CONDITIONAL (STRONG) DIVISIBILITY SEQUENCES

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Abstract. A conditional recurrence sequence $\{q_n\}$ is one in which the recurrence satisfied by q_n depends on the residue of n modulo some integer $r \geq 2$. If a conditional sequence $\{q_n\}$ is a (strong) divisibility sequence then we define it as a *conditional (strong) divisibility sequence*. In this paper, we find some families of the conditional (strong) divisibility sequences for $r = 2$. These sequences are a generalization of the best known (strong) divisibility sequences in the literature, such as the Fibonacci sequence, the Lucas sequence, the Lehmer sequence, etc. Also, they contain some new fourth-order linear divisibility sequences which are different from the ones in the literature. An open problem is to determine the conditional (strong) divisibility sequences for $r > 2$.

1. Introduction

A sequence of rational integers $\{a_n\}$ is said to be a *divisibility sequence (DS)* if $m \mid n$ whenever $a_m \mid a_n$ and it is said to be a *strong divisibility sequence (SDS)* if $\gcd(a_m, a_n) = a_{\gcd(m,n)}$. These sequences are of particular interest because of their remarkable factorization properties and usage in applications, such as factorization problem, primality testing, etc. The best known examples are the Fibonacci sequence, the Lucas sequence, the Lehmer sequence, Vandermonde sequences, resultant sequences and their divisors, elliptic divisibility sequences, etc.

Kimberling [5] asked which recurrent sequences $\{a_n\}$ are divisibility or strong divisibility sequences. Lucas studied second order divisibility sequences of integers in [7]. Lehmer extended it to some fourth-order linear divisibility sequences in [6]. Williams and Guy found some other fourth-order linear divisibility sequences in [12].

Let $\{a_{i,j}\}$ be rational numbers for $0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant s$, and define a sequence $\{q_n\}$ with given initial terms q_i, $0 \leq i \leq s-1$, and for $n \geqslant s$

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + \cdots + a_{0,s}q_{n-s}, & \text{if } n \equiv 0 \pmod{r}; \\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + \cdots + a_{1,s}q_{n-s}, & \text{if } n \equiv 1 \pmod{r}; \\ \vdots & \vdots \\ a_{r-1,1}q_{n-1} + a_{r-1,2}q_{n-2} + \cdots + a_{r-1,s}q_{n-s}, & \text{if } n \equiv r-1 \pmod{r}. \end{cases} \tag{1.1}$$

Daniel et al. called such a sequence a "*general conditional recurrence sequence*" in [10]. In this paper, we ask Kimberling's question for conditional recurrence sequences, that is, we ask which conditional sequence $\{q_n\}$ are (strong) divisibility sequences. We consider the case of $(r, s) = (2, 2)$ and find some families of conditional (strong) divisibility sequences. If we take $(r, s) = (2, 2)$ in (1.1) then we obtain the sequence $\{f_n\}$ with initial conditions $f_0 = 0$, $f_1 = 1$ and for $n \geq 2$,

$$f_n = \begin{cases} a_1 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even}; \\ a_2 f_{n-1} + b_2 f_{n-2}, & \text{if } n \text{ is odd}. \end{cases} \tag{1.2}$$

From now on, we assume that a_1, a_2, b_1, and b_2 are nonzero integers. We will obtain some families of the conditional (strong) divisibility from the sequence $\{f_n\}$.

For $(r,s)=(2,2)$, the (strong) conditional divisibility sequences obtained from $\{f_n\}$ can be considered as a generalization of the second order (strong) divisibility sequences. Also, they contain some new families of fourth order linear (strong) divisibility sequences which are different from ones in [6] and [12]. The following are some special conditional (strong) divisibility examples obtained from the sequence $\{f_n\}$.

(1) If $a_1=b_1=a_2=b_2=1$, we obtain the Fibonacci sequence [$A117567$] which is SDS.
(2) If $a_1=a_2=k$ and $b_1=b_2=1$, we obtain the k-generalized Fibonacci sequence which is SDS.
(3) If $a_1=a_2=1$ and $b_1=b_2=2$, we obtain the Jacobsthal sequence which is SDS.
(4) If $a_1=a_2=2$ and $b_1=b_2=1$, we obtain the Pell sequence which is SDS.
(5) If $b_1=b_2=1$ and a_1,a_2 are non-zero numbers, we obtain a SDS sequence which is studied in [4].

In addition, Lehmer numbers are the fourth order strong divisibility sequence and they are also a special case of conditional strong divisibility sequences obtained from $\{f_n\}$. In the following table we give some more SDS examples, which appear in Sloane's *On-Line Encyclopedia of Integer Sequences*, for $(r,s)=(2,2)$.

(a_1,b_1,a_2,b_2)	Sequence	Name
$(1,1,2,1)$	[$A002530$]	Lehmer numbers with parameters $R=2$ and $Q=-1$
Even indices of $(1,2,3,4)$	[$A023001$]	$\frac{8^n-1}{7}$ (see Example 2.8)
$(2,1,0,1)$	[$A124625$]	Even numbers sandwiched between 1's
$(2,-1,0,-1)$	[$A009531$]	Expansion of the e.g.f. $\sin(x)(1+x)$
$(2,3,1,3)$	[$A174988$]	
$(1,3,2,3)$	[$A002536$]	(see Example 2.15)

In Section 2, we study the sequence $\{f_n\}$ and determine when $\{f_n\}$ is a divisibility or strong divisibility sequence so that we get some families of divisibility sequences and strong divisibility sequences from the sequence $\{f_n\}$. In Section 3, we define the sequence $\{l_n\}$ by changing the initial terms in $\{f_n\}$ and we similarly obtain some other families of divisibility and strong divisibility sequences from the conditional sequence $\{l_n\}$.

2. The Fibonacci-Like Conditional Sequence $\{f_n\}$

We can get the following results for the sequence $\{f_n\}$ by taking $r=2$ in Theorems 5, 6, and 9 in [9].

For $n\geq 4$,

$$f_n=Af_{n-2}-Bf_{n-4} \tag{2.1}$$

where $A:=a_1a_2+b_1+b_2$ and $B:=b_1b_2$.

The generating function of the sequence $\{f_n\}$ is

$$F(x)=\frac{x+a_1x^2-b_1x^3}{1-Ax^2+Bx^4}. \tag{2.2}$$

By using (2.2), the Binet's formulas for the sequence $\{f_n\}$ are given:

$$f_{2m}=a_1\frac{\alpha^m-\beta^m}{\alpha-\beta}, \tag{2.3}$$

$$f_{2m+1}=(a_1a_2+b_2)\frac{\alpha^m-\beta^m}{\alpha-\beta}-(b_1b_2)\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}, \tag{2.4}$$

where $\alpha = \frac{A+\sqrt{A^2-4B}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4B}}{2}$ that is, α and β are the roots of the polynomial $p(z) = z^2 - Az + B$.

We will get some divisibility properties of the conditional sequences $\{f_n\}$ and we find some conditional (strong) divisiblity sequences by using these properties. We will study two cases for the sequences $\{f_n\}$. In the first case, we only consider the even indices of the sequences $\{f_n\}$ and in the second case, we take $b_1 = b_2$ in $\{f_n\}$.

2.1. **The case of even indices of $\{f_n\}$.** Along this subsection, we consider even number indices of $\{f_n\}$ unless otherwise stated.

Corollary 2.1. *If m is a positive even integer, then $a_1 \neq 0$ always divides f_m.*

Proof. Since f_m and $a_1 \neq 0$ are integers, we can clearly get the result by using (2.3). □

Lemma 2.2. *The terms of the sequence $\{f_n\}$ satisfy*

$$a_1 f_{m+n} = f_m f_{n+2} - b_1 b_2 f_n f_{m-2}$$

for any positive even integers m and n.

Proof. Since m and n are even integers, we have $m = 2k$ and $n = 2s$ for some integer k and s. By using the Binet-like formula for even indices of the sequence $\{f_n\}$ and the identity $\alpha\beta = b_1 b_2$, we get

$$\begin{aligned}
f_m f_{n+1} - b_1 b_2 f_n f_{m-1} &= a_1^2 \frac{\left(\alpha^k - \beta^k\right)\left(\alpha^{s+1} - \beta^{s+1}\right)}{(\alpha-\beta)^2} - b_1 b_2 a_1^2 \frac{\left(\alpha^s - \beta^s\right)\left(\alpha^{k-1} - \beta^{k-1}\right)}{(\alpha-\beta)^2} \\
&= a_1^2 \frac{\alpha^{k+s+1} - \alpha^{s+1}\beta^k - \alpha^s\beta^{s+1} + \beta^{k+s+1}}{(\alpha-\beta)^2} \\
&\quad - a_1^2 \frac{\alpha\beta\left(\alpha^{k+s-1} - \alpha^{k-1}\beta^s - \alpha^s\beta^{k-1} + \beta^{k+s-1}\right)}{(\alpha-\beta)^2} \\
&= a_1^2 \frac{\alpha^{k+s+1} - \alpha\beta^{k+s} - \alpha^{k+s}\beta + \beta^{k+s+1}}{(\alpha-\beta)^2} \\
&= a_1^2 \frac{\alpha^{k+s} - \beta^{k+s}}{\alpha-\beta} = a_1 f_{m+n}.
\end{aligned}$$

□

Theorem 2.3. *If m and n are even positive integers, then we have*

$$m \mid n \Rightarrow f_m \mid f_n.$$

Proof. If m and n is even then $m = 2a$ and $n = 2b$ for an integer a, b. If m divides n then we can write $2b = 2ak$ for some positive integer k. We shall use induction on k to prove the theorem. First we need to show that the statement of the theorem holds for $k = 1$. If $k = 1$ then, $2b = 2a$ so that $f_m \mid f_n$. Assume that it holds for positive integer k, that is we have $f_m \mid f_{mk}$. By Lemma 2.2, we get

$$f_{2a(k+1)} = f_{2ak+2a} = f_{2ak} \frac{f_{2(a+1)}}{a_1} - b_1 b_2 f_{2a} \frac{f_{2(ak-1)}}{a_1}.$$

Here $\frac{f_{2(ak-1)}}{a_1}$ and $\frac{f_{2(a+1)}}{a_1}$ are integers due to Corollary 2.1. So, we get the desired result

$$f_m \mid f_{m(k+1)}$$

by using $f_m \mid f_n$ and $f_m \mid f_{mk}$ in the above equation. □

By Theorem 2.3, the even indices of a conditional sequence $\{f_n\}$ is a divisibility sequence (DS).

Theorem 2.4. *If m is even, then*

$$\gcd(f_{m+2}, f_m) \mid a_1\lambda^{\omega}$$

for some non-negative integer ω, where $\lambda = \gcd(A, B)$.

Proof. Let m be an even integer such that $m = 2n$. We have $f_0 = 0$, $f_2 = a_1$ and

$$f_{2(n+1)} = Af_{2n} - Bf_{2(n-1)} \tag{2.5}$$

for positive integer n by (2.1). Assume that p is a prime and a common divisor of $f_{2(n+1)}$ and f_{2n} such that $\gcd(p, \lambda) = 1$. We prove the result by breaking the proof into two cases.

Case 1. $\gcd(p, B) = 1$. We have $p|f_{2(n+1)}$ and $p|f_{2n}$, so we get $p|Bf_{2(n-1)}$ by (2.5). Also, since p and B are relatively prime, we get $p|f_{2(n-1)}$. We have

$$f_{2n} = Af_{2(n-1)} - Bf_{2(n-2)} \tag{2.6}$$

by (2.1). Since $p|f_{2n}$ and $p|f_{2(n-1)}$ we get $p|Bf_{2(n-2)}$ by (2.6). Similarly, we get $p|f_{2(n-2)}$ by $\gcd(p, B) = 1$. If we continue in this way then we ultimately get $p|f_2 = a_1$. Also we have $a_1|f_{2n}$ for a non-negative integer n by Corollary 2.1. As a result $\gcd(f_{m+2}, f_m) = a_1\lambda^{\omega}$ for some non-negative integer ω.

Case 2. $\gcd(p, B) \neq 1$. In this case, we have $p|B$, so $p|Bf_{2(n-2)}$. Since $p|f_{2n}$ and $p|Bf_{2(n-2)}$ we get $p|Af_{2(n-1)}$ by (2.6). Since $\gcd(p, \lambda) = 1$ and $\gcd(p, B) \neq 1$ we obtain $\gcd(p, A) = 1$. Now we have $p|Af_{2(n-1)}$ and $\gcd(p, A) = 1$, so we get $p|f_{2(n-1)}$. Similarly, we have

$$f_{2(n-1)} = Af_{2(n-1)} - Bf_{2(n-3)}$$

by (2.1). We have $p|f_{2(n-1)}$ and $p|Bf_{2(n-3)}$ (since $p|B$) so $p|Af_{2(n-2)}$. Since $\gcd(p, A) = 1$ we get $p|f_{2(n-2)}$. If we continue in this way, we get $p|f_2 = a_1$. As a result, we get the desired result. □

Corollary 2.5. *If $a_1 = 1$ and $\gcd(A, B) = 1$, then*

$$\gcd(f_m, f_{m+2}) = 1$$

for positive even integer m.

Proof. The desired result is obtained by taking $a_1 = 1$ and $\gcd(A, B) = 1$ in Theorem 2.4 □

The following theorem is trivial by (2.3), it follows from the paper of Carmichael in [3]. But we will provide a proof of it.

Theorem 2.6. *If $a_1 = 1$ and $\gcd(A, B) = 1$ then*

$$\gcd(f_m, f_n) = f_{\gcd(m,n)}$$

for positive even integers m and n.

Proof. Let $d_1 = \gcd(m, n)$ and $d_2 = \gcd(f_m, f_n)$ for positive even integers m and n. Since $d_1|m$ and $d_1|n$, we get $f_{d_1}|f_m$ and $f_{d_1}|f_n$ by the fact that $\{f_n\}$ is a divisibility sequence for even indices. So, we get

$$f_{d_1}|d_2 \tag{2.7}$$

by the definition of greatest common divisor. Since $d_1 = \gcd(m, n)$, there exists integers a and b such that $d_1 = am + bn$. Since d_1, m, and n are positive integers, we must have either $a \leq 0$ or $b \leq 0$. Assume without loss of generality $a \leq 0$. There exists a $k \geq 0$ such that

$a = -k$. So we can rearrange the above equality as $bn = d_1 + km$. Now if we use Lemma 2.2 by taking $bn = d_1 + km$ and $a_1 = 1$, we get

$$f_{d_1+km} = f_{d_1} f_{km+2} - b_1 b_2 f_{km} f_{d_1-2}. \tag{2.8}$$

Since $d_2 = \gcd(f_m, f_n)$, we have $d_2|f_m$ and $d_2|f_n$. Also, since even indices of $\{f_n\}$ are a divisibility sequence, we have $f_m|f_{mk}$ and $f_n|f_{bn}$. Thus, we obtain

$$d_2|f_m : \text{and} : f_m|f_{km} \Rightarrow d_2|f_{km}$$

and

$$d_2|f_n : \text{and} : f_n|f_{bn} \Rightarrow d_2|f_{bn}.$$

Now, we have $d_2|f_{km}, d_2|f_{bn}$ and $\gcd(A, B) = 1$, so we get

$$d_2|f_{d_1} \tag{2.9}$$

by using Corollary 2.5 and equation 2.8. By using (2.7) and (2.9), we get the desired result

$$d_2 = f_{d_1} \Rightarrow \gcd(f_m, f_n) = f_{\gcd(m,n)}.$$

□

Recall that the generalized Fibonacci sequence $\{U_n\} = \{U_n(P, Q)\}$ is defined by parameters $P, Q \in \mathbb{Z}$ with initial conditions $U_0 = 0$, $U_1 = 1$, and for $n \geq 2$

$$U_n = U_n(P, Q) = PU_{n-1} - QU_{n-2}.$$

Note that as is well-known, if $gcd(P, Q) = 1$, then the sequence $\{U_n\}$ is strong divisibility sequences [7]. Indeed,

$$f_{2n} = a_1 U_n(A, B),$$

so $\{U_n\}$ is a special case of the sequence f_n.

Corollary 2.7. *If $gcd(A, B) = 1$, then the even indices of conditional sequence $\{f_n\}$ is a strong divisibility sequence.*

Proof.

$$\begin{aligned}\gcd(f_{2m}, f_{2n}) &= \gcd(a_1 U_m, a_1 U_n) = a_1 \gcd(U_m, U_n)\\ &= a_1 U_{\gcd(m,n)} = f_{2\gcd(m,n)} = f_{\gcd(2m,2n)}.\end{aligned}$$

□

Example 2.8. *If we take $a_1 = 1$, $b_1 = 2$, $a_2 = 3$, $b_2 = 4$ in $\{f_n\}$, we get*

$$f_n = \begin{cases} f_{n-1} + 2q_{n-2}, & \textit{if } n \textit{ is even;} \\ 3f_{n-1} + 4q_{n-2}, & \textit{if } n \textit{ is odd.} \end{cases}$$

In the following table we give the terms of the sequence for $1 \leq n \leq 10$.

n	***0***	*1*	***2***	*3*	***4***	*5*	***6***	*7*	***8***	*9*	***10***	...
f_n	***0***	*1*	***1***	*7*	***9***	*55*	***73***	*439*	***585***	*3511*	***4681***	....

Let the even indices of this sequence 0, 1, 9, 73, 585, 4681, ... *gives* [A023001]. *Since*

$$A = a_1 a_2 + b_1 + b_2 = 3 + 2 + 4 = 9$$

and

$$B = b_1 b_2 = 8,$$

we have $\gcd(A, B) = \gcd(9, 8) = 1$. *Also, since $a_1 = 1$, this sequence is a strong divisibility sequence by Theorem 2.6.*

Example 2.9. *If we take* $a_1 = 1$, $b_1 = 3$, $a_2 = 2$, $b_2 = 4$ *in* $\{f_n\}$, *we obtain*

$$f_n = \begin{cases} f_{n-1} + 3f_{n-2}, & \text{if } n \text{ is even;} \\ 2f_{n-1} + 4f_{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

So the terms of the sequence are

n	**0**	1	**2**	3	**4**	5	**6**	7	**8**	9	**10**	...
f_n	**0**	1	**1**	6	**9**	42	**69**	306	**513**	2250	**3789**	....

Let the even indices of this sequence 0, 1, 6, 9, 69, 513, 3789, ... *Since* $\gcd(f_4, f_6) = 3 \neq 1 = f_2 = f_{\gcd(6,4)}$, *it is not a strong divisibility sequence. Note that the hypothesis of Theorem 2.6 is not satisfied since* $A = 2+3+4 = 9$ *and* $B = 12$ *so that* $\gcd(A, B) = \gcd(9, 12) = 3 \neq 1$.

2.2. **The Case** $b_1 = b_2$ **in** $\{f_n\}$**.** In this subsection, the indices of $\{f_n\}$ are non-negative integers with $b_1 = b_2$.

Lemma 2.10. *Assume that* $a_1 \neq 0$. *The terms of the sequences* $\{f_n\}$ *satisfy the following.*

(i) *If* a *and* b *are odd then*

$$f_{a+b} = f_a f_{b+1} + b_1 f_{a-1} f_b.$$

(ii) *If* a *is odd and* b *is even then*

$$f_{a+b} = f_a f_{b+1} + \frac{b_1 a_2}{a_1} f_{a-1} f_b.$$

(iii) *If* a *is even and* b *is odd then*

$$f_{a+b} = \frac{a_2}{a_1} f_a f_{b+1} + b_1 f_{a-1} f_b.$$

(iv) *If* a *and* b *are even then*

$$f_{a+b} = f_a f_{b+1} + b_1 f_{a-1} f_b.$$

Proof. We only prove the identity (i). The other identities can be proven similarly.

(i) If a and b are odd then $a = 2k + 1$ and $b = 2s + 1$ for some integer k and s, respectively. By using identities $a_1 a_2 + b_1 = \alpha + \beta - b_1, \alpha\beta = b_1 b_2$ and $b_1 = b_2$, we get

$$\begin{aligned} f_a &= (\alpha + \beta - b_1) \frac{\alpha^k - \beta^k}{\alpha - \beta} - \alpha\beta \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}, \\ f_{b+1} &= a_1 \frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta}, \\ f_{a-1} &= a_1 \frac{\alpha^k - \beta^k}{\alpha - \beta}, \\ f_b &= (\alpha + \beta - b_1) \frac{\alpha^s - \beta^s}{\alpha - \beta} - \alpha\beta \frac{\alpha^{s-1} - \beta^{s-1}}{\alpha - \beta}. \end{aligned}$$

We want to prove the identity

$$f_{a+b} = f_a f_{b+1} + b_1 f_{a-1} f_b$$

for odd a and b. We denote the right-hand side of this identity by RHS.

$$\begin{aligned} RHS = &\left(\frac{(\alpha+\beta)\left(\alpha^k-\beta^k\right)}{\alpha-\beta} - \frac{\alpha\beta\left(\alpha^{k-1}-\beta^{k-1}\right)}{\alpha-\beta}\right) a_1 \frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta} \\ &- \frac{a_1 b_1\left(\alpha^k-\beta^k\right)\left(\alpha^{s+1}-\beta^{s+1}\right)}{(\alpha-\beta)^2} - \frac{a_1 b_1^2\left(\alpha^k-\beta^k\right)\left(\alpha^s-\beta^s\right)}{(\alpha-\beta)^2} \\ &+ \frac{a_1 b_1\left(\alpha^k-\beta^k\right)}{\alpha-\beta}\left(\frac{(\alpha+\beta)\left(\alpha^s-\beta^s\right)}{\alpha-\beta} - \frac{\alpha\beta\left(\alpha^{s-1}-\beta^{s-1}\right)}{\alpha-\beta}\right). \end{aligned}$$

If we expand and arrange the RHS, we get

$$\begin{aligned} RHS &= a_1 \frac{\alpha^{k+1}\alpha^{s+1} + \beta^{k+1}\beta^{s+1} - \beta\alpha^{k+1}\alpha^s - \alpha\beta^{k+1}\beta^s}{(\alpha-\beta)^2} \\ &= a_1 \frac{\alpha\alpha^{k+s+1} + \beta\beta^{k+s+1} - \beta\alpha^{k+s+1} - \alpha\beta^{k+s+1}}{(\alpha-\beta)^2} \\ &= a_1 \frac{(\alpha-\beta)\left(\alpha^{k+s+1}-\beta^{k+s+1}\right)}{(\alpha-\beta)^2} \\ &= a_1 \frac{\left(\alpha^{k+s+1}-\beta^{k+s+1}\right)}{\alpha-\beta} \\ &= f_{a+b}. \end{aligned}$$

□

Theorem 2.11. *If $m, n \in \mathbb{Z}^+$, then we have*

$$m \mid n \Rightarrow f_m \mid f_n.$$

Proof. In order to show this, we break it into 4 cases.

(1) m and n are even.
By using Theorem 2.3, we obtain the desired result.

(2) m is odd and n is even.
If $m \mid n$, m odd and n even then we can write $n = m(2k)$ for some integer k. We want to show that $f_m \mid f_{m(2k)} = f_n$. Since

$$f_n = f_{m(2k)} = f_{mk+mk} = f_{mk} f_{mk+1} + b_1 f_{mk-1} f_{mk}$$

by part (i) of Lemma 2.10 (if k is odd) or part (iv) of Lemma 2.10 (if k is even). So, we have $f_{mk} \mid f_{m(2k)}$ for all positive inetegers k. That is, we have

$$f_m \mid f_{2m}, \quad f_{2m} \mid f_{4m}, \quad f_{3m} \mid f_{6m}, \quad \ldots.$$

Thus, we obtain the desired result from $f_m \mid f_n = f_{m(2k)}$.

