ON THE D(4)-DIOPHANTINE TRIPLES OF FIBONACCI NUMBERS

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Abstract. Let F_m be the mth Fibonacci number. We prove that if $F_{2n+6}F_k + 4$ and $4F_{2n+4}F_k + 4$ are both perfect squares, then $k = 2n$ for $n \geq 1$, except in the case $n = 1$, in which we can additionally have $k = 1$.

1. Introduction

The sequence $\{F_n\}_{n\geq 1}$ of Fibonacci numbers is given by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 1.$$

A set of m positive integers $\{a_1, \ldots, a_m\}$ is called *$D(n) - m$-tuple* (or *a Diophantine m-tuple with the property $D(n)$*) if $a_i a_j + n$ is a perfect square, for all $i \neq j$ in $\{1, \ldots, m\}$. In 1993, Dujella [2] proved that $\{F_{2n}, F_{2n+6}, 4F_{2n+4}, 4F_{2n+2}F_{2n+3}F_{2n+5}\}$ is a $D(4)$-Diophantine quadruple. In 2010, Filipin, He, and Togbé [4] proved that if d is a positive integer such that $\{F_{2n}, F_{2n+6}, 4F_{2n+4}, d\}$ is a $D(4)$-Diophantine quadruple, then $d = 4F_{2n+2}F_{2n+3}F_{2n+5}$. In this paper, we fix the positive integer n and look at positive integers k such that $\{F_{2n+6}, 4F_{2n+4}, F_k\}$ is a $D(4)$-Diophantine triple. Our result is the following:

Theorem 1.1. *If $\{F_{2n+6}, 4F_{2n+4}, F_k\}$ is a $D(4)$-Diophantine triple, then $k = 2n$, except in the case $n = 1$, in which we have the additional solution $k = 1$.*

The exception $k = 1$ in case $n = 1$ comes from $F_1 = F_2$. A similar result was obtained by He, Luca, and Togbé (see [5]). The technique will be similar and we will organize this paper as follows. In Section 2, we recall some results useful for the remaining sections. Sections 3 through 6 help us prepare the proof of Theorem 1.1 using a combination of results on a linear form in three logarithms and a linear form in two logarithms to reduce the bounds of the parameters. The last section is devoted to the proof of Theorem 1.1.

2. Useful Lemmas

In this section, we will recall some results that will be useful in the next sections.

For any non-zero algebraic number γ of degree d over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}(X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log a + \sum_{j=1}^{d} \log \max(1, |\gamma^{(j)}|\right)$$

its absolute logarithmic height. We will use the following result due to Matveev [8].

Lemma 2.1. *Let Λ be a linear form in logarithms of multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_N$ with rational integer coefficients $b_1, \ldots, b_N$ ($b_N \neq 0$). Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \leq j \leq N$. Define the numbers*

D, A_j $(1 \le j \le N)$ and E by $D := [\mathbb{Q}(\alpha_1, \ldots, \alpha_N) : \mathbb{Q}]$, $A_j = \max\{Dh(\alpha_j), |\log \alpha_j|\}$, $E = \max\{1, \max\{|b_j|A_j/A_N; 1 \le j \le N\}\}$. *Then*

$$\log|\Lambda| > -C(N)C_0W_0D^2\Omega,$$

where

$$\begin{aligned} C(N) &:= \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1}, \\ C_0 &:= \log(e^{4.4N+7}N^{5.5}D^2\log(eD)), \\ W_0 &:= \log(1.5eED\log(eD)), \quad \Omega = A_1 \cdots A_N. \end{aligned}$$

We recall also the following result of Laurent [7].

Lemma 2.2. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \ge \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\} \text{ For } i = 1, 2, \quad b' \ge \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then

$$\log|\Lambda| \ge -17.9 \cdot D^4 \left(\max\left\{\log b' + 0.38, \frac{30}{D}, 1\right\}\right)^2 h_1h_2.$$

The following lemma is a slight modification of the original version of Baker-Davenport reduction method. (See [3], Lemma 5a).

Lemma 2.3. *Assume that κ and μ are real numbers and M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = ||\mu Q|| - M \cdot ||\kappa Q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(AQ/\eta)}{\log B} \le j \le M.$$

3. The Related Sequences

For any fixed positive integer n, we assume that there exist positive integers k, x, y such that

$$F_{2n+6}F_k + 4 = x^2, \quad 4F_{2n+4}F_k + 4 = y^2. \tag{3.1}$$

We eliminate F_k to obtain the norm form equation

$$F_{2n+6} \cdot y^2 - 4F_{2n+4} \cdot x^2 = 4(F_{2n+6} - 4F_{2n+4}). \tag{3.2}$$

To generate all solutions of this equation, we need the following lemma, which follows from Lemma 1 of [1].

Lemma 3.1. *Let a and b be positive integers satisfying $ab + 4 = r^2$ and $a < b < 5a$. All positive solutions of the equation*

$$ay^2 - bx^2 = 4(a-b)$$

are given by

$$y\sqrt{a} + x\sqrt{b} = (\pm 2\sqrt{a} + 2\sqrt{b})\left(\frac{r+\sqrt{ab}}{2}\right)^j, j \geq 0.$$

Since $4F_{2n+4}F_{2n+6} + 4 = (2F_{2n+5})^2$ and $F_{2n+6} < 4F_{2n+4} < 5F_{2n+6}$, Lemma 3.1 implies that all solutions of equation (3.2) are given by

$$y\sqrt{F_{2n+6}} + 2x\sqrt{F_{2n+4}} = (\pm 2\sqrt{F_{2n+6}} + 4\sqrt{F_{2n+4}})(F_{2n+5} + \sqrt{F_{2n+6}F_{2n+4}})^j, \quad j \geq 0.$$

Now, we define the sequence $(U_j)_{j\geq 0}$ and $(V_j)_{j\geq 0}$ by

$$V_j + U_j\sqrt{F_{2n+6}F_{2n+4}} := (F_{2n+5} + \sqrt{F_{2n+6}F_{2n+4}})^j.$$

Thus, we get

$$x = x_j = 2V_j \pm F_{2n+6}U_j. \tag{3.3}$$

Substituting (3.3) into the first equation of (3.1), we obtain

$$F_k = \pm 4V_jU_j + (F_{2n+6} + 4F_{2n+4})\,U_j^2. \tag{3.4}$$

This is the main equation that we will solve. Put

$$C_j^{(\pm)} := \pm 4V_jU_j + (F_{2n+6} + 4F_{2n+4})\,U_j^2, \quad \text{for } j = 1, 2, \ldots. \tag{3.5}$$

Therefore, we have to solve the equation

$$C_j^{(\pm)} = F_k, \tag{3.6}$$

for some positive integers j and k. One can notice that the above equation has the solution

$$C_1^{(-)} = F_{2n}. \tag{3.7}$$

That is exactly the solution stated in Theorem (1.1). Our aim will be to show that there are no other solutions. Since

$$F_{2n+9} < C_1^{(+)} = 5F_{2n+6} < F_{2n+10}$$

then, to get a contradiction, we will assume that $j \geq 2$ for the $+$ and $-$ cases.

Put

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\alpha} = \frac{1-\sqrt{5}}{2}.$$

Using Binet formula, we see that

$$F_k = \frac{\alpha^k - \overline{\alpha}^k}{\sqrt{5}}, \quad \text{for all } k \geq 1. \tag{3.8}$$

Put

$$\beta_n := F_{2n+5} + \sqrt{F_{2n+5}^2 - 1},$$

and

$$V_j := \frac{\beta_n^j + \beta_n^{-j}}{2}, \quad U_j := \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{F_{2n+5}^2 - 1}}. \tag{3.9}$$

It is obvious that V_j and U_j depend on n, but we will assume that n is fixed throughout the argument. Define

$$\gamma_n^{(\pm)} := \pm\frac{1}{\sqrt{F_{2n+5}^2 - 1}} + \frac{F_{2n+6} + 4F_{2n+4}}{4(F_{2n+5}^2 - 1)}. \tag{3.10}$$

We use formula (3.5) to deduce that

$$\begin{aligned} C_j^{(\pm)} &= \pm\frac{\beta_n^{2j} - \beta_n^{-2j}}{\sqrt{F_{2n+5}^2 - 1}} + (F_{2n+6} + 4F_{2n+4}) \cdot \frac{\beta_n^{2j} - 2 + \beta_n^{-2j}}{4(F_{2n+5}^2 - 1)} \\ &= \beta_n^{2j}\gamma_n^{(\pm)} - \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} + \beta_n^{-2j}\gamma_n^{(\mp)}. \end{aligned} \tag{3.11}$$

Therefore, the next equation comes from equation (3.6) when we use (3.8) and (3.11)

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} + \beta_n^{-2j}\gamma_n^{(\mp)} = \frac{\alpha^k - \overline{\alpha}^k}{\sqrt{5}}. \tag{3.12}$$

4. The Use of a Linear Form in Three Logarithms

In this section, we will introduce a linear form in three logarithms and determine some lower and upper bounds. But, we will start by giving some bounds for $\gamma_n^{(+)}$ and $\gamma_n^{(-)}$.

Lemma 4.1. *We have*

(i) $2.79\alpha^{-2n-4} < \gamma_n^{(+)} < 2.85\alpha^{-2n-4}$,
(ii) $0.02\alpha^{-2n-4} < \gamma_n^{(-)} < 0.04\alpha^{-2n-4}$.

Proof. The definition of $\gamma_n^{(\pm)}$ gives

$$\begin{aligned} \sqrt{\gamma_n^{(\pm)}} &= \frac{1}{\sqrt{F_{2n+6}}} \pm \frac{1}{2\sqrt{F_{2n+4}}} \\ &= 5^{1/4}\alpha^{-n-2}\left(\frac{1}{\alpha\sqrt{1 - 1/\alpha^{4n+12}}} \pm \frac{1}{2\sqrt{1 - 1/\alpha^{4n+8}}}\right). \end{aligned} \tag{4.1}$$

As the Taylor series of $(1-x)^{-1/2}$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots,$$

implies

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} < 1 + \frac{x}{2(1-x)} \quad \text{for } x \in (0,1),$$

we see that

$$\frac{1}{\alpha} + 0.5 < \frac{1}{\alpha\sqrt{1 - 1/\alpha^{4n+12}}} + \frac{1}{2\sqrt{1 - 1/\alpha^{4n+8}}} < \frac{1}{\alpha} + 0.51, \tag{4.2a}$$

$$\frac{1}{\alpha} - 0.51 < \frac{1}{\alpha\sqrt{1 - 1/\alpha^{4n+12}}} - \frac{1}{2\sqrt{1 - 1/\alpha^{4n+8}}} < \frac{1}{\alpha} - 0.5. \tag{4.2b}$$

We use (4.1) and (4.2) to obtain

$$\frac{1}{\alpha} + 0.5 < \frac{\sqrt{\gamma_n^{(+)}}}{5^{1/4}\alpha^{-n-2}} < \frac{1}{\alpha} + 0.51,$$

and

$$\frac{1}{\alpha} - 0.51 < \frac{\sqrt{\gamma_n^{(-)}}}{5^{1/4}\alpha^{-n-2}} < \frac{1}{\alpha} - 0.5.$$

Straightforward calculations give the results (i) and (ii) in the lemma completing its proof. □

Let us define the following linear form in three logarithms:

$$\Lambda := 2j \log \beta_n - k \log \alpha + \log(\sqrt{5} \cdot \gamma_n^{(\pm)}). \tag{4.3}$$

In the next result, we will determine an upper bound for Λ.

Lemma 4.2. *If $j \geq 2$, then $0 < \Lambda < 100\beta_n^{-2j}$.*

Proof. We use equation (3.12) to obtain

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} = \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} - \beta_n^{-2j}\gamma_n^{(\mp)} - \frