FURTHER CLOSED FORMS FOR FINITE SUMS OF WEIGHTED PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. The finite sum

$$\sum_{i=1}^{n} 2^{-i} F_{i-1} = 1 - \frac{F_{n+2}}{2^n},$$

occurs in Section 9.1 of Knott [1], and is the inspiration for the present paper. We refer to the term 2^{-i} in the summand as the weight term. Here, we present seven families of such finite sums that we believe to be new. In each of these seven families, the product that defines the summand can be made arbitrarily long. The sequences that we employ are generalizations of the Fibonacci/Lucas sequences.

1. Introduction

Let a and b be integers with $(a,b) \neq (0,0)$. For any non-zero integer p, we define, for all integers n, the integer sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n(a,b,p) = W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b, \tag{1.1}$$

and

$$\overline{W}_n(a,b,p) = \overline{W}_n = W_{n-1} + W_{n+1}.$$

With $\Delta = p^2 + 4$, it follows that

$$\overline{\overline{W}}_n = \Delta W_n. \tag{1.2}$$

Identity (1.2) is used, for instance, if we take $\{H_n\}$ to be $\{L_n\}$ in S_6 or S_7 (see Section 2).

For $(a,b,p) = (0,1,1)$, we have $\{W_n\} = \{F_n\}$, and $\{\overline{W}_n\} = \{L_n\}$, which are the Fibonacci and Lucas sequences, respectively. Taking $(a,b) = (0,1)$, and allowing p to remain arbitrary, we write $\{W_n(p)\} = \{U_n\}$, and $\{\overline{W}_n(p)\} = \{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas sequences, respectively.

When we specialize (1.1) by taking $p = 1$, and allow a and b to remain arbitrary, we write $\{W_n\}$=$\{H_n\}$. Thus, $\{H_n\}$ and $\{\overline{H}_n\}$ satisfy the same recurrence as $\{F_n\}$, and are generalizations of $\{F_n\}$ and $\{L_n\}$, respectively.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the Binet forms for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

Before proceeding, we introduce some familiar notation to make the statement of our formulas more succinct. Throughout our presentation, we employ i as the dummy variable, and take, for instance, $[F_{ki}]_m^n$ to mean $F_{kn} - F_{km}$.

The finite sum

$$\sum_{i=1}^{n} 2^{-i} F_{i-1} = 1 - \frac{F_{n+2}}{2^n} = -\left[\frac{F_{i+2}}{2^i}\right]_0^n \tag{1.5}$$

occurs in Section 9.1 of Knott [1], where other references for this sum are given. Prompted by (1.5), we undertook a search for analogous sums where, in each case, the summand consists of a lengthier product. Our search resulted in seven multi-parameter analogues of (1.5), which are the topic of this paper. In all cases, the summand involves a product of terms from the sequences that we defined earlier. We present seven main results, each of which can be specialized to the Fibonacci/Lucas numbers. Furthermore, in each of our main results, the length of the product that defines the summand is governed by the parameter j, which can be arbitrarily large.

When conducting our research, we began by searching for results that hold for the sequences $\{H_n\}$ and $\{\overline{H}_n\}$. From these results, we determined those that hold for the more general sequences $\{W_n\}$ and $\{\overline{W}_n\}$. The distinction between these two types of results is determined by the validity of certain key identities that we list in Section 3, and that are required for the proofs. For instance, the final four identities in the array (3.15) hold for sequences generated by (1.1) with $p = 1$, but do not hold for arbitrary values of p.

In Section 2, we define the seven finite sums of weighted products of generalized Fibonacci numbers that are the topic of this paper. In Section 3, we present our main results, which are the closed forms of the seven sums defined in Section 2. We also provide a sample proof. In Section 4, we give several special cases of our main results for the Fibonacci/Lucas numbers. We conclude with Section 5, where we present a limited number of similar sums that involve squared factors.

2. The Finite Sums

For all the finite sums that we define, the upper limit of summation is n, a positive integer. Furthermore, for the remainder of this paper, $j \geq 1$ and $k \neq 0$ are assumed to be integers. We now define seven finite sums of weighted products whose closed forms we give in the next section. The first three finite sums involve sequences generated by the recurrence given in (1.1), in which $p \neq 0$ is an arbitrary integer. These finite sums are

$$\begin{aligned}
S_1(n,j,k) &= \sum_{i=1}^{n} U_{jk-1}^{-i} W_{ki} \cdots W_{k(i+j-2)} W_{k(i-1)+1}, \ jk \neq 1, \\
S_2(n,j,k) &= \sum_{i=1}^{n} U_{jk+1}^{-i} W_{ki} \cdots W_{k(i+j-2)} W_{k(i-1)-1}, \ jk \neq -1, \quad \text{and} \\
S_3(n,j,k) &= \sum_{i=1}^{n} V_{jk}^{-i} W_{ki} \cdots W_{k(i+j-2)} W_{k(i-j-1)}.
\end{aligned}$$

Next, we define four finite sums that involve sequences generated by the recurrence given in (1.1), in which $p = 1$. These finite sums are

$$S_4(n,j,k) = \sum_{i=1}^{n} (-1)^i F_{jk-2}^{-i} H_{ki} \cdots H_{k(i+j-2)} H_{k(i-1)+2},\ jk \neq 2,$$
$$S_5(n,j,k) = \sum_{i=1}^{n} F_{jk+2}^{-i} H_{ki} \cdots H_{k(i+j-2)} H_{k(i-1)-2},\ jk \neq -2,$$
$$S_6(n,j,k) = \sum_{i=1}^{n} (-1)^i L_{jk-1}^{-i} H_{ki} \cdots H_{k(i+j-2)} \overline{H}_{k(i-1)+1}, \quad \text{and}$$
$$S_7(n,j,k) = \sum_{i=1}^{n} L_{jk+1}^{-i} H_{ki} \cdots H_{k(i+j-2)} \overline{H}_{k(i-1)-1}.$$

Each of the finite sums that we define above has a so-called weight term. For instance, for $S_1(n,j,k)$ the weight term is U_{jk-1}^{-i}. Excluding the weight term, the length of the product in each of the finite sums $S_i(n,j,k)$, $1 \leq i \leq 7$, is j. When $j \geq 2$, it is straightforward to write down each summand. For example, when $j=2$ the summand of $S_1(n,j,k)$ is $U_{2k-1}^{-i} W_{ki} W_{k(i-1)+1}$.

When $j=1$, the summand of each of the $S_i(n,j,k)$ is to be interpreted as the product of the weight term, and the *last* factor in the product that defines the summand. For instance, for $j=1$ the summand in $S_2(n,j,k)$ is to be interpreted as $U_{k+1}^{-i} W_{k(i-1)-1}$.

3. The Closed Forms and a Sample Proof

In this section, in Theorems 3.1 and 3.2, we give the closed forms for each of the finite sums defined in Section 2. At the end of this section, we also provide a sample proof. Our first theorem gives the closed forms for S_1, S_2, and S_3.

Theorem 3.1. *Suppose $j \geq 1$ and $k \neq 0$ are integers. Then, with the constraints on j and k given in the definitions of S_1, S_2, and S_3, we have*

$$S_1(n,j,k) = \frac{1}{U_{jk}} \left[\frac{W_{ki} \cdots W_{k(i+j-1)}}{U_{jk-1}^{i}} \right]_0^n, \tag{3.1}$$
$$S_2(n,j,k) = \frac{1}{U_{jk}} \left[\frac{W_{ki} \cdots W_{k(i+j-1)}}{U_{jk+1}^{i}} \right]_0^n, \quad \text{and} \tag{3.2}$$
$$S_3(n,j,k) = (-1)^{jk+1} \left[\frac{W_{ki} \cdots W_{k(i+j-1)}}{V_{jk}^{i}} \right]_0^n. \tag{3.3}$$

Our next theorem gives the closed forms for S_4, S_5, S_6, and S_7.

Theorem 3.2. *Suppose $j \geq 1$ and $k \neq 0$ are integers. Then, with the constraints on j and k given in the definitions of S_4, S_5, S_6, and S_7, we have*

$$S_4(n,j,k) = \frac{1}{F_{jk}} \left[\frac{(-1)^i H_{ki} \cdots H_{k(i+j-1)}}{F^i_{jk-2}} \right]_0^n, \tag{3.4}$$

$$S_5(n,j,k) = -\frac{1}{F_{jk}} \left[\frac{H_{ki} \cdots H_{k(i+j-1)}}{F^i_{jk+2}} \right]_0^n, \tag{3.5}$$

$$S_6(n,j,k) = \frac{1}{F_{jk}} \left[\frac{(-1)^i H_{ki} \cdots H_{k(i+j-1)}}{L^i_{jk-1}} \right]_0^n, \quad \text{and} \tag{3.6}$$

$$S_7(n,j,k) = -\frac{1}{F_{jk}} \left[\frac{H_{ki} \cdots H_{k(i+j-1)}}{L^i_{jk+1}} \right]_0^n. \tag{3.7}$$

We conclude this section with a proof of (3.1). The method of proof that we employ can also be used to prove each of (3.2)-(3.7).

A key identity that we require for the proof of (3.1) is

$$W_{k(n+j)} - U_{jk-1} W_{kn} = U_{jk} W_{kn+1}, \tag{3.8}$$

which is true for all integers j, k, and n. To prove (3.8), we transpose the product on the right to the left side, substitute the Binet forms, then expand and factor the expression that results to obtain

$$-(\alpha\beta + 1) U_{jk-1} \left(b U_{kn} + a U_{kn-1} \right). \tag{3.9}$$

Since $\alpha\beta = -1$, (3.8) follows from (3.9).

Denoting the right side of (3.1) by $r(n,j,k)$, we see, after some calculations, that the difference $r(n+1,j,k) - r(n,j,k)$ is given by

$$\begin{array}{l} \dfrac{1}{U_k} \left(\dfrac{W_{k(n+1)} - U_{k-1} W_{kn}}{U^{n+1}_{k-1}} \right), \; j = 1 \\ \dfrac{W_{k(n+1)} \cdots W_{k(n+j-1)}}{U_{jk}} \times \dfrac{W_{k(n+j)} - U_{jk-1} W_{kn}}{U^{n+1}_{jk-1}}, \; j \geq 2. \end{array} \tag{3.10}$$

Then, with the use of (3.8), we see that the quantities in (3.10) become

$$\begin{aligned} \frac{W_{kn+1}}{U^{n+1}_{k-1}} &= S_1(n+1,j,k) - S_1(n,j,k), \; j = 1 \\ \frac{W_{k(n+1)} \cdots W_{k(n+j-1)} W_{kn+1}}{U^{n+1}_{jk-1}} &= S_1(n+1,j,k) - S_1(n,j,k), \; j \geq 2. \end{aligned} \tag{3.11}$$

Together, (3.10) and (3.11) show that

$$r(n+1,j,k) - r(n,j,k) = S_1(n+1,j,k) - S_1(n,j,k), \; j \geq 1. \tag{3.12}$$

In light of (3.12), to complete the proof of (3.1) it is enough to prove that

$$r(1,j,k) = S_1(1,j,k), \; j \geq 1. \tag{3.13}$$

We consider the cases $j = 1$ and $j \geq 2$ separately. In each of these cases, we write down the left and right sides of (3.13) and simplify. Accordingly, we are required to prove that

$$\begin{aligned} W_k - U_{k-1} W_0 &= U_k W_1, \; j = 1, \\ W_{jk} - U_{jk-1} W_0 &= U_{jk} W_1, \; j \geq 2. \end{aligned} \tag{3.14}$$

Both of the equalities in (3.14) follow from (3.8), and this completes the proof of (3.1).

For the proof of each of (3.2)-(3.7), we require an identity that is analogous to (3.8). To assist the interested reader, we record these identities below. They are, respectively,

$$\begin{aligned}
W_{k(n+j)} - U_{jk+1}W_{kn} &= U_{jk}W_{kn-1},\\
W_{k(n+j)} - V_{jk}W_{kn} &= (-1)^{jk+1}W_{k(n-j)},\\
H_{k(n+j)} + F_{jk-2}H_{kn} &= F_{jk}H_{kn+2},\\
H_{k(n+j)} - F_{jk+2}H_{kn} &= -F_{jk}H_{kn-2},\\
H_{k(n+j)} + L_{jk-1}H_{kn} &= F_{jk}\overline{H}_{kn+1}, \quad \text{and}\\
H_{k(n+j)} - L_{jk+1}H_{kn} &= -F_{jk}\overline{H}_{kn-1}.
\end{aligned} \tag{3.15}$$

4. Special Cases of (3.1)–(3.7)

Keeping in mind the conventions we outlined in the final two paragraphs of Section 2, we now consider some special cases of our main results.

In (3.2), take $\{W_n\}$ to be $\{F_n\}$, and in (3.5) and (3.7) take $\{H_n\}$ to be $\{F_n\}$. In each of these cases, let $(j,k) = (2,1)$. Then (3.2), (3.5), and (3.7) become, respectively,

$$\begin{aligned}
\sum_{i=1}^{n} 2^{-i}F_iF_{i-2} &= \frac{F_nF_{n+1}}{2^n},\\
\sum_{i=1}^{n} 3^{-i}F_iF_{i-3} &= -\frac{F_nF_{n+1}}{3^n}, \quad \text{and}\\
\sum_{i=1}^{n} 4^{-i}F_iL_{i-2} &= -\frac{F_nF_{n+1}}{4^n}.
\end{aligned}$$

In (3.4), take $\{H_n\}$ to be $\{F_n\}$, and let $(j,k) = (3,2)$. Then (3.4) becomes

$$\sum_{i=1}^{n}(-1)^i 3^{-i}F_{2i}^2F_{2i+2} = \frac{(-1)^n}{8} \times \frac{F_{2n}F_{2n+2}F_{2n+4}}{3^n}.$$

In (3.5), setting $j = k = 1$ and taking $\{H_n\}$ to be $\{F_n\}$, we obtain

$$\sum_{i=1}^{n} 2^{-i}F_{i-3} = -\frac{F_n}{2^n}, n \geq 1. \tag{4.1}$$

In (4.1), we replace the running variable i by $i+2$, then multiply both sides of the resulting sum by 4 to obtain

$$\sum_{i=-1}^{n-2} 2^{-i}F_{i-1} = -\frac{F_n}{2^{n-2}}, n \geq 1. \tag{4.2}$$

Now, transposing the first two terms of the sum on the left side of (4.2) to the right side, we have

$$\sum_{i=1}^{n-2} 2^{-i}F_{i-1} = 1 - \frac{F_n}{2^{n-2}}, n \geq 3.$$

Then, replacing n by $n+2$, we obtain (1.5), valid for $n \geq 1$. Similar manipulations yield the Lucas counterpart of (1.5), which is

$$\sum_{i=1}^{n} 2^{-i} L_{i-1} = 3 - \frac{L_{n+2}}{2^n}, n \geq 1.$$

Finally for this section, in (3.6), take $\{H_n\}$ to be $\{F_n\}$, and let $(j,k) = (2,1)$. Then (3.6) becomes

$$\sum_{i=1}^{n} (-1)^i F_{2i} = (-1)^n F_n F_{n+1}.$$

5. Concluding Comments

Results analogous to those presented above, where the summand or the closed form involve squared factors, seem to be rare. We have discovered only two such results. These results are

$$\begin{aligned} \sum_{i=1}^{n} 2^{-2i} H_{i-3} \overline{H}_i &= -\left[\frac{H_i^2}{2^{2i}}\right]_0^n \quad \text{and} \\ \sum_{i=1}^{n} 3^{-2i} H_i^2 H_{i+1}^2 H_{i-2} \overline{H}_i &= \frac{1}{4}\left[\frac{H_i^2 H_{i+1}^2 H_{i+2}^2}{3^{2i}}\right]_0^n. \end{aligned} \tag{5.1}$$

The key identities required for the proofs of the sums in (5.1) are, respectively,

$$\begin{aligned} H_{n+2}^2 - 9H_{n-1}^2 &= 4H_{n-2}\overline{H}_n \quad \text{and} \\ 4H_{n-1}^2 - H_n^2 &= H_{n-3}\overline{H}_n. \end{aligned}$$

When we replace $\{H_n\}$ by $\{F_n\}$, the two sums in (5.1) become, respectively,

$$\begin{aligned} \sum_{i=1}^{n} 2^{-2i} F_{i-3} L_i &= -\frac{F_n^2}{2^{2n}} \quad \text{and} \\ \sum_{i=1}^{n} 3^{-2i} F_i^2 F_{i+1}^2 F_{i-2} L_i &= \frac{1}{4} \times \frac{F_n^2 F_{n+1}^2 F_{n+2}^2}{3^{2n}}. \end{aligned} \tag{5.2}$$

Readers interested in the results presented here may wish to consult the recent paper [4]. In [4], we present closed forms for products similar to those in the present paper, but which involve the weight terms $U_{jk\pm1}^{i-1}$, V_{jk}^{i-1}, $F_{jk\pm2}^{i-1}$, and $L_{jk\pm1}^{i-1}$. The papers [2, 3, 5, 6], with the references that they contain, are concerned with the broad topic of closed forms for finite sums of products of Fibonacci/generalized Fibonacci numbers.

Finally, for all the results contained in this paper, our process of discovery involved some guessing, and the examination of numerical data.

Acknowledgment

I would like to express my gratitude to the referee for a careful reading of the manuscript, and for suggestions that have improved the presentation of this paper. The tone of this referee's comments was uplifting and supportive.

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MSC2010: 11B39, 11B37.

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