ON THE RATE OF p-ADIC CONVERGENCE OF ALTERNATING SUMS OF POWERS OF BINOMIAL COEFFICIENTS

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Abstract. Let $m \geq 1$ be an integer and p be an odd prime. We study alternating sums and lacunary sums of mth powers of binomial coefficients from the point of view of arithmetic properties. We develop new congruences and prove the p-adic convergence of some subsequences and that in every step we gain at least three more p-adic digits of the limit. These gains are exact under some explicitly given condition. The main tools are congruential and divisibility properties of the binomial coefficients.

1. Introduction

In this paper $\mathbb{N}$ and $\mathbb{Z}^+$ denote the nonnegative and positive numbers and p denotes an odd prime. We study certain sums of binomial coefficients. For $m, n \in \mathbb{Z}^+$, we introduced the Franel-like numbers

$$F(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m$$

in [8]. In this papers we discuss its alternating version

$$G(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m (-1)^k. \tag{1.1}$$

The numbers $F(n,3)$, $n \in \mathbb{Z}^+$, are called Franel numbers and are considered to be the generalization of the numbers $F(n,1) = 2^n$ and $F(n,2) = \binom{2n}{n}$. It is known that there is no closed form for $F(n,3)$, cf. [10, Theorem 8.8.1, p160] while $G(n,3)$ does have one.

Let $r \in \mathbb{Z}^+$ and split the summation in (1.1) into lacunary sums or subsums by

$$l^*(n,r,m,i) = \sum_{j \equiv i \pmod{r}}^{n} \binom{n}{j}^m (-1)^j \tag{1.2}$$

with integers $0 \leq i \leq r-1$, in a similar fashion to the definition of

$$l(n,r,m,i) = \sum_{j \equiv i \pmod{r}}^{n} \binom{n}{j}^m$$

in [8]. Therefore, we have

$$G(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m (-1)^k = \sum_{i=0}^{r-1} \sum_{j \equiv i \pmod{r}} \binom{n}{j}^m (-1)^j = \sum_{i=0}^{r-1} l^*(n,r,m,i).$$

Our main focus is on particular subsequences of $\{G(n,m)\}_{n \geq 1}$. We consider

$$G(ap^n, m) = \sum_{k=0}^{ap^n} \binom{ap^n}{k}^m (-1)^k = \sum_{i=0}^{p-1} l^*(ap^n, p, m, i) \tag{1.3}$$

with $a \in \mathbb{Z}^+$.

For an integer n, the p-adic order $\nu_p(n)$ of n is the highest power of prime p which divides n. We set $\nu_p(0) = \infty$ and $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$ if both m and n are integers.

We develop congruences for $G(ap^n, m)$ via congruences for the lacunary alternating mth power sums of binary coefficients in Theorems 2.6 and 2.12, in order to establish the convergence of $G(ap^n, m)$ and its rate, $\nu_p\big(G(ap^n, m) - G(ap^{n-1}, m)\big)$, as $n \to \infty$.

In Section 2 we present the main results. Section 3 is devoted to preparation for the proofs that are included in Section 4. Our main results are Theorems 2.6 and 2.12. Further results are presented in Theorems 2.2 and 2.8.

2. Main results

In [6] and [11] we studied the non-alternating lacunary sums of binomial coefficients and derived

Theorem 2.1 (Theorem 2, [6])**.** *For any odd prime p and $q \in \mathbb{Z}^+$, and $i, 0 \le i \le p^q - 1$, we have that $\nu_p(l(n, p^q, 1, i)) = \nu_p\left(\binom{n}{i}\right)$.*

In its proof, we also derived a congruence for $l(ap^n + s, p, 1, i)$, for any odd prime p, $1 \le i \le p - 1$, integer $s, 0 \le s \le i - 1$, and $a \in \mathbb{Z}^+$:

$$l(ap^n + s, p, 1, i) \equiv -a2^{a-1}s!(i!(p+s-i)!)^{p-2}p^n \pmod{p^{n+1}}. \tag{2.1}$$

The proof uses Theorem 3.5 by Anton, Stickelberger, and Hensel (see identity (2) in [5]) to determine $\frac{1}{p^q}\binom{N}{M} \pmod p$ with p^q being the exact power of p dividing $\binom{N}{M}$. For a general odd exponent m, after taking the mth powers, a slight modification of the argument in the proof of Theorem 2.1, which heavily relies on Theorem 3.5, results in

$$l(ap^n, p, m, i) \equiv a^m F(a-1, m) \left(\frac{1}{p}\binom{p}{i}\right)^m p^{mn} \pmod{p^{mn+1}}$$

for $1 \le i \le p - 1$. (Note that we discuss Franel-like numbers with even exponents in [9] and obtain higher p-adic orders.)

In the case of the alternative Franel-like sequences $G(ap^n, m)$ we drop the assumption on the parity of m and note that a similar derivation does not result in a congruence that reveals the p-adic order but only a lower bound as given in the next

Theorem 2.2. *For any odd prime p, exponent $m \ge 1$, and $a, n \in \mathbb{Z}^+$ such that $(a, p) = 1$, we have that*

$$\nu_p\left(\sum_{\substack{k=0\\ p\nmid k}}^{ap^n} \binom{ap^n}{k}^m (-1)^k\right) = \nu_p\left(G(ap^n, m) - \sum_{\substack{k=0\\ p\mid k}}^{ap^n} \binom{ap^n}{k}^m (-1)^k\right) \ge mn.$$

Remark 2.3. *The above lower bound will suffice in the proof of Theorem 2.12; although, the p-adic orders seem larger. For instance, in the calculations as compared to those in the derivation of* (2.1)*, we have an extra factor $(-1)^j = (-1)^i \prod_r (-1)^{t_r}$ where $j = i + pt$ with $t = (\ldots, t_2, t_1, t_0)_p$ in base p. The second power of -1 can be redistributed among the factors in the proof. Note that, e.g., $t_0, 0 \le t_0 \le p - 1$, changes its parity in every step as j increases.*

We get $\sum_{t_0=0}^{p-1}\binom{p-1}{t_0}^m(-1)^{t_0} \equiv \sum_{t_0=0}^{p-1}((-1)^{t_0})^m(-1)^{t_0} = \sum_{t_0=0}^{p-1}1 \equiv 0 \pmod p$ *if* m *is odd since* $\binom{p-1}{t_0} \equiv (-1)^{t_0} \pmod p$ *for* $0 \le t_0 \le p-1$*. It follows that*

$$l^*(ap^n,p,m,i) = \sum_{j\equiv i \pmod p}^{ap^n} \binom{ap^n}{j}^m (-1)^j \equiv 0 \pmod{p^{mn+1}};$$

and thus, the p*-adic order is at least* $mn+1$*.*

We introduce the following assumption under which the exact p-adic order can be determined.

Assumption 2.4. *For the odd prime* p*,* $m \ge 3$ *and* $a \ge 2$ *with* $(a,p)=1$*, we assume that*

$$\nu_p\left(\sum_{j=1}^{a-1} aj(a-j)\binom{a}{j}^m(-1)^j\right) < m-2.$$

Remark 2.5. *We write*

$$A = \sum_{j=1}^{a-1} aj(a-j)\binom{a}{j}^m(-1)^j \tag{2.2}$$

and observe that A *can be rewritten as*

$$a^2(a-1)\sum_{j=1}^{a-1}\binom{a-2}{j-1}\binom{a}{j}^{m-1}(-1)^j.$$

It follows that if $a \equiv 1 \pmod{p^M}$ *with* $M \ge m-2$ *then Assumption 2.4 cannot hold.*

One of our main results is presented in the following theorem, which guarantees that the p-adic limit $\lim_{n\to\infty} G(ap^n,m)$ exists.

