A FORMULA FOR AN INFINITE FAMILY OF FIBONACCI-WORD SEQUENCES

MARTIN GRIFFITHS

Abstract. We define here an infinite family of mathematical objects whereby each such object is a sequence based on the structure of the Fibonacci word. Subsequently a formula for the nth term of each of these sequences is obtained.

1. Introduction and Preliminary Definitions

The well-known *Fibonacci word*, which we denote by W_∞, is an infinite string constructed from an alphabet comprising two letters, $\{a, b\}$ say, via the morphism

$$\varphi : \quad b \to ba, \quad a \to b.$$

Starting with the single-letter string b, we obtain

$$W_\infty = babbababbabba \cdots .$$

The *lower Wythoff sequence* and the *upper Wythoff sequence* [7] are given by $(\lfloor j\phi \rfloor)_{j \geq 1}$ and $\left(\lfloor j\phi^2 \rfloor\right)_{j \geq 1}$, respectively, where $\lfloor x \rfloor$ is the *floor function*, denoting the largest integer not exceeding x, and ϕ is the *golden ratio*, given by

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

They are each examples of *Beatty sequences* [1, 6], and possess an association with W_∞ in a sense we now describe.

Let $\mathcal{L}$ and $\mathcal{U}$ denote the sets of numbers in the lower and upper Wythoff sequences, respectively. Then $\mathcal{L}$ and $\mathcal{U}$ comprise a pair of complementary sets [1, 7], by which we mean that $\mathcal{L} \cap \mathcal{U} = \emptyset$ and $\mathcal{L} \cup \mathcal{U} = \mathbb{N}$. In other words, these sets have no elements in common, yet contain all the positive integers between them. To illustrate this point, we may obtain the first few terms of each of the sequences. Indeed, we have $(\lfloor j\phi \rfloor)_{j \geq 1} = (1, 3, 4, 6, 8, 9, 11, 12, \ldots)$ and $\left(\lfloor j\phi^2 \rfloor\right)_{j \geq 1} = (2, 5, 7, 10, 13, \ldots)$.

We are now in a position to define an infinite family of *Fibonacci-word sequences*. As will be seen, these are infinite sequences of non-decreasing positive integers constructed via Definitions 1.1, 1.2, and 1.3 below.

Definition 1.1. *For $k \geq 1$ and $n \geq 0$, let $\mathcal{F}_k(n)$ denote the finite sequence comprising F_k copies of n given by*

$$(\underbrace{n, n, \ldots, n}_{F_k \text{ copies of } n}),$$

where F_k is the kth Fibonacci number.

Definition 1.2. *Let $S_1 = (p_1, \ldots, p_i)$ and $S_2 = (q_1, \ldots, q_j)$ be finite sequences of integers of length $i \geq 1$ and $j \geq 1$, respectively. We define the concatenation S_1S_2 of S_1 and S_2 as the sequence of length $i + j$ given by*

$$S_1S_2 = (p_1, \ldots, p_i, q_1, \ldots, q_j).$$

Definition 1.3. *For any fixed $k \geq 0$, we define the kth Fibonacci-word sequence $(A_k(n))_{n\geq1}$ by way of*

$$(A_k(n))_{n\geq1} = \mathcal{F}_{k+2}(1)\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(3)\mathcal{F}_{k+2}(4)\mathcal{F}_{k+1}(5)\cdots, \tag{1.1}$$

where the finite sequence $\mathcal{F}_j(n)$ has subscript $j = k + 2$ when n appears in the lower Wythoff sequence and subscript $j = k + 1$ when n appears in the upper Wythoff sequence.

Let us use Definition 1.3 to construct the first four Fibonacci-word sequences. The initial sequence, corresponding to $k = 0$, is given by

$$\begin{aligned}(A_0(n))_{n\geq1} &= \mathcal{F}_2(1)\mathcal{F}_1(2)\mathcal{F}_2(3)\mathcal{F}_2(4)\mathcal{F}_1(5)\cdots \\ &= (1, 2, 3, 4, 5, \ldots),\end{aligned} \tag{1.2}$$

which is a linear sequence.

The second sequence,

$$\begin{aligned}(A_1(n))_{n\geq1} &= \mathcal{F}_3(1)\mathcal{F}_2(2)\mathcal{F}_3(3)\mathcal{F}_3(4)\mathcal{F}_2(5)\cdots \\ &= (1, 1, 2, 3, 3, 4, 4, 5, \ldots),\end{aligned} \tag{1.3}$$

is known as the *Hofstadter G-sequence*. It is given by $(\lfloor (j+1)/\phi \rfloor)_{j\geq1}$ [2, 3, 5], and appears as sequence A005206 in [4], where the reader can find many relevant comments, links, and formulas.

Next, we have

$$\begin{aligned}(A_2(n))_{n\geq1} &= \mathcal{F}_4(1)\mathcal{F}_3(2)\mathcal{F}_4(3)\mathcal{F}_4(4)\mathcal{F}_3(5)\cdots \\ &= (1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, \ldots),\end{aligned}$$

which can be found, shifted by two terms, as A060144 in [4].

The fourth Fibonacci-word sequence is given by

$$\begin{aligned}(A_3(n))_{n\geq1} &= \mathcal{F}_5(1)\mathcal{F}_4(2)\mathcal{F}_5(3)\mathcal{F}_5(4)\mathcal{F}_4(5)\cdots \\ &= (1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, \ldots).\end{aligned}$$

This appears as sequence A192002 in [4] shifted by three terms. Various formulas can be found for this and the previous sequence in [4].

Definition 1.4. *We define $B_k(n)$ by way of*

$$B_k(n) = (-1)^{k+1}\left(\sum_{j=1}^{F_k} \lfloor (n + F_{k+1} + j - 1)\,\phi \rfloor - nF_{k+1} - \binom{F_{k+2}}{2}\right),$$

noting that when $k = 0$, the sum that appears on the right side above is defined to be empty, and

$$\binom{F_{k+2}}{2} = \binom{F_2}{2} = \binom{1}{2} = 0.$$

Our aim in this paper is to show that for any fixed $k \geq 0$, it is the case that $(A_k(n))_{n\geq1} = (B_k(n))_{n\geq1}$.

2. An Intermediate Result

In this section, we prove a lemma concerning $B_k(n)$. This will go some way towards showing that, for each $k \geq 0$, $(A_k(n))_{n\geq 1} = (B_k(n))_{n\geq 1}$.

Lemma 2.1. *For any fixed $k \geq 0$,*

$$(B_{k+2}(n - F_{k+2}))_{n\geq F_{k+2}+1} = (B_k(n))_{n\geq F_{k+2}+1} - (B_{k+1}(n))_{n\geq F_{k+2}+1}.$$

Proof. Suppose that $k \geq 0$ and $n \geq F_{k+2} + 1$. First, from Definition 1.4, we have

$$B_{k+1}(n) = (-1)^{k+2} \left(\sum_{j=1}^{F_{k+1}} \lfloor (n + F_{k+2} + j - 1)\, \phi \rfloor - nF_{k+2} - \binom{F_{k+3}}{2} \right) \tag{2.1}$$

and

$$\begin{aligned} &B_{k+2}(n - F_{k+2}) \\ &\quad = (-1)^{k+3} \left(\sum_{j=1}^{F_{k+2}} \lfloor (n - F_{k+2} + F_{k+3} + j - 1)\, \phi \rfloor - (n - F_{k+2})\, F_{k+3} - \binom{F_{k+4}}{2} \right) \\ &\quad = (-1)^{k+3} \left(\sum_{j=1}^{F_{k+2}} \lfloor (n + F_{k+1} + j - 1)\, \phi \rfloor - (n - F_{k+2})\, F_{k+3} - \binom{F_{k+4}}{2} \right). \end{aligned} \tag{2.2}$$

Next, from definition 1.4 and (2.1) we obtain

$$\begin{aligned} B_k(n) - B_{k+1}(n) &= (-1)^{k+1} \left(\sum_{j=1}^{F_k} \lfloor (n + F_{k+1} + j - 1)\, \phi \rfloor + \sum_{j=1}^{F_{k+1}} \lfloor (n + F_{k+2} + j - 1)\, \phi \rfloor \right) \\ &\qquad - G(k,n) \\ &= (-1)^{k+1} \sum_{j=1}^{F_{k+2}} \lfloor (n + F_{k+1} + j - 1)\, \phi \rfloor - G(k,n), \end{aligned} \tag{2.3}$$

where

$$G(k,n) = nF_{k+1} + \binom{F_{k+2}}{2} + nF_{k+2} + \binom{F_{k+3}}{2}.$$

Finally, some straightforward manipulations show that

$$G(k,n) = (n - F_{k+2})\, F_{k+3} + \binom{F_{k+4}}{2},$$

which in turn demonstrates via (2.2) and (2.3) that

$$B_{k+2}(n - F_{k+2}) = B_k(n) - B_{k+1}(n)$$

for any $k \geq 0$ and $n \geq F_{k+2} + 1$. That the lemma is true follows from this. □

3. A Result Concerning Fibonacci-Word Sequences

In this section, we show that there is a corresponding result for the Fibonacci-word sequences. First, to help clarify the proof of this result, we provide further terms of (1.1) explicitly to give

$$(A_k(n))_{n\geq 1} = \mathcal{F}_{k+2}(1)\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(3)\mathcal{F}_{k+2}(4)\mathcal{F}_{k+1}(5)\mathcal{F}_{k+2}(6)\mathcal{F}_{k+1}(7)\mathcal{F}_{k+2}(8)\mathcal{F}_{k+2}(9)\mathcal{F}_{k+1}(10)\mathcal{F}_{k+2}(11)\cdots,$$

From the method of construction of the Fibonacci-word sequence, the positions of the finite sequences with subscripts $k+2$ and $k+1$ correspond to the positions of the bs and as in W_∞, respectively.

On applying the morphism φ twice in succession to W_∞, we have

$$\underbrace{bab}\,\underbrace{ba}\,\underbrace{bab}\,\underbrace{bab}\,\underbrace{ba}\cdots.$$

This partitions the Fibonacci word in such a way that bab and ba correspond to the positions of the bs and as in W_∞, respectively. A similar structure occurs for the Fibonacci-word sequence, as follows:

$$\underbrace{\mathcal{F}_{k+2}(1)\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(3)}\,\underbrace{\mathcal{F}_{k+2}(4)\mathcal{F}_{k+1}(5)}\,\underbrace{\mathcal{F}_{k+2}(6)\mathcal{F}_{k+1}(7)\mathcal{F}_{k+2}(8)}\,\underbrace{\mathcal{F}_{k+2}(9)\mathcal{F}_{k+1}(10)\mathcal{F}_{k+2}(11)}\cdots.$$

A shift then gives

$$\mathcal{F}_{k+2}(1)\,\underbrace{\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(3)\mathcal{F}_{k+2}(4)}\,\underbrace{\mathcal{F}_{k+1}(5)\mathcal{F}_{k+2}(6)}\,\underbrace{\mathcal{F}_{k+1}(7)\mathcal{F}_{k+2}(8)\mathcal{F}_{k+2}(9)}\,\underbrace{\mathcal{F}_{k+1}(10)\mathcal{F}_{k+2}(11)\mathcal{F}_{k+2}(12)}\cdots,$$

where the underbraces containing finite sequences with subscripts $k+1$, $k+2$ and $k+2$ correspond to the bs in W_∞, whereas those containing finite sequences with subscripts $k+1$ and $k+2$ correspond to the as.

The following definition is used in the proof of Lemma 3.2. It concerns the difference between two sequences.

Definition 3.1. *Let $(x_n)_{n\geq 1} = (x_1, x_2, x_3, \ldots)$ and $(y_n)_{n\geq 1} = (y_1, y_2, y_3, \ldots)$ each be an infinite sequence. We define the difference of $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ to be the infinite sequence given by*

$$(x_n)_{n\geq 1} - (y_n)_{n\geq 1} = (x_1 - y_1, x_2 - y_2, x_3 - y_3, \ldots).$$

In other words, the subtraction is carried out term-wise.