(3) m and n are odd.
If m divides n then we can write $n = m(2k+1)$ for some integer k. Since $m(2k)$ is even and m is odd, we get

$$f_n = f_{m(2k+1)} = f_{m(2k)+m} = \frac{a_2}{a_1} f_{m(2k)} f_{m+1} + b_1 f_{m(2k)-1} f_m.$$

Now, using a_1 divides f_{m+1} since $m+1$ is even by Corollary 2.1, $f_m \mid f_m$ and $f_m \mid f_{m(2k)}$ (we proved this in part 2) in the above equation, we get the desired result $f_m \mid f_n = f_{m(2k+1)}$.

(4) m is even and n is odd.
This case is not possible since m divides n.

□

According to Theorem 2.11, if $b_1 = b_2$ then the sequence $\{f_n\}$ is a conditional divisibility sequence. So we find a family of divisibility sequences.

Theorem 2.12. *If n is a positive integer, then we have*

$$\gcd(f_n, f_{n-1}) \mid \lambda^{\omega}$$

for some non-negative integer ω, where $\lambda = \gcd(a_1a_2, b_1)$.

Proof. By the definition of the sequence $\{f_n\}$ and $b_1 = b_2$, we have

$$f_n = \begin{cases} a_1 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even;} \\ a_2 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.10}$$

Assume that p is a prime such that $p \mid f_n$ and $p \mid f_{n-1}$ with $gcd\,(p, \lambda) = 1$. We prove the theorem breaking it into two cases.

(i) $gcd\,(p, b_1) = 1$. Since $p \mid f_n$ and $p \mid f_{n-1}$, we get $p \mid b_1 f_{n-2}$ by (2.10). Then, we get $p \mid f_{n-2}$, since p and b_1 are relatively prime. By the definition of the sequence $\{f_n\}$, we also have

$$f_{n-1} = \begin{cases} a_1 f_{n-2} + b_1 f_{n-3}, & \text{if } n-1 \text{ is even;} \\ a_2 f_{n-2} + b_1 f_{n-3}, & \text{if } n-1 \text{ is odd.} \end{cases} \tag{2.11}$$

Now, we have $p \mid f_{n-1}$ and $p \mid f_{n-2}$, so we can get $p \mid b_1 f_{n-3}$ by (2.11). Similarly we get $p \mid f_{n-3}$, since $gcd\,(p, b_1) = 1$. If we continue the process in this way, we get $p \mid f_1 = 1$. Thus, we obtain the desired result.

(ii) $gcd\,(p, b_1) \neq 1$.
Since $gcd\,(p, \lambda) = 1$ and $gcd\,(p, b_1) \neq 1$, we must have $gcd\,(p, a_1a_2) = 1$. This means that $gcd\,(p, a_1) = 1$ and $gcd\,(p, a_2) = 1$. We have $p \mid f_{n-1}$ and $p \mid b_1 f_{n-3}$ (since $p \mid b_1$), so we get $p \mid a_1 f_{n-2}$ or $p|a_2 f_{n-2}$ according to the sign of $n-1$ by (2.11). Now, we have $gcd\,(p, a_1) = gcd\,(p, a_2) = 1$, so $p \mid f_{n-2}$ for both cases. Then, since

$$f_{n-2} = \begin{cases} a_1 f_{n-3} + b_1 f_{n-4}, & \text{if } n-2 \text{ is even;} \\ a_2 f_{n-3} + b_2 f_{n-4}, & \text{if } n-2 \text{ is odd;} \end{cases}$$

we can similarly get $p \mid f_{n-3}$. If we continue in this way, we finally get $p \mid f_1 = 1$. As a result, we get the desired result.

□

Corollary 2.13. *If $\gcd(a_1a_2, b_1) = 1$, then*

$$\gcd(f_m, f_{m+1}) = 1$$

for any non-negative integer m.

Proof. The desired result is obtained by taking $\gcd(a_1a_2, b_1) = 1$ in Theorem 2.12. □

Theorem 2.14. *If $\gcd(a_1a_2, b_1) = 1$ and $a_1 = 1$, then*

$$\gcd(f_m, f_n) = f_{\gcd(m,n)}$$

for non-negative integers m and n.

Proof. Let $d_1 = \gcd(m,n)$ and $d_2 = \gcd(f_m, f_n)$ for positive even integers m and n. Since $d_1|m$ and $d_1|n$, we get $f_{d_1}|f_m$ and $f_{d_1}|f_n$ by Theorem 2.11. So, we obtain

$$f_{d_1}|d_2 \tag{2.12}$$

by the definition of the greatest common divisor. Since $d_1 = \gcd(m,n)$, there exists integers a and b such that $d_1 = am + bn$. Since d_1, m, and n are positive integers, we must have either $a \leq 0$ or $b \leq 0$. Assume without loss of generality $a \leq 0$. There exists a $k \geq 0$ such that $a = -k$. So, we can rearrange the above equality as $bn = d_1 + km$. Now, we break it into four cases according to whether d_1 and km are either even or odd.

(i) d_1 is odd and km is odd.

If we use Lemma 2.10 by taking $bn = d_1 + km$, we get

$$f_{d_1+km} = f_{d_1} f_{km+1} + b_1 f_{d_1-1} f_{km}. \tag{2.13}$$

Since $d_2 = \gcd(f_m, f_n)$, we have $d_2|f_m$ and $d_2|f_n$. Also, we have $f_m|f_{mk}$ and $f_n|f_{bn}$ Theorem 2.11. So, we get

$$d_2|f_m \text{ and } f_m|f_{km} \Rightarrow d_2|f_{km}$$

and

$$d_2|f_n \text{ and } f_n|f_{bn} \Rightarrow d_2|f_{bn}.$$

Now, we have $d_2|f_{km}$ and $d_2|f_{bn}$ so that $d_2|f_{d_1} f_{km+1}$ by equation 2.13. Also, since $\gcd(a_1 a_2, b_1) = 1$, so we must have

$$d_2|f_{d_1} \tag{2.14}$$

by using Corollory 2.13. Note that $\gcd(f_{km+1}, f_{km}) = 1$. By using (2.12) and (2.14), we get the desired result as follows:

$$d_2 = f_{d_1} \Rightarrow \gcd(f_m, f_n) = f_{\gcd(m,n)}.$$

(ii) d_1 is odd and km is even. If we take $a_1 = 1$ then we can prove it similar to case (i).

(iii) d_1 is even and km are odd. If km is odd then, k and m are both odd. If m is odd then $d_1 = \gcd(m,n)$ is odd, that is contradiction. So, this case is not possible.

(iv) d_1 and km are even.

It is clear, since the even indices of the sequence $\{f_n\}$ is a conditional strong divisibility sequence by previous subsection.

□

Now recall that the Lehmer sequence $\{u_n\} = \{u_n(R,Q)\}$ satisfies the second order linear recurrence equations with initial conditions $u_0 = 0$, $u_1 = 1$, and for $n \geq 2$

$$u_n = u_n(R,Q) = \begin{cases} u_{n-1} - Qu_{n-2}, & \text{if } n \text{ is even;} \\ Ru_{n-1} - Qu_{n-2}, & \text{if } n \text{ is odd;} \end{cases}$$

and the associated Lehmer sequence $\{v_n\} = \{v_n(R,Q)\}$ satisfies the second order linear recurrence equations with initial conditions $v_0 = 2$, $v_1 = 1$, and for $n \geq 2$

$$v_n = v_n(R,Q) = \begin{cases} Rv_{n-1} - Qv_{n-2}, & \text{if } n \text{ is even;} \\ v_{n-1} - Qv_{n-2}, & \text{if } n \text{ is odd} \end{cases}$$

If we take $R = a_1 a_2$ and $Q = -b_1$ in Lehmer numbers, we get

$$f_n = \begin{cases} a_1 u_n(a_1 a_2, -b_1), & \text{if } n \text{ is even;} \\ u_n(a_1 a_2, -b_1), & \text{if } n \text{ is odd.} \end{cases}$$

It is well-known that both Lehmer and associated Lehmer sequences are linear recurrence sequences of order at most four. If $\gcd(R,Q)=1$, Lehmer sequence $\{u_n(R,Q)\}$ is a strong divisibility sequence [6]. Using this fact, Bala showed that if $\gcd(a_1a_2,b_1)=1$, then $\{f_n\}$ with $b_1=b_2$ is a strong divisibility sequence in [1]. The Lehmer sequence is a special case of $\{f_n\}$.

Example 2.15. *If we take $a_1=1$, $a_2=2$, $b_1=b_2=3$ in $\{f_n\}$, we get*

$$f_n=\begin{cases} f_{n-1}+3q_{n-2}, & \text{if } n \text{ is even;}\\ 2f_{n-1}+3q_{n-2}, & \text{if } n \text{ is odd.}\end{cases}$$

In the following table we give the terms of the sequence for $1\le n\le 10$

n	0	1	2	3	4	5	6	7	8	9	10	...
f_n	0	1	1	5	8	31	55	203	368	1345	2449	....

The terms of the sequence 0, 1, 1, 5 ,8, 31, 55, 203, 368, 1345, ... *gives* [A002536]. *Since we have* $\gcd(a_1a_2,b_1)=\gcd(2,3)=1$ *and* $a_1=1$, *this sequence is a strong divisibility sequence by Theorem 2.14.*

3. The Lucas-Like Conditional Sequence $\{l_n\}$

We define the *Lucas conditional sequence* $\{l_n\}$ with the initial conditions $l_0=2$, $l_1=a_2$, and for $n\ge 2$,

$$l_n=\begin{cases} a_1l_{n-1}+b_1l_{n-2}, & \text{if } n \text{ is even;}\\ a_2l_{n-1}+b_2l_{n-2}, & \text{if } n \text{ is odd;}\end{cases} \tag{3.1}$$

by changing initial values in (1.2). We can get the following results for the sequence $\{l_n\}$ by taking $r=2$ in Theorems 5, 6, and 9 in [9].

For $n\ge 4$,

$$l_n=Al_{n-2}-Bl_{n-4} \tag{3.2}$$

where $A:=a_1a_2+b_1+b_2$ and $B:=b_1b_2$.

The generating function of the sequence $\{l_n\}$ is

$$L(x)=\frac{2+a_2x-(a_1a_2+2b_2)x^2+b_1a_2x^3}{1-Ax^2+Bx^4}. \tag{3.3}$$

By using (3.3), the Binet's formulas for the sequence $\{l_n\}$ are given:

$$l_{2m}=(a_1a_2+2b_1)\frac{\alpha^m-\beta^m}{\alpha-\beta}-2(b_1b_2)\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta} \tag{3.4}$$

$$l_{2m+1}=a_2\left((a_1a_2+2b_1+b_2)\frac{\alpha^m-\beta^m}{\alpha-\beta}-b_1b_2\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\right) \tag{3.5}$$

where $\alpha=\frac{A+\sqrt{A^2-4B}}{2}$ and $\beta=\frac{A-\sqrt{A^2-4B}}{2}$ that is, α and β are the roots of the polynomial $p(z)=z^2-Az+B$.

Corollary 3.1. *If n is a odd positive integer, then $a_2\neq 0$ always divides l_n.*

Proof. Since l_n and $a_2\neq 0$ are integers, we can clearly get the result by using (3.5). □

Now, we will obtain some divisibility properties of the Lucas-like sequence $\{l_n\}$ and we will show some relations between $\{f_n\}$ and $\{l_n\}$. In this section we take $b_1=b_2$ in the Lucas-like sequence $\{l_n\}$.

Lemma 3.2. *Let $m>n\ge 0$ and $e:=\min\{m-n,n\}$. The terms of the sequences $\{l_n\}$ satisfy the following:*

(i) *If m and n are even then*
$$l_m = l_{m-n}l_n - b_1^e l_{|m-2n|}.$$
(ii) *If m and n are odd then*
$$l_m = l_{m-n}l_n \pm b_1^e l_{|m-2n|},$$
where $+$ sign is used if and only if $m - 2n \geq 0$.

(iii) *If m is odd and n is even then*
$$l_m = l_{m-n}l_n \pm b_1^e l_{|m-2n|},$$
where $+$ sign is used if and only if $m - 2n < 0$.

(iv) *If m is even and n is odd and $a_2 \neq 0$ then*
$$l_m = \frac{a_1}{a_2} l_{m-n}l_n + b_1^e l_{|m-2n|}.$$

Proof. Here we only prove identity (i). Other identities can be proven similarly using $\alpha + \beta = a_1a_2 + 2b_1$ and $\alpha\beta = b_1^2$.

(i) Let $m := 2k$, $n := 2s$; for some $k, s \in \mathbb{Z}^+$. If $e := \min\{m - n, n\} = n$ then $m - 2n \geq 0$ and $|m - 2n| = m - 2n$.
$$\begin{aligned}
l_m &= l_{2k} = \alpha^k + \beta^k \\
l_n &= l_{2s} = \alpha^s + \beta^s \\
l_{m-n} &= l_{2(k-s)} = \alpha^{k-s} + \beta^{k-s} \\
l_{m-2n} &= l_{2(k-2s)} = \alpha^{k-2s} + \beta^{k-2s} \\
b_1^n l_{m-2n} &= b_1^{2s} l_{2(k-2s)} = (\alpha\beta)^s \left(\alpha^{k-2s} + \beta^{k-2s}\right) \\
&= \alpha^{k-s}\beta^s + \alpha^s\beta^{k-s}.
\end{aligned}$$
Therefore,
$$l_m = l_{m-n}l_n - b_1^n l_{m-2n}.$$
If $e := \min\{m - n, n\} = m - n$ then $m - 2n < 0$ and $|m - 2n| = 2n - m$.
$$\begin{aligned}
b_1^{m-n} l_{2n-m} &= b_1^{2(k-s)} l_{2(2s-k)} = (\alpha\beta)^{k-s} \left(\alpha^{2s-k} + \beta^{2s-k}\right) \\
&= \alpha^{k-s}\beta^s + \alpha^s\beta^{k-s}
\end{aligned}$$
Therefore,
$$l_m = l_{m-n}l_n - b_1^{m-n} l_{2n-m}.$$
□

Theorem 3.3. *If $m \geq 2$ and n are positive integers, then we have*
$$m \mid n \Rightarrow l_m \mid l_n,$$
where $n = (2k - 1)\, m$.

Proof. We break the proof into two cases.
Case 1. m is even. Since $n = (2k - 1)\, m$, n is even. Now we shall use induction on k. If $k = 1$ then $n = m$ so that $l_m \mid l_n$. Assume that the induction hypothesis holds for $2k - 1$, that is $l_m \mid l_{(2k-1)m}$. By part (i) of Lemma 3.2 (i), we have
$$l_{(2k+1)m} = l_{2km+m} = l_{2km}l_m - b_1^m l_{(2k-1)m}.$$

Since $l_m \mid l_m$ and $l_m \mid l_{(2k-1)m}$, we get the desired result $l_m \mid l_{(2k+1)m}$ by using the above equation.

Case 2. m is odd. Since $n = (2k-1)\,m$, n is odd. Similarly we use induction on k. If $k = 1$ then $n = m$ so that $l_m \mid l_n$. Assume that it holds for odd integer $2k-1$, so we have $l_m \mid l_{(2k-1)m}$. By part (ii) of Lemma 3.2 (iii), we have

$$l_{(2k+1)m} = l_{2km+m} = l_{2km} l_m + b_1^m l_{(2k-1)m}.$$

Since $l_m \mid l_m$ and $l_m \mid l_{(2k-1)m}$, we get the desired result $l_m \mid l_{(2k+1)m}$. □

Corollary 3.4. *If $m \geq 2$ and n are positive integers, then we have*

$$\gcd(l_m, l_n) = l_{\gcd(m,n)},$$

where $n = (2k-1)\,m$.

Proof. Since $n = (2k-1)\,m$, we get $\gcd(m,n) = m$. We have $l_{\gcd(m,n)} \mid l_n$ by Theorem 3.3 and $l_{\gcd(m,n)} \mid l_m$ since $\gcd(m,n) = m$. So, $l_{\gcd(m,n)} \mid \gcd(l_m, l_n)$. Also, since we have $\gcd(l_n, l_m) \mid l_{m=\gcd(m,n)}$, we get the desired result. □

Theorem 3.5. *Let $m = 2^a m'$, $n = 2^b n'$, m' and n' odd, a and $b \geq 0$, and let $d = \gcd(m,n)$. If $\gcd(a_1 a_2, b_1) = 1$ then*

$$\gcd(l_m, l_n) = \begin{cases} l_d, & \text{if } a = b; \\ 1 \text{ or } 2, & \text{if } a \neq b. \end{cases}$$

Proof. To prove the theorem, we need a result due to McDaniel [8]. As noted in [8, page 28], the formula remains valid for associated Lehmer sequences. Indeed, the associated Lehmer sequences $\{v_n(R,Q)\}$ with parameters $R = a_1 a_2$ and $Q = -b_1$ is a special case of the conditional sequence $\{l_n\}$for the case of $b_1 = b_2$, that is

$$l_n = \begin{cases} v_n(a_1 a_2, -b_1), & \text{if } n \text{ is even;} \\ a_2 v_n(a_1 a_2, -b_1), & \text{if } n \text{ is odd.} \end{cases}$$

There are two cases when $a = b$.

Case 1. m and n are odd. We have

$$\begin{aligned} \gcd(l_n, l_m) &= \gcd(a_2 v_n, a_2 v_m) \\ &= a_2 \gcd(v_n, v_m) \\ &= a_2 v_{\gcd(n,m)} \\ &= l_{\gcd(n,m)} \text{ since } \gcd(n,m) \text{ is odd.} \end{aligned}$$

Case 2. m and n are even. We have

$$\begin{aligned} \gcd(l_n, l_m) &= \gcd(v_n, v_m) \\ &= v_{\gcd(n,m)} \\ &= l_{\gcd(n,m)} \text{ since } \gcd(n,m) \text{ is even.} \end{aligned}$$

When $a \neq b$, the proof depends on the parities of n and m. Let us show that

$$\gcd(l_m, l_n) = 1 \text{ or } 2$$

in the case when m is odd and n is even. The remaining cases are similar.

First we prove

$$\gcd(v_{2n}, a_2) = 1 \text{ or } 2, \text{ for all positive integers } n$$

for the sequence $\{v_n(a_1 a_2, b_1)\}$, where a_2 is prime to v_n when n is even.

The proof is by induction. This is clearly true when $n = 1$ since $v_2 = a_1a_2 + 2b_1$. Assume that

$$\gcd(v_{2n}, a_2) = 1 \text{ or } 2 \text{ for some } n.$$

Using the recurrence relation of $\{v_n\}$, we find

$$\begin{aligned}\gcd(v_{2n+2}, a_2) &= \gcd(a_1a_2v_{2n+1} + b_1v_{2n}, a_2)\\ &= \gcd(b_1v_{2n}, a_2)\\ &= \gcd(v_{2n}, a_2) \text{ since by assumption } a_2 \text{is relatively prime to } b_1\\ &= 1 \text{ or } 2\end{aligned}$$

and the induction goes through.

By the choices of a_1a_2 and b_1, we get

$$\begin{aligned}\gcd(l_n, l_m) &= \gcd(v_n, a_2v_m)\\ &= \gcd(v_n, v_m), \text{ since } \gcd(v_n, a_2) = 1 \text{ or } 2 \text{ for } n \text{ is even}\\ &= 1 \text{ or } 2.\end{aligned}$$

□

3.1. **Some more divisibility properties.** The following lemma is a generalization of a famous identity.

Lemma 3.6. *If n is even, then $f_{2n} = f_nl_n$, otherwise $f_{2n} = \frac{a_1}{a_2}f_nl_n$.*

Proof. If n is even then $n = 2m$ for an integer m. We have

$$f_{2m}\left(\frac{b_2 - b_1}{a_1}f_{2m} + l_{2m}\right) = \frac{a_1(\alpha^m - \beta^m)\left(A(\alpha^m - \beta^m) - 2B\left(\alpha^{m-1} - \beta^m\right)\right)}{(\alpha - \beta)^2}$$

by using (2.3) and (3.4) where $A := a_1a_2 + b_1 + b_2$ and $B := b_1b_2$. If we substitute $\alpha + \beta = a_1a_2 + b_1 + b_2$ and $\alpha\beta = b_1b_2$ in the above equation, we get

$$\begin{aligned}f_{2m}\left(\frac{b_2 - b_1}{a_1}f_{2m} + l_{2m}\right) &= \frac{a_1(\alpha^m - \beta^m)\left(\alpha^{m+1} + \beta\alpha^m - \alpha\beta^m - \beta^{m+1} - 2\beta\alpha^m + 2\alpha\beta^m\right)}{(\alpha - \beta)^2}\\ &= \frac{a_1(\alpha^m - \beta^m)(\alpha - \beta)(\alpha^m + \beta^m)}{(\alpha - \beta)^2}\\ &= \frac{a_1\left(\alpha^{2m} - \beta^{2m}\right)}{(\alpha - \beta)}\\ &= f_{4m}.\end{aligned}$$

Since $b_1 = b_2$ we obtain the desired result

$$f_{2n} = f_nl_n.$$

Now assume that n is odd, so $n = 2m + 1$ for some integer m. If we use (2.3), (2.4), and (3.5) and identities $\alpha + \beta = a_1a_2 + b_1 + b_2$, $\alpha\beta = b_1b_2$, and $b_1 = b_2$ we obtain the desired result as follows:

$$\begin{aligned}\frac{a_1}{a_2}f_{2m+1}l_{2m+1} &= a_1\frac{\alpha^{2m+2} - \alpha^{2m+1}\beta - \alpha\beta^{2m+1} + \beta^{2m+2}}{(\alpha - \beta)^2}\\ &= a_1\frac{\alpha^{2m+1} - \beta^{2m+1}}{\alpha - \beta}\\ &= f_{4m+2}.\end{aligned}$$

□

Theorem 3.7. *If m is even then*

$$m \,|n \,\Rightarrow l_m \mid f_n$$

else

$$m \,|n \,\Rightarrow l_m \mid f_n \quad \left(\frac{a_1}{a_2} \textit{ is integer}\right)$$

where $n = 2km$ and $k \geq 1$.

Proof. Since m is even, we have $f_{2m} = f_m l_m$ by Lemma 3.6. So, $l_m \mid f_{2m}$. Now, we have $n = 2mk$ for $k \geq 1$. We shall use induction on k. If $k = 1$ then $n = 2m$ and $l_m \mid f_{2m}$, that is, it holds for $k = 1$. Assume that it holds for positive integer k, that is we have $l_m|f_{2mk}$. By Lemma 2.2, we have

$$f_{2m(k+1)} = f_{2mk+2m} = f_{2mk}\frac{f_{2(m+1)}}{a_1} - b_1 b_2 f_{2m}\frac{f_{2(mk-1)}}{a_1}.$$

Here, $\frac{f_{2(m+1)}}{a_1}$ and $\frac{f_{2(mk-1)}}{a_1}$ are integers by Corollary 2.1. Also, we have $l_m \mid f_{2m}$ and $l_m \mid f_{2mk}$, so we get the desired result $l_m \mid f_{2m(k+1)}$. □

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