Theorem 2.6. *For any odd prime* p*,* $a \in \mathbb{N}$*,* $m \in \mathbb{Z}^+$*, and* $n \ge 2$*, we have*

$$G(ap^n,m) \equiv G(ap^{n-1},m) \pmod{p^n}.$$

Now we turn to the rate of p-adic convergence in Theorem 2.6, which is determined in Theorem 2.12. Let $B_n, n \ge 0$, be the nth Bernoulli number. We set

$$C_p = \begin{cases} 45, & \text{if } p=3, \\ -p^3B_{p-3}/3, & \text{if } p \ge 5, \end{cases}$$

which plays an important role in the Jacobstahl–Kazandzidis congruence, cf. Theorem 3.5.

Remark 2.7. *It is well known that* $\nu_p(B_n) \ge -1$ *by the von Staudt–Clausen theorem. If* $\nu_p(B_{p-3}) \ge 1$ *then the prime* p *is called a Wolstenholme prime, e.g.,* $p = 16843$ *and* 2124679*. Note that* $\nu_3(C_3) = 2$*. For a prime* $p \ge 5$*, we have* $\nu_p(C_p) \ge 2$ *while* $\nu_p(C_p) \ge 3$ *if* $\nu_p(B_{p-3}) \ge 0$*, and* $\nu_p(C_p) \le 3$ *exactly if* p *is not a Wolstenholme prime.*

We define

$$e_n(a,p,m) = \nu_p\left(\sum_{k=0}^{ap^{n-1}}\left(\binom{ap^n}{kp}^m(-1)^{kp} - \binom{ap^{n-1}}{k}^m(-1)^k\right)\right)$$

and obtain

Theorem 2.8. *For any odd prime p which is not a Wolstenholme prime, a even with $(a,p)=1$, integers $m \geq 4$ and $n \geq 2$, and under Assumption 2.4, we have that*

$$\begin{aligned}&\sum_{k=0}^{ap^{n-1}} \left(\binom{ap^n}{kp}^m (-1)^{kp} - \binom{ap^{n-1}}{k}^m (-1)^k \right) \\ &\equiv p^{3n-3} amC_p \sum_{j=1}^{a-1} j(a-j) \binom{a}{j}^m (-1)^j \pmod{p^{1+e_n(a,p,m)}},\end{aligned}$$

which already guarantees that $e_n(a,p,m) \geq 3n-1$.

Remark 2.9. *Note that for $m \geq 0$ and odd integer $a \geq 1$ the left hand side sum is simply 0 by*

Lemma 2.10 (Remark 2.2, [9])**.** *For any odd prime p, integer $n \geq 1$, exponent $m \geq 0$, and odd integer $a \geq 1$, we have*

$$\sum_{k=0}^{ap^n} \binom{ap^n}{k}^m (-1)^k = \sum_{\substack{k=0 \\ p|k}}^{ap^n} \binom{ap^n}{k}^m (-1)^k = 0.$$

For $m = 2$ and 3 we need

Lemma 2.11. *For $n \geq 0$ even and $m = 2$ or 3 we have*

$$\sum_{k=0}^{n} \binom{n}{k}^m (-1)^k = (-1)^{n/2} \binom{\frac{mn}{2}}{\frac{n}{2} \cdots \frac{n}{2}}$$

where the last factor is a multinomial coefficient.

Finally, we state our main theorem on the rate of p-adic convergence.

Theorem 2.12. *Assume that p is an odd prime, $a, m \in \mathbb{Z}^+$ with $(a,p) = 1$, and $n \geq 2$. We rewrite the difference*

$$\begin{aligned}&G(ap^n, m) - G(ap^{n-1}, m) \qquad (2.3)\\ &= \sum_{\substack{k=0 \\ p\nmid k}}^{ap^n} \binom{ap^n}{k}^m (-1)^k + \sum_{k=0}^{ap^{n-1}} \left(\binom{ap^n}{kp}^m (-1)^{kp} - \binom{ap^{n-1}}{k}^m (-1)^k \right).\end{aligned}$$

If a is odd or $m = 1$ then $G(ap^n, m) = 0$. For $a \geq 2$ even we have the following cases.

If $m = 2$ then we obtain that

$$G(ap^n, 2) - G(ap^{n-1}, 2) \equiv (-1)^{a/2} \frac{a^3}{4} C_p \binom{a}{\frac{a}{2}} p^{3n-3} \pmod{p^{3n-2+\nu_p(C_p)+\nu_p(\binom{a}{a/2})}},$$

and $\nu_p(G(ap^n, 2) - G(ap^{n-1}, 2)) = 3n - 3 + \nu_p(C_p) + \nu_p(\binom{a}{a/2})$ if p is not a Wolstenholme prime.

If $m = 3$ then we obtain that

$$G(ap^n, 3) - G(ap^{n-1}, 3)$$
$$\equiv (-1)^{a/2} a^3 C_p \binom{a}{\frac{a}{2}} \binom{\frac{3a}{2}}{\frac{a}{2}} p^{3n-3} \pmod{p^{3n-2+\nu_p(C_p)+\nu_p(\binom{a}{a/2}\binom{3a/2}{a/2})}},$$

and $\nu_p(G(ap^n, 3) - G(ap^{n-1}, 3)) = 3n - 3 + \nu_p(C_p) + \nu_p(\binom{a}{a/2}\binom{3a/2}{a/2})$ if p is not a Wolstenholme prime.

If $a \geq 2$, $m \geq 4$, and Assumption 2.4 is satisfied then the first sum's p-adic order is at least mn and it is bigger than that of the second sum, which is at most $3n - 3 + \nu_p(mC_p) + m - 3 \leq 3n + \nu_p(m) + m - 3$ if p is not a Wolstenholme prime. We have that

$$G(ap^n, m) - G(ap^{n-1}, m) \equiv p^{3n-3} amC_p \sum_{j=1}^{a-1} j(a-j) \binom{a}{j}^m (-1)^j \pmod{p^{1+e_n(a,p,m)}}.$$

Remark 2.13. *If $m = 1$ then for any $1 \leq i \leq p-1$, according to an application of Theorem 3 in* [4], *we obtain*

$$D = D_i = \sum_{\substack{k=0 \\ k \equiv i \pmod p}}^{ap^n} \binom{ap^n}{k} (-1)^k \equiv (-1)^{\frac{ap^n}{p-1}-1} p^{\frac{ap^n}{p-1}-1} \pmod{p^{\frac{ap^n}{p-1}}}$$

if $p-1|a$ (and thus, a is even), and $\nu_p(D) \geq \lfloor \frac{ap^n}{p-1} \rfloor$, otherwise. In these cases D has a surprisingly high p-adic order.

Since $G(ap^n, 1) = 0$ for $n \geq 0$, then after summation for $i = 1, 2, \ldots, p-1$, and by (2.3) *we get that*

$$\sum_{\substack{k=0 \\ p \nmid k}}^{ap^n} \binom{ap^n}{k} (-1)^k = - \sum_{k=0}^{ap^{n-1}} \left(\binom{ap^n}{kp} (-1)^k - \binom{ap^{n-1}}{k} (-1)^k \right)$$
$$\equiv (-1)^{\frac{ap^n}{p-1}} p^{\frac{ap^n}{p-1}-1} \pmod{p^{\frac{ap^n}{p-1}}}$$

if $p-1|a$.

3. Preparation

The following four theorems comprise the basic facts regarding divisibility and congruence properties of the binomial coefficients. We assume that $0 \leq k \leq n$.

Theorem 3.1 (Kummer, 1852)**.** *The power of a prime p that divides the binomial coefficient $\binom{n}{k}$ is given by the number of carries when we add k and $n-k$ in base p.*

From now on M and N will denote integers such that $0 \leq M \leq N$.

Theorem 3.2 (Lucas, 1877)**.** *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d$ and $M = m_0 + m_1 p + \cdots + m_d p^d$ with $0 \leq n_i, m_i \leq p-1$ for each i, be the base p representations of N and M, respectively. Then*

$$\binom{N}{M} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \cdots \binom{n_d}{m_d} \pmod{p}.$$

Lucas' theorem is further extended in

Theorem 3.3 (Anton, 1869, Stickelberger, 1890, Hensel, 1902)**.** *Let* $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d, M = m_0 + m_1 p + \cdots + m_d p^d$ *and* $R = N - M = r_0 + r_1 p + \cdots + r_d p^d$ *with* $0 \le n_i, m_i, r_i \le p-1$ *for each* i*, be the base* p *representations of* $N, M,$ *and* $R = N - M$*, respectively. Then with* $q = \nu_p\big(\binom{N}{M}\big)$*,*

$$(-1)^q \frac{1}{p^q}\binom{N}{M} \equiv \left(\frac{n_0!}{m_0! r_0!}\right)\left(\frac{n_1!}{m_1! r_1!}\right)\cdots\left(\frac{n_d!}{m_d! r_d!}\right) \pmod{p}.$$

Davis and Webb (1990) and Granville (1995) have independently generalized Lucas' theorem and its extension Theorem 3.3 to prime power moduli. Their theorem implies the following congruence in

Corollary 3.4. [1, Corollary 2.35]

$$\binom{ap^r}{bp^s} \equiv \binom{ap^{r-1}}{bp^{s-1}} \pmod{p^{\epsilon_0+q}} \tag{3.1}$$

with $a, b, q, r, s \in \mathbb{Z}^+$*,* $(a,p) = (b,p) = 1$*,* $r \ge q, s \ge q$*,* $\epsilon_0 \le \nu_p\big(\binom{ap^r}{bp^s}\big)$ *and* $r, s \ge 1$*.*

It follows that for $0 \le k \le ap^{n-1}$, we have that $\binom{ap^n}{kp} \equiv \binom{ap^{n-1}}{k} \pmod{p}$. Note that it can be further improved by an application of Theorem 3.5 although (3.1) suffices in the proof of Theorem 2.6.

We also use

Theorem 3.5 (Jacobstahl–Kazandzidis congruence, cf. Corollary 11.6.22, [2])**.** *Let* M *and* N *such that* $0 \le M \le N$ *and* p *prime. We have*

$$\binom{pN}{pM} \equiv \begin{cases} \left(1 - \frac{B_{p-3}}{3} p^3 NM(N-M)\right)\binom{N}{M} & \left(\bmod\ p^4 NM(N-M)\binom{N}{M}\right),\ \text{if } p \ge 5, \\ (1 + 45NM(N-M))\binom{N}{M} & \left(\bmod\ p^4 NM(N-M)\binom{N}{M}\right),\ \text{if } p = 3, \\ (-1)^{M(N-M)} P(N,M)\binom{N}{M} & \left(\bmod\ p^4 NM(N-M)\binom{N}{M}\right),\ \text{if } p = 2, \end{cases}$$

where $P(N,M) = 1 + 6NM(N-M) - 4NM(N-M)(N^2 - NM + M^2) + 2(NM(N-M))^2$ *and* B_n *stands for the* n*th Bernoulli number.*

4. Proofs

Proof of Theorem 2.6. By congruence (3.1) and $n \ge 2$, we obtain that for all $a, s \in \mathbb{Z}^+$, $n \ge s$, and $(j,p) = 1$

$$\binom{ap^n}{jp^s}(-1)^{jp^s} \equiv \binom{ap^{n-1}}{jp^{s-1}}(-1)^{jp^{s-1}} \pmod{p^n}$$

since we can set $\epsilon_0 = \nu_p(\binom{ap^n}{jp^s}) \geq n-s$ and $q = s$. Summing up for $s = 1, 2, \ldots, n$ and applying Theorem 2.2, it follows that

$$\begin{aligned} G(ap^n, m) \equiv \sum_{j=0}^{ap^{n-1}} \binom{ap^n}{jp}^m (-1)^{jp} &= \sum_{s=1}^{n} \left(\sum_{\substack{j=0 \\ p\nmid j}}^{ap^{n-s}} \binom{ap^n}{jp^s}^m (-1)^{jp^s} \right) \\ &\equiv \sum_{s=1}^{n} \left(\sum_{\substack{j=0 \\ p\nmid j}}^{ap^{n-s}} \binom{ap^{n-1}}{jp^{s-1}}^m (-1)^{jp^{s-1}} \right) \\ &= \sum_{j=0}^{ap^{n-2}} \binom{ap^{n-1}}{jp}^m (-1)^{jp} \\ &\equiv G(ap^{n-1}, m) \pmod{p^n}. \end{aligned}$$

□

Proof of Theorem 2.8. We can leave out terms with $k = 0$ and ap^{n-1}. We apply Theorem 3.5 with $N = ap^{n-1}$ and $M = k$, and deduce from

$$\binom{pN}{pM} \equiv \binom{N}{M} + C_p NM(N-M)\binom{N}{M} \pmod{p^4 NM(N-M)\binom{N}{M}}$$

that

$$\binom{pN}{pM}^m \equiv \binom{N}{M}^m + mC_p NM(N-M)\binom{N}{M}^m \pmod{mp^4 NM(N-M)\binom{N}{M}^m}$$

and thus,

$$\begin{aligned} &\binom{pN}{pM}^m (-1)^{pM} - \binom{N}{M}^m (-1)^M \\ &\equiv mC_p NM(N-M)\binom{N}{M}^m (-1)^M \pmod{mp^4 NM(N-M)\binom{N}{M}^m} \end{aligned} \tag{4.1}$$

by binomial expansion. This yields that

$$\begin{aligned} B = \nu_p\left(\binom{ap^n}{kp}^m (-1)^{kp} - \binom{ap^{n-1}}{k}^m (-1)^k\right) &= \nu_p(mC_p) + n - 1 + \nu_p(k) \\ &\quad + \nu_p(ap^{n-1} - k) + m\nu_p\left(\binom{ap^{n-1}}{k}\right) \end{aligned} \tag{4.2}$$

for $1 \leq k \leq ap^{n-1} - 1$ if $\nu_p(C_p) \leq 3$ (otherwise, after replacing $\nu_p(C_p)$ with 2, we have only a lower bound on the p-adic order of the difference on the left hand side since $\nu_p(C_p) \geq 2$ by Remark 2.7).

If $a \geq 2$, $(a,p) = 1$, and $m \geq 4$ then we have three cases according to $\nu_p(k)$. In each case we find a lower bound on the expression in (4.2).

First we assume that $\nu_p(k) = n-1$. We have that $k = jp^{n-1}$ with some j so that $1 \leq j \leq a-1$ and $(j,p) = 1$. The combined contribution of these terms to the right hand side of (4.1) is

$$p^{3n-3} mC_p \sum_{\substack{j=1 \\ (j,p)=1}}^{a-1} aj(a-j)\binom{a}{j}^m (-1)^j \pmod{p^{1+e_n(a,p,m)}}.$$

If $\nu_p(k) \geq n$, i.e., $k = jp^{n-1}$ with $(j,p) > 1$, then $B \geq \nu_p(mC_p) + n - 1 + n + n - 1 = \nu_p(mC_p) + 3(n-1) + 1$, so we can include these terms even if it is unnecessary.

In the remaining case we have $\nu_p(k) < n-1$ and therefore, $B \geq \nu_p(mC_p) + (m+1)(n-1) - (m-2)\nu_p(k) > \nu_p(mC_p) + 3(n-1)$. These terms can still contribute to the congruence if the terms with $\nu_p(k) \geq n-1$ result in a sum which is divisible with p, however, Assumption 2.4 helps in eliminating the terms with $\nu_p(k) < n-1$. In fact, let us assume that in fact, $\nu_p(k) = n-1-t$ with $t \in \mathbb{Z}^+$. Then $B \geq \nu_p(mC_p) + (m+1)(n-1) - (m-2)(n-1-t) = \nu_p(mC_p) + 3(n-1) + (m-2)t$. If Assumption 2.4 is satisfied then by (2.2) we have $B > \nu_p(mC_p) + 3(n-1) + \nu_p(A)$ since $t \geq 1$ and $m-2 > \nu_p(A)$; thus, these terms can be ignored. $\square$

Proof of Lemma 2.11. If $m = 2$ then we consider $(1-x)^n(1+x)^n$ and find the coefficient of the middle term with x^n. The identity with $m = 3$ is called Dixon's identity (and it can be proven, for example, by the Zeilberger–Wilf algorithm). $\square$

Proof of Theorem 2.12. If a is odd then clearly,

$$\binom{ap^n}{k}^m (-1)^k = -\binom{ap^n}{ap^n - k}^m (-1)^{ap^n - k},$$

therefore, the terms of the sum in $G(ap^n, m)$ cancel each other in pairs. If $m = 1$ then $\sum_{k=0}^{ap^n} \binom{ap^n}{k}(-1)^k = (1-1)^{ap^n} = 0$ by binomial expansion.

From now on we deal only with even integers $a \geq 2$. If $m = 2$ then

$$G(ap^n, 2) = (-1)^{a/2}\binom{ap^n}{\frac{ap^n}{2}}$$

by Lemma 2.11. In general, the p-adic analysis of the difference of central binomial coefficients given by Theorem 2.1 in [7] provides the result. To handle both cases with $m = 2$ and 3 in a similar fashion, we use Lemma 2.11 and then repeatedly apply the Jacobstahl–Kazandzidis congruences, cf. Theorem 3.5, first with $d = a/2$, $N = 2dp^{n-1}$ and $M = dp^{n-1}$. In fact, if $m = 2$ then we obtain that

$$G(ap^n, 2) - G(ap^{n-1}, 2) \equiv (-1)^{a/2}\frac{a^3}{4}C_p\binom{a}{\frac{a}{2}}p^{3n-3} \pmod{p^{3n-2+\nu_p(C_p)+\nu_p(\binom{a}{a/2})}}.$$

The second application concerns the setting $N = 3dp^{n-1}$ and the above M. If $m = 3$ then we obtain that

$$\begin{aligned} &G(ap^n, 3) - G(ap^{n-1}, 3) \\ &\equiv (-1)^{a/2}a^3 C_p \binom{a}{\frac{a}{2}}\binom{\frac{3a}{2}}{\frac{a}{2}}p^{3n-3} \pmod{p^{3n-2+\nu_p(C_p)+\nu_p(\binom{a}{a/2}\binom{3a/2}{a/2})}}. \end{aligned}$$

If $m \geq 4$ then we evaluate the two sums in (2.3). The p-adic order of the first sum on the right hand side is at least mn by Theorem 2.2.

The second sum in (2.3) needs a more refined approach. From the second sum we can leave out the terms with $k = 0$ and ap^{n-1} again, and then it can be evaluated via the Jacobstahl–Kazandzidis congruence, and we complete the proof by invoking Theorem 2.8.

Note that if $a \geq 2$, $(a,p) = 1$, $m \geq 4$, $n \geq 2$, and p is not a Wolstenholme prime, then under Assumption 2.4 we have that $mn > 3n + \nu_p(m) + m - 3 \geq 3n - 3 + \nu_p(mC_p) + m - 3 \geq 3n - 3 + \nu_p(mC_p) + \nu_p(A)$ by (2.2). Indeed, the first inequality follows from the fact that $(m-3)(n-1) > \nu_p(m)$ since $n \geq 2 > \nu_p(m)/(m-3) + 1$. $\square$

Remark 4.1. *We note that Dixon* [3] *determined the p-adic order of multinomial coefficients in terms of carry counting which might come handy in the cases with $m = 2$ and 3. Clearly, the p-adic order of $B_n = \binom{bp^n}{c_1p^n \ldots c_kp^n}$ with $c_1 + c_2 + \cdots + c_k = b, c_1, \ldots, c_k, b \in \mathbb{Z}^+$ does not depend on n. To find the p-adic order of the differences $B_n - B_{n-1}$ we need a more detailed analysis via the Jacobstahl–Kazandzidis congruence.*

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