The difference between two finite sequences, $(x_1, x_2, \ldots, x_i)$ and $(y_1, y_2, \ldots, y_i)$, each of length $i \geq 1$, is given by $(x_1 - y_1, x_2 - y_2, \ldots, x_i - y_i)$.

Lemma 3.2. *For $k \geq 0$,*

$$(A_{k+2}(n - F_{k+2}))_{n\geq F_{k+2}+1} = (A_k(n))_{n\geq F_{k+2}+1} - (A_{k+1}(n))_{n\geq F_{k+2}+1}.$$

Proof. Let $k \geq 0$. Referring to Table 1, the first line is the Fibonacci-word sequence $(A_k(n))_{n\geq 1}$. Regarding the arrows initially as empty sequences, the second line gives the Fibonacci-word sequence $(A_{k+1}(n))_{n\geq 1}$. Note that $\mathcal{F}_{k+3}(1)$, the first sequence in this line and currently situated in block 0, can be split up so that $\mathcal{F}_{k+2}(1)$ remains in block 0 while $\mathcal{F}_{k+1}(1)$ now appears at the beginning of block 1. Similarly, the final sequence in the second line of block 1, $\mathcal{F}_{k+3}(3)$, can be split up so that $\mathcal{F}_{k+2}(3)$ remains in block 1 while $\mathcal{F}_{k+1}(3)$ now appears at the beginning of block 2. This process continues indefinitely to produce the third line.

A key point here is that the length and position of each finite sequence appearing in the first line of Table 1 matches precisely the length and position of the corresponding sequence in the third line. It is thus straightforward to subtract the third line from the first to give the fourth line. By way of block 0, we have

$$\mathcal{F}_{k+2}(1) - \mathcal{F}_{k+2}(1) = \mathcal{F}_{k+2}(0).$$

Then from block 1, we obtain

$$\begin{aligned}\left(\mathcal{F}_{k+1}(2)-\mathcal{F}_{k+1}(1)\right)\left(\mathcal{F}_{k+2}(3)-\mathcal{F}_{k+2}(2)\right)\left(\mathcal{F}_{k+2}(4)-\mathcal{F}_{k+2}(3)\right) &= \mathcal{F}_{k+1}(1)\mathcal{F}_{k+2}(1)\mathcal{F}_{k+2}(1)\\ &= \mathcal{F}_{k+4}(1),\end{aligned}$$

from block 2, we have

$$\begin{aligned}\left(\mathcal{F}_{k+1}(5)-\mathcal{F}_{k+1}(3)\right)\left(\mathcal{F}_{k+2}(6)-\mathcal{F}_{k+2}(4)\right) &= \mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(2)\\ &= \mathcal{F}_{k+3}(2),\end{aligned}$$

and so on. Disregarding block 0, this results, for a fixed $k \geq 0$, in the infinite sequence

$$\begin{aligned}\left(A_k(n)\right)_{n\geq F_{k+2}+1}-\left(A_{k+1}(n)\right)_{n\geq F_{k+2}+1} &= \mathcal{F}_{k+4}(1)\mathcal{F}_{k+3}(2)\mathcal{F}_{k+4}(3)\mathcal{F}_{k+4}(4)\mathcal{F}_{k+3}(5)\cdots\\ &= \left(A_{k+2}\left(n-F_{k+2}\right)\right)_{n\geq F_{k+2}+1},\end{aligned}$$

as required. □

Block 0	Block 1	Block 2	Block 3	Block 4
$\mathcal{F}_{k+2}(1)$	$\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(3)\mathcal{F}_{k+2}(4)$	$\mathcal{F}_{k+1}(5)\mathcal{F}_{k+2}(6)$	$\mathcal{F}_{k+1}(7)\mathcal{F}_{k+2}(8)\mathcal{F}_{k+2}(9)$	$\mathcal{F}_{k+1}(10)\mathcal{F}_{k+2}(11)\mathcal{F}_{k+2}(12)\cdots$
$\mathcal{F}_{k+3}(1)$	$\longrightarrow\ \mathcal{F}_{k+2}(2)\mathcal{F}_{k+3}(3)$	$\longrightarrow\ \mathcal{F}_{k+3}(4)$	$\longrightarrow\ \mathcal{F}_{k+2}(5)\mathcal{F}_{k+3}(6)$	$\longrightarrow\ \mathcal{F}_{k+2}(7)\ \mathcal{F}_{k+3}(8)\cdots$
$\mathcal{F}_{k+2}(1)$	$\mathcal{F}_{k+1}(1)\mathcal{F}_{k+2}(2)\mathcal{F}_{k+2}(3)$	$\mathcal{F}_{k+1}(3)\mathcal{F}_{k+2}(4)$	$\mathcal{F}_{k+1}(4)\mathcal{F}_{k+2}(5)\mathcal{F}_{k+2}(6)$	$\mathcal{F}_{k+1}(6)\ \mathcal{F}_{k+2}(7)\ \mathcal{F}_{k+2}(8)\cdots$
$\mathcal{F}_{k+2}(0)$	$\mathcal{F}_{k+1}(1)\mathcal{F}_{k+2}(1)\mathcal{F}_{k+2}(1)$	$\mathcal{F}_{k+1}(2)\mathcal{F}_{k+2}(2)$	$\mathcal{F}_{k+1}(3)\mathcal{F}_{k+2}(3)\mathcal{F}_{k+2}(3)$	$\mathcal{F}_{k+1}(4)\ \mathcal{F}_{k+2}(4)\ \mathcal{F}_{k+3}(4)\cdots$
	$\mathcal{F}_{k+4}(1)$	$\mathcal{F}_{k+3}(2)$	$\mathcal{F}_{k+4}(3)$	$\mathcal{F}_{k+4}(4)$

TABLE 1. A depiction of the manipulations involving Fibonacci-word sequences that are used in Lemma 3.2.

4. Putting the Pieces Together

Theorem 4.1. *For any fixed $k \geq 0$, the corresponding Fibonacci-word sequence is given by*

$$\left(A_k(n)\right)_{n\geq 1}=\left((-1)^{k+1}\left(\sum_{j=1}^{F_k}\lfloor (n+F_{k+1}+j-1)\,\phi\rfloor - nF_{k+1}-\binom{F_{k+2}}{2}\right)\right)_{n\geq 1}.$$

Proof. Recalling from Definition 1.4 that when $k=0$ we define the sum appearing on the right side above to be empty, and

$$\binom{F_{k+2}}{2}=\binom{F_2}{2}=\binom{1}{2}=0,$$

we have

$$\left(B_0(n)\right)_{n\geq 1}=\left(-\left(-nF_1\right)\right)_{n\geq 1}=(n)_{n\geq 1}=(1,2,3,4,5,\ldots).$$

Also

$$\begin{aligned}
(B_1(n))_{n\geq 1} &= \left((-1)^2\left(\lfloor (n+F_2+1-1)\,\phi\rfloor - nF_2 - \binom{F_3}{2}\right)\right)_{n\geq 1}\\
&= (\lfloor (n+1)\phi\rfloor - n - 1)_{n\geq 1}\\
&= \left(\left\lfloor \frac{n+1}{\phi}\right\rfloor\right)_{n\geq 1}\\
&= (1,1,2,3,3,4,4,5,\ldots).
\end{aligned}$$

From the above, in addition to (1.2) and (1.3), we see that $(B_0(n))_{n\geq 1} = (A_0(n))_{n\geq 1}$ and $(B_1(n))_{n\geq 1} = (A_1(n))_{n\geq 1}$.

Using these results in conjunction with Lemmas 2.1 and 3.2, a straightforward induction argument completes the proof of the theorem. □

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Department of Mathematical Sciences, University of Essex, Colchester, CO4 3SQ, United Kingdom

E-mail address: griffm@essex.ac.uk