + L_k$ for every $k \geq 0$ remains the same. Hence, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, $L_7 = 29$, $L_8 = 47$, $L_9 = 76$, etc.

The Fibonacci and Lucas numbers are special cases of the following sequences of numbers.

Let s, t, p and q be complex numbers such that $\Delta \neq 0$, where $\Delta = \sqrt{p^2 + 4\,q}$. Let $\alpha = \frac{p+\Delta}{2}$, $\beta = \frac{p-\Delta}{2}$, $\alpha_0 = t - s\,\beta$, $\beta_0 = t - s\,\alpha$,

$$w_n = w_n(s,t;p,q) = \frac{\alpha_0\,\alpha^n - \beta_0\,\beta^n}{\Delta},$$

$$x_n = x_n(s,\, t;\, p,\, q) = \alpha_0\,\alpha^n + \beta_0\,\beta^n,$$

$y_n = w_n(0,\, 1;\, p,\, q)$ and $z_n = x_n(0,\, 1;\, p,\, q)$.

The numbers w_n are known as Horadam numbers (see [4]) and the numbers x_n as the associated (general) Lucas numbers. It is obvious that $w_n(0,\, 1;\, 1,\, 1) = F_n$ and $x_n(0,\, 1;\, 1,\, 1) = L_n$ for every integer n. Let $\delta = -\alpha_0\,\beta_0 = q\,s^2 - t^2 + p\,s\,t$.

3. Candido identities for Horadam and Lucas numbers

Let m and n be integers. Let $\overline{q} = -q$, $\mu = \overline{q}^{\,m}$, $\nu = \overline{q}^{\,n}$, $n' = n+m$ and $n'' = n + 2\,m$. Let

$$a = \mu\,w_n,\;\; b = z_m\,w_{n'},\;\; c = w_{n''}. \tag{5}$$

$$a = \mu\,x_n,\;\; b = z_m\,x_{n'},\;\; c = x_{n''}. \tag{6}$$

$$a = \mu\,w_n,\;\; b = y_m\,x_{n'},\;\; c = w_{n''}. \tag{7}$$

$$a = \mu\,x_n,\;\; b = \Delta^2\,y_m\,w_{n'},\;\; c = x_{n''}. \tag{8}$$

Lemma 1. *(i) If either (5) or (6) is true, then $a + c = b$.*

(ii) If either (7) or (8) is true, then $a + b = c$.

Proof. Let (5) holds. Then

$$a + c = (-q)^m\,\tfrac{\alpha_0\,\alpha^n - \beta_0\,\beta^n}{\Delta} + \tfrac{\alpha_0\,\alpha^{n+2\,m} - \beta_0\,\beta^{n+2\,m}}{\Delta} = \tfrac{\alpha_0\,\alpha^n}{\Delta}\left((-q)^m + \alpha^{2\,m}\right) - \tfrac{\beta_0\,\beta^n}{\Delta}\left((-q)^m + \beta^{2\,m}\right) = \tfrac{\alpha_0\,\alpha^n}{\Delta}\left((\alpha\,\beta)^m + \alpha^{2\,m}\right) - \tfrac{\beta_0\,\beta^n}{\Delta}\left((\alpha\,\beta)^m + \beta^{2\,m}\right) = \tfrac{\alpha_0\,\alpha^{n+m}}{\Delta}\,(\beta^m + \alpha^m) - \tfrac{\beta_0\,\beta^{n+m}}{\Delta}\,(\alpha^m + \beta^m) = z_m\,\tfrac{\alpha_0\,\alpha^{n+m} - \beta_0\,\beta^{n+m}}{\Delta} = b.$$

The other parts of this lemma have similar proofs. □

For an integer k, let $N_k = a^k + b^k + c^k$, $M_k = (b\,c)^k + (c\,a)^k + (a\,b)^k$, $\sigma = N_1 = a + b + c$, $\tau = M_1 = b\,c + c\,a + a\,b$ and $\pi = a\,b\,c$. Using this notation, the identities (1) and (2) are much shorter

$$(1) \qquad N_2^2 - 2\,N_4 = Q, \qquad\qquad (2) \qquad 4\,M_2 - N_2^2 = Q.$$

Hence, Lemma 1 implies the following versions of Candido identity for Horadam and (general) Lucas numbers.

Theorem 3.1. *If either (5), (6), (7) or (8) hold, then* $N_2^2 = 2\,N_4 = 4\,M_2$.

