FURTHER CLOSED FORMS FOR FINITE SUMS OF WEIGHTED PRODUCTS OF THE SINE AND COSINE FUNCTIONS

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Abstract. In this paper, we present closed forms for six families of finite sums of weighted products of the sine/cosine functions. In each finite sum that we define, the summand contains a product of trigonometric functions, and the length of this product can be made as large as we please.

A special case of one of our main results is the sum

$$\sum_{i=1}^{n} \left(\frac{1}{2\cos 2} \right)^i \cos i \cos(i-3) = \cos 1 - \frac{\cos n \cos(n+1)}{2^n \cos^n 2}.$$

Here the weight term in the summand is $\left(\frac{1}{2\cos 2}\right)^i$.

1. Introduction

In this paper, we give closed forms for six families of finite sums of weighted products of the sine/cosine functions. Two of the finite sums that we consider contain only the integer parameter $j \geq 1$. In addition to the parameter j, the remaining four finite sums that we consider contain the parameter k, which is a rational number. In each of the six finite sums in question, the parameter j can be arbitrarily large, and governs the length of the product that defines the summand.

To give the reader an indication of the nature of the results that we present here, we now give an example of one of our main results. Putting $j = 3$ in (3.2), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{1}{2\cos 3} \right)^i \cos i \cos(i+1) \cos(i-4) \\ &= \cos 1 \cos 2 - \frac{\cos n \cos(n+1) \cos(n+2)}{2^n \cos^n 3}. \end{aligned}$$

By comparison, we have

$$\begin{aligned} &\sum_{i=1}^{n} (2\cos 3)^{i-1} \cos i \cos(i+1) \cos(i+5) \\ &= 2^n \cos^n 3 \cos n \cos(n+1) \cos(n+2) - \cos 1 \cos 2, \end{aligned}$$

a result that occurs in Section 1 of [1]. Indeed, it was a desire to obtain further results of the kind in [1] that prompted us to pursue the research that led to the present paper.

In Section 2, we define the six finite sums that are the topic of this paper. In Section 3, we give the closed form of each of these six sums, and provide a sample proof. In Section 4, we give several special cases of our main results.

2. The Finite Sums

Throughout this paper, the upper limit of summation is an integer $n \geq 1$. In all the sums that we define, the parameter $j \geq 1$ is taken to be an integer. We now define the six finite sums of weighted products whose closed forms we give in the next section.

The two finite sums S_1 and S_2, defined below, contain only the parameter j, and are

$$S_1(n,j) = \sum_{i=1}^{n} \left(\frac{1}{2\cos j}\right)^i \sin i \cdots \sin(i+j-2)\sin(i-j-1), \textit{and}$$
$$S_2(n,j) = \sum_{i=1}^{n} \left(\frac{1}{2\cos j}\right)^i \cos i \cdots \cos(i+j-2)\cos(i-j-1).$$

The four finite sums S_3, S_4, S_5, and S_6, which we define below, contain the additional parameter k. To avoid zero denominators, we impose restrictions on k. In S_3 and S_4, $k \neq 0$ is assumed to be a rational number with $jk+1 \neq 0$. In S_5 and S_6, $k \neq 0$ is assumed to be a rational number with $jk-1 \neq 0$.

$$S_3(n,j,k) = \sum_{i=1}^{n} \left(\frac{\sin 1}{\sin(jk+1)}\right)^i \sin(ki)\cdots\sin(k(i+j-2))\sin(k(i-1)-1),$$
$$S_4(n,j,k) = \sum_{i=1}^{n} \left(\frac{\sin 1}{\sin(jk+1)}\right)^i \cos(ki)\cdots\cos(k(i+j-2))\cos(k(i-1)-1),$$
$$S_5(n,j,k) = \sum_{i=1}^{n} \left(-\frac{\sin 1}{\sin(jk-1)}\right)^i \sin(ki)\cdots\sin(k(i+j-2))\sin(k(i-1)+1), \textit{and}$$
$$S_6(n,j,k) = \sum_{i=1}^{n} \left(-\frac{\sin 1}{\sin(jk-1)}\right)^i \cos(ki)\cdots\cos(k(i+j-2))\cos(k(i-1)+1).$$

Each of the finite sums that we define above has a weight term. In S_1 and S_2, the weight term is $\left(\frac{1}{2\cos j}\right)^i$. In S_3 and S_5, the weight terms are $\left(\frac{\sin 1}{\sin(jk+1)}\right)^i$ and $\left(-\frac{\sin 1}{\sin(jk-1)}\right)^i$, respectively. Excluding the weight term, the length of the product that defines the summand in each of the sums S_i, $1 \leq i \leq 6$, is j.

It is easy to write down each summand when $j \geq 2$. For instance, when $j = 2$ the summand of S_1 is $\left(\frac{1}{2\cos 2}\right)^i \sin i \sin(i-3)$. When $j = 1$, the summand of each of the S_i is to be interpreted as the product of the weight term and the rightmost factor. For instance, for $j = 1$ the summand of S_3 is to be interpreted as $\left(\frac{\sin 1}{\sin(k+1)}\right)^i \sin(k(i-1)-1)$.

At this point, we remark that in [1] we give closed forms for eight similarly defined sums that involve the weight terms $(2\cos j)^{i-1}$, $\left(\frac{\sin(jk+1)}{\sin 1}\right)^{i-1}$, and $\left(\frac{\sin(jk-1)}{\sin 1}\right)^{i-1}$.

3. The Closed Forms and a Sample Proof

In this section, we give the closed forms for each of the finite sums defined in Section 2. We present these closed forms in two theorems. Throughout, we take the running variable i to be the dummy variable, so that, for instance, $[\sin(ki)]_m^n$ is taken to mean $\sin(kn) - \sin(km)$. At the end of this section, we also provide a sample proof. In our first theorem, j is the only parameter.

Theorem 3.1. *Let $j \geq 1$ be an integer. Then*

$$S_1(n,j) = -\frac{\sin n \cdots \sin(n+j-1)}{2^n \cos^n(j)}, \textit{ and} \tag{3.1}$$

$$S_2(n,j) = -\left[\frac{\cos i \cdots \cos(i+j-1)}{2^i \cos^i(j)}\right]_0^n. \tag{3.2}$$

In our second theorem, the parameters are j and k.

Theorem 3.2. *Let $j \geq 1$ be an integer. Then, with the restrictions imposed on k in the definitions of S_3, S_4, S_5, and S_5, we have*

$$S_3(n,j,k) = -\frac{1}{\sin(jk)} \frac{\sin^{n+1} 1 \sin(kn) \cdots \sin(k(n+j-1))}{\sin^n(jk+1)}, \tag{3.3}$$

$$S_4(n,j,k) = -\frac{1}{\sin(jk)} \left[\frac{\sin^{n+1} 1 \cos(ki) \cdots \cos(k(i+j-1))}{\sin^i(jk+1)}\right]_0^n, \tag{3.4}$$

$$S_5(n,j,k) = \frac{1}{\sin(jk)} \frac{(-1)^n \sin^{n+1} 1 \sin(kn) \cdots \sin(k(n+j-1))}{\sin^n(jk-1)}, \textit{ and} \tag{3.5}$$

$$S_6(n,j,k) = \frac{1}{\sin(jk)} \left[\frac{(-1)^i \sin^{i+1} 1 \cos(ki) \cdots \cos(k(i+j-1))}{\sin^i(jk-1)}\right]_0^n. \tag{3.6}$$

Each of (3.1)-(3.6) can be proved in the same manner. To illustrate the method, we now give a proof of (3.3). To proceed, we require the following two identities from elementary trigonometry:

$$\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}, \textit{ and} \tag{3.7}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{3.8}$$

A key identity that we require for the proof of (3.3) is

$$\sin(jk+1)\sin(kn) - \sin 1 \sin(k(n+j)) = \sin(jk)\sin(kn-1), \tag{3.9}$$

which is true for all real numbers j, k, and n. To prove (3.9), we apply (3.7) to each product on the left of (3.9), and then apply (3.8) to the result. We leave the details to the interested reader.

Keeping in mind our discussion at the end of Section 2 regarding how the summand is to be interpreted, we consider the cases $j = 1$ and $j > 1$ separately. Denote the right side of (3.3) by $r(n,j,k)$. Then, after a little algebra, we see that

$$\begin{aligned}
&r(n+1,1,k) - r(n,1,k) \\
&= \left(\frac{\sin 1}{\sin(k+1)}\right)^{n+1} \frac{\sin(k+1)\sin(kn) - \sin 1 \sin(k(n+1))}{\sin k} \\
&= \left(\frac{\sin 1}{\sin(k+1)}\right)^{n+1} \sin(kn-1) \text{ by (3.9)} \\
&= S_3(n+1,1,k) - S_3(n,1,k).
\end{aligned} \tag{3.10}$$

Furthermore, it is immediate that

$$r(1,1,k) = S_3(1,1,k) = -\frac{\sin^2 1}{\sin(k+1)}. \tag{3.11}$$

Together, (3.10) and (3.11) prove (3.3) for $j = 1$.

For convenience, denote the left side of (3.9) by $d(n, j, k)$. For $j > 1$, we have

$$\begin{aligned}
&r(n+1, j, k) - r(n, j, k) \\
&= \left(\frac{\sin 1}{\sin(jk+1)}\right)^{n+1} \frac{\sin(k(n+1)) \cdots \sin(k(n+j-1)) d(n, j, k)}{\sin(jk)} \\
&= \left(\frac{\sin 1}{\sin(k+1)}\right)^{n+1} \sin(k(n+1)) \cdots \sin(k(n+j-1)) \sin(kn-1) \text{ by } (3.9) \\
&= S_3(n+1, j, k) - S_3(n, j, k).
\end{aligned} \tag{3.12}$$

Also, for $j > 1$,

$$r(1, j, k) = S_3(1, j, k) = -\frac{\sin^2 1}{\sin(jk+1)} \sin k \cdots \sin(k(j-1)). \tag{3.13}$$

Together, (3.12) and (3.13) complete the proof of (3.3).

The proofs of (3.1), (3.2), (3.4), (3.5), and (3.6) follow along similar lines. In each case, we require an identity that is analogous to (3.9). To aid the interested reader, we record each of these five identities below. They are, respectively,

$$\begin{aligned}
2\cos j \sin n - \sin(n+j) &= \sin(n-j), \\
2\cos j \cos n - \cos(n+j) &= \cos(n-j), \\
\sin(jk+1)\cos(kn) - \sin 1 \cos(k(n+j)) &= \sin(jk)\cos(kn-1), \\
\sin(jk-1)\sin(kn) + \sin 1 \sin(k(n+j)) &= \sin(jk)\sin(kn+1), \textit{and} \\
\sin(jk-1)\cos(kn) + \sin 1 \cos(k(n+j)) &= \sin(jk)\cos(kn+1).
\end{aligned} \tag{3.14}$$

Although the identities in (3.14) are probably not original, their proofs could serve as exercises for students studying elementary trigonometry.

4. Special Cases of (3.1)–(3.6)

In this section, we consider certain special cases of (3.1)-(3.6).

Let $j = 1$. Then (3.1) and (3.2) become, respectively,

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\sin(i-2)}{2^i \cos^i 1} &= -\frac{\sin n}{2^n \cos^n 1}, \textit{and} \\
\sum_{i=1}^{n} \frac{\cos(i-2)}{2^i \cos^i 1} &= -\left[\frac{\cos i}{2^i \cos^i 1}\right]_0^n.
\end{aligned}$$

Staying with (3.1) and (3.2), and setting $j = 2$, we obtain, respectively,

$$\sum_{i=1}^{n} \frac{\sin i \sin(i-3)}{2^i \cos^i 2} = -\frac{\sin n \sin(n+1)}{2^n \cos^n 2}, \textit{and} \tag{4.1}$$

$$\sum_{i=1}^{n} \frac{\cos i \cos(i-3)}{2^i \cos^i 2} = -\left[\frac{\cos i \cos(i+1)}{2^i \cos^i 2}\right]_0^n. \tag{4.2}$$

Let $(j,k)=(3,1)$. Then (3.3) and (3.4) become, respectively,

$$\sum_{i=1}^{n}\left(\frac{\sin 1}{\sin 4}\right)^{i}\sin i\sin(i+1)\sin(i-2)=-\frac{\sin 1}{\sin 3}\left(\frac{\sin 1}{\sin 4}\right)^{n}\sin n\sin(n+1)\sin(n+2),\mathit{and}$$

$$\sum_{i=1}^{n}\left(\frac{\sin 1}{\sin 4}\right)^{i}\cos i\cos(i+1)\cos(i-2)=-\frac{\sin 1}{\sin 3}\left[\left(\frac{\sin 1}{\sin 4}\right)^{i}\cos i\cos(i+1)\cos(i+2)\right]_{0}^{n}.$$

Finally for this section, with $(j,k)=(3,1)$, (3.5) and (3.6) become, respectively,

$$\sum_{i=1}^{n}\left(-\frac{\sin 1}{\sin 2}\right)^{i}\sin^{2} i\sin(i+1)=\frac{\sin 1}{\sin 3}\left(-\frac{\sin 1}{\sin 2}\right)^{n}\sin n\sin(n+1)\sin(n+2),\mathit{and}$$

$$\sum_{i=1}^{n}\left(-\frac{\sin 1}{\sin 2}\right)^{i}\cos^{2} i\cos(i+1)=\frac{\sin 1}{\sin 3}\left[\left(-\frac{\sin 1}{\sin 2}\right)^{i}\cos i\cos(i+1)\cos(i+2)\right]_{0}^{n}.$$

5. Concluding Comments

To present this paper succinctly, we have chosen to present all our results in abbreviated form. We now indicate how our results can be expressed in their most general form. Let θ be any real number that is not a rational multiple of π. This condition on θ eliminates the possibility of vanishing denominators. Then this entire paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ. For instance, the generalized forms of the sums (4.1) and (4.2) are, respectively,

$$\sum_{i=1}^{n}\frac{\sin(i\theta)\sin((i-3)\theta)}{2^{i}\cos^{i}(2\theta)}=-\frac{\sin(n\theta)\sin((n+1)\theta)}{2^{n}\cos^{n}(2\theta)},\mathit{and}$$

$$\sum_{i=1}^{n}\frac{\cos(i\theta)\cos((i-3)\theta)}{2^{i}\cos^{i}(2\theta)}=-\left[\frac{\cos(i\theta)\cos((i+1)\theta)}{2^{i}\cos^{i}(2\theta)}\right]_{0}^{n}.$$

References

[1] R. S. Melham, *Closed forms for finite sums of weighted products of the sine and cosine functions*, The Fibonacci Quarterly, **55.2** (2017), 123–128.

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