Let the middle terms of the above four identities be denoted by $Q_1, \ldots, Q_4$. The following four identities show that $\frac{Q_1}{4}, \ldots, \frac{Q_4}{4}$ are also complete squares. Let $A = z_m^2 - \mu$ and $B = z_m^2 - 3\,\mu$.

$$\left(A\,w_{n'}^2 + \delta\,y_m^2\,\mu\,\nu\right)^2 = \left(A\,w_n\,w_{n''} + \delta\,y_{2\,m}^2\,\nu\right)^2 = \tfrac{Q_1}{4}. \tag{9}$$

$$\left(A\,x_{n'}^2 - \delta\,\Delta^2\,y_m^2\,\mu\,\nu\right)^2 = \left(A\,x_n\,x_{n''} - \delta\,\Delta^2\,y_{2\,m}^2\,\nu\right)^2 = \tfrac{Q_2}{4}. \tag{10}$$

$$\left(B\,w_{n'}^2 + 3\,\delta\,y_m^2\,\mu\,\nu\right)^2 = \left(B\,w_n\,w_{n''} + \delta\,y_{2\,m}^2\,\nu\right)^2 = \tfrac{Q_3}{4}. \tag{11}$$

$$\left(B\,x_{n'}^2 - 3\,\delta\,\Delta^2\,y_m^2\,\mu\,\nu\right)^2 = \left(B\,x_n\,x_{n''} - \delta\,\Delta^2\,y_{2\,m}^2\,\nu\right)^2 = \tfrac{Q_4}{4}. \tag{12}$$

For an integer j, let κ_j be $z_{2\,m} + j\,\mu$. For the numbers y_n and z_n the above four identities have the following simpler form.

$$2\,(\kappa_1\,y_{2\,n'} + \kappa_4\,\mu\,\nu)^2 = \mu^4\,y_n^4 + z_m^4\,y_{n'}^4 + y_{n''}^4. \tag{13}$$

$$2\,(\kappa_1\,z_{2\,n'} + \kappa_4\,\mu\,\nu)^2 = \mu^4\,z_n^4 + z_m^4\,z_{n'}^4 + z_{n''}^4. \tag{14}$$

$$2\,(\kappa_{-1}\,z_{2\,n'} + \kappa_{-4}\,\mu\,\nu)^2 = \Delta^4\left(\mu^4\,y_n^4 + y_m^4\,z_{n'}^4 + y_{n''}^4\right). \tag{15}$$

$$2\,(\kappa_{-1}\,z_{2\,n'} - \kappa_{-4}\,\mu\,\nu)^2 = \mu^4\,z_n^4 + \Delta^8\,y_m^4\,y_{n'}^4 + z_{n''}^4. \tag{16}$$

The following are also corollaries of Lemma 1.

Corollary 1. *(i) If either (5) or (6) holds, then*

$$\sigma^2 - 4\,a\,b = 2\left(N_2 - 2\,a^2\right).$$

(ii) If either (7) or (8) holds, then

$$\sigma^2 - 4\,a\,b = 2\,N_2.$$

Proof. This follows from Lemma 1 and the algebraic identities

$$\sigma^2 - 4\,a\,b - 2\left(N_2 - 2\,a^2\right) = (3\,a + b - c)(a - b + c)$$

and

$$4\,a\,b + 2\,N_2 - \sigma^2 = (a + b - c)^2.$$

□

Corollary 2. *If either (5), (6), (7) or (8) holds, then*

$$N_1^3 = 4\,N_3 + 12\,\pi.$$

Proof. This follows from Lemma 1 and the algebraic identity

$$N_1^3 - 4\,N_3 - 12\,\pi = \frac{3\,Q}{\sigma}.$$

□

Since $N_1^6 - 16\,N_6 + 240\,\pi^2 - 32\,M_1^3$ contains $\frac{Q}{\sigma}$ as a factor, we can add the identity

$$N_1^6 + 240\,\pi^2 = 16\left(N_6 + 2\,M_1^3\right)$$

to Corollary 2.

4. Candido like identity for products $w_n\, x_n$

The following is the version of Candido identity for the products $r_n = w_n\, x_n$ of Horadam and (general) Lucas numbers. Let $A = 2\,\delta\, y_{2\,m}\, q^{n'}$.

Theorem 4.1. *If either $a = \mu^2\, r_n,\ b = z_m^2\, r_{n'},\ c = r_{n''}$ or $a = \mu^2\, r_n,\ b = \Delta^2\, y_m^2\, r_{n'},\ c = r_{n''}$, then $N_1^2 + A^2 = 2\, N_2$ and $N_1^2 - A^2 = 4\, M_1$.*

Proof. Let us compute the difference $2\, N_2 - N_1^2$ using

$$w_n = \tfrac{1}{\Delta}\left(\alpha_0\, \alpha^n - \tfrac{\beta_0\, \nu}{\alpha^n}\right) \quad \text{and} \quad x_n = \alpha_0\, \alpha^n + \tfrac{\beta_0\, \nu}{\alpha^n}. \tag{17}$$

After simplification of a long expression collecting terms that contain powers of $\overline{q}$, we discover that

$$2\, N_2 - N_1^2 = \tfrac{4}{\Delta^2}\, {\alpha_0}^2 {\beta_0}^2 (\mu\, \nu)^2 \left(\alpha^m - \tfrac{\mu}{\alpha^m}\right)^2 \left(\alpha^m + \tfrac{\mu}{\alpha^m}\right)^2 = A^2.$$

Similarly we get $2\, N_2 - 4\, M_1 = 2\, A^2$ that implies $N_1^2 - A^2 = 4\, M_1$. □

5. Candido like identities for products $w_n\, w_{n+5\,m}, \dots$

In this section we shall consider analogues of Candido identity for products $w_n\, w_{n+5\,m}$, $w_{n+m}\, w_{n+4\,m}, \dots$.

Theorem 5.1. *If either*

(i) $a = w_n\, w_{n+5\,m},\ b = w_{n+m}\, w_{n+4\,m},\ c = w_{n+2\,m}\, w_{n+3\,m}$,
$B = \delta\, q^n\, y_m\, y_{2\,m}$ and $D = z_{2\,m}^2 + \mu\, z_{2\,m} + \mu^2$,

(ii) $a = x_n\, x_{n+5\,m},\ b = x_{n+m}\, x_{n+4\,m},\ c = x_{n+2\,m}\, x_{n+3\,m}$,
$B = \Delta^2\, \delta\, q^n\, y_m\, y_{2\,m}$ and $D = z_{2\,m}^2 + \mu\, z_{2\,m} + \mu^2$,

(iii) $a = w_n\, w_{n+6\,m},\ b = w_{n+m}\, w_{n+5\,m},\ c = w_{n+2\,m}\, w_{n+4\,m}$,
$B = \delta\, q^n\, y_m$ and $D = y_m^2\, z_{4\,m}^2 + 3\,\mu\, y_{3\,m}\, y_{5\,m}$,

or

(iv) $a = x_n\, x_{n+6\,m},\ b = x_{n+m}\, x_{n+5\,m},\ c = x_{n+2\,m}\, x_{n+4\,m}$,
$B = \Delta^2\, \delta\, q^n\, y_m$ and $D = y_m^2\, z_{4\,m}^2 + 3\,\mu\, y_{3\,m}\, y_{5\,m}$,

then $N_1^2 + 2\, B^2\, D = 3\, N_2$.

Proof. In order to prove (i), we again use (17) and compute the difference $3\, N_2 - N_1^2$. This time we get

$$3\, N_2 - N_1^2 = \tfrac{2}{\Delta^4}\left[\left(\alpha^{2\,m} + \tfrac{\mu^2}{\alpha^{2\,m}}\right)^2 + \mu\left(\alpha^{2\,m} + \tfrac{\mu^2}{\alpha^{2\,m}}\right) + \mu^2\right] \left(\alpha^m - \tfrac{\mu}{\alpha^m}\right)^4 \left(\alpha^m + \tfrac{\mu}{\alpha^m}\right)^2 \delta^2\, \nu^2 = 2\, B^2\, D.$$

The other parts have similar proofs. These computations and all others in this paper have been checked by a computer in Maple V (version 9.5). □

6. Candido like identity for products with two terms

For products of two and three Horadam and (general) Lucas numbers there are analogues of Candido identity with only two terms.

Theorem 6.1. *If either*

(i) $a = w_n\, w_{n+3\,m},\ b = w_{n+m}\, w_{n+2\,m}$ and $B = \delta\, q^n\, y_m\, y_{2\,m}$,

(ii) $a = x_n\, x_{n+3\,m},\ b = x_{n+m}\, x_{n+2\,m}$ and $B = \Delta^2\, \delta\, q^n\, y_m\, y_{2\,m}$,

(iii) $a = w_n\, w_{n+4\,m}\, w_{n+5\,m},\ b = w_{n+m}\, w_{n+2\,m}\, w_{n+6\,m}$ and $B = \delta\, q^n\, y_m\, y_{2\,m}\, y_{3\,m}\, x_{n+3\,m}$,

or

(iv) $a = x_n\, x_{n+4\,m}\, x_{n+5\,m}$, $b = x_{n+m}\, x_{n+2\,m}\, x_{n+6\,m}$ *and* $B = \Delta^4\,\delta\, q^n\, y_m\; y_{2\,m}\, y_{3\,m}\, w_{n+3\,m}$, *then*

$$(a+b)^2 + B^2 = 2\left(a^2+b^2\right).$$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. We verify that the difference $2\left(a^2+b^2\right) - (a+b)^2$ is precisely the square of B. □

7. Candido like identity for N_2^3 and $\left(N_2 - 4\,b^2\right)^3$

The algebraic identities

$$N_2^3 + 12\,\pi^2 - 4\,N_6 = 3\,Q\,N_2$$

$$\left(N_2 - 4\,b^2\right)^3 + 60\,\pi^2 - 4\left(N_6 - 4\,b^6\right) = 3\,Q\left(N_2 + 4\,b^2\right)$$

imply the following identities for the cubes of N_2 and $N_2 - 4\,b^2$.

Theorem 7.1. *If either (5), (6), (7) or (8) hold, then*

$$N_2^3 + 12\,\pi^2 = 4\,N_6$$

and

$$\left(N_2 - 4\,b^2\right)^3 + 60\,\pi^2 = 4\left(N_6 - 4\,b^6\right).$$

8. Candido like identity for N_2^k $(k = 4,\,5,\,6)$

The algebraic identity $N_2^4 - 4\,N_8 - 8\,M_4 = Q(3\,N_4 + 2\,M_2)$ implies that $N_2^4 = 4\,N_8 + 8\,M_4$ hold when either (5), (6), (7) or (8) is true. We also have the following analogous result.

Theorem 8.1. *(i) If either (5) or (6) hold, then*

$$N_2^4 + 64\left(b^2 - c\,a\right)\pi^2 = 8\,N_8.$$

(ii) If either (7) or (8) hold, then

$$N_2^4 + 64\left(b^2 + c\,a\right)\pi^2 = 8\,N_8.$$

Proof. This follows from Lemma 1 and the fact that

$$N_2^4 + 64\left(b^2 - c\,a\right)\pi^2 - 8\,N_8 \tag{18}$$

and

$$N_2^4 + 64\left(b^2 + c\,a\right)\pi^2 - 8\,N_8 \tag{19}$$

contain $a - b + c$ and $a + b - c$ as factors, respectively. □

Analogous results hold also for the fifth and the sixth powers of N_2. In this cases instead of (18) and (19) we have

$$N_2^5 + 240\left(b^2 - c\,a\right)^2\pi^2 - 16\,N_{10}, \tag{18_5}$$

$$N_2^5 + 240\left(b^2 + c\,a\right)^2\pi^2 - 16\,N_{10} \tag{19_5}$$

and

$$N_2^6 + 96\left(8\,c^6 + 24\,\pi\,c^3 + 25\,\pi^2 + 8\,a^3\,b^3\right)\pi^2 - 32\,N_{12}, \tag{18_6}$$

$$N_2^6 + 96\left(8\,c^6 - 24\,\pi\,c^3 + 25\,\pi^2 - 8\,a^3\,b^3\right)\pi^2 - 32\,N_{12}. \tag{19_6}$$

Their proofs are similar to the above proof for the fourth power of N_2.

9. More Candido like identities for products

Theorem 9.1. *If either*

(i) $A = \mu$, $B = 2\, z_{2\,m} + \mu$ *and* $a = w_n\, w_{n+5\,m}$, $b = w_{n+m}\, w_{n+4\,m}$, $c = w_{n+2\,m}\, w_{n+3\,m}$,

or

(ii) $A = \mu\, y_{3\,m}$, $B = y_m \left[2\, z_m^4 - \ \mu \left(5\, z_m^2 - \mu\right)\right]$ *and* $a = w_n\, w_{n+6\,m}$, $b = w_{n+m}\, w_{n+5\,m}$, $c = w_{n+2\,m}\, w_{n+4\,m}$,

then

$$A\left(A\, a^2 - B\, b^2 + B\, c^2\right) = \left(A\, a - B\, b + B\, c\right)^2.$$

The above theorem holds also when in a, b and c we replace Horadam with (general) Lucas numbers.

Proof of (i) for the (general) Lucas numbers. We shall find expressions $V = A\, a - B\, b + B\, c$ and $U = A\left(A\, a^2 - B\, b^2 + B\, c^2\right)$ and check that $U = V^2$.

In order to make long expressions shorter, we shall use the following notation. Let $u = \alpha_0$ and $v = \beta_0$. For an integer k, let $\langle k\rangle = \alpha^{k\,m}$ and $[k] = \alpha^{k\,n}$. Let R and P denote polynomials

$$\mu^4 - 2\langle 2\rangle\mu^3 + \langle 4\rangle\mu^2 - 2\langle 6\rangle\mu + \langle 8\rangle,$$

$$\mu^{10} - 2\langle 2\rangle\mu^9 - \langle 4\rangle\mu^8 + 8\langle 10\rangle\mu^5 + \langle 12\rangle\mu^4 - \langle 16\rangle\mu^2 - 2\langle 18\rangle\mu + \langle 20\rangle.$$

Using the representations (17), one can show that U is

$$k_{u^4}\, u^4 + k_{u^3\,v}\, u^3\, v + k_{u^2\,v^2}\, u^2\, v^2 + k_{u\,v^3}\, u\, v^3 + k_{v^4}\, v^4,$$

where $k_{u^4} = \langle 10\rangle\, [4]\, \mu^2$, $k_{u^3\,v} = -2[2]\, \mu^2\, \nu(\mu + \langle 2\rangle)R$, $k_{u^2\,v^2} = \langle -10\rangle\mu^2\ \nu^2\, P$, $k_{u\,v^3} = -2\langle -10\rangle\, [-2]\mu^7\, \nu^3(\mu + \langle 2\rangle)R$ and $k_{v^4} = \langle -10\rangle\, [-4]\, \mu^{12}\, \nu^4$. On the other hand, V is equal

$$\langle 5\rangle\, [2]\, \mu\, u^2 - [-5]\, \mu\, \nu(\mu + \langle 2\rangle)R\, u\, v + \langle -5\rangle\, [-2]\, \mu^6\, \nu^2\, v^2.$$

Comparing coefficients of corresponding powers of u and v, we see that $U = V^2$.

The other cases of Theorem 7 have similar proofs. ☐

10. Four terms Candido like identity for products

Let $A = \mu^2\, y_{3\,m}\, y_{4\,m}$, $B = \mu\, y_m \left(2\, y_{8\,m} + 4\, \mu\, y_{6\,m} + 5\, \mu^2\, y_{4\,m}\right)$,

$$C = y_m \left(y_{10\,m} + 4\, \mu\, y_{8\,m} + 13\, \mu^2\, y_{6\,m} + 19\, \mu^3\, y_{4\,m} + 16\, \mu^4\, y_{2\,m}\right)$$

and $D = y_m \left(y_{10\,m} + 6\, \mu\, y_{8\,m} + 2\, \mu^2\, y_{6\,m} + 9\, \mu^3\, y_{4\,m} + \mu^4\, y_{2\,m}\right)$.

Theorem 10.1. *If* $a = w_n\, w_{n+7\,m}$, $b = w_{n+m}\, w_{n+6\,m}$, $c = w_{n+2\,m}\, w_{n+5\,m}$ *and* $d = w_{n+3\,m}\, w_{n+4\,m}$, *then*

$$A\, a^2 + B\, b^2 + C\, c^2 - D\, d^2 = A\left(a + b + c + d\right)^2.$$

The above theorem holds also when in a, b, c and d we replace Horadam with (general) Lucas numbers.

Proof. As in the above proof of Theorem 7, we shall find

$$U = A\, a^2 + B\, b^2 + C\, c^2 - D\, d^2 \quad \text{and} \quad V = a + b + c + d$$

and check that $U = A\, V^2$.

These tasks are straightforward and technically complicated so that we leave them to the dedicated readers as a challenge. ☐

11. Summary and Future Work

A brief summary of this paper is as follows. We first recalled definitions of Horadam (w_n) and (general) Lucas numbers (x_n) together with their important special cases (y_n) and (z_n). After the proof of a simple Lemma 1, we established Candido identities for Horadam and (general) Lucas numbers (Theorem 1) and several of their variations (identities (9)–(16)) and two corollaries. For products $w_n\,x_n$ the analogue of Candido identity required adding or subtracting a certain complete square (Theorem 2). In Theorem 3 we have given four cases of Candido like identities for three products of either two Horadam or two (general) Lucas numbers with equal sums of indices. The next Theorem 4 considered only two such products but both for products of two and three terms. In Theorems 5 and 6 the third and the fourth, fifth and sixth powers of the sum of squares have given us nice looking formulas. Finally, in Theorems 7 and 8, we returned to three and four products of two terms from either Horadam or (general) Lucas numbers.

In the future we would like to discover and prove Candido like identities with more terms (similar to the Melham identity in the Introduction) and for more kinds of numbers (tribonacci, polygonal,...).

12. Acknowledgement

We are grateful to a referee who has made some useful improvements of this paper and pointed out that a recent article [5] also considers the Candido identity and develops a method that can make a search for this kind of identities somewhat simpler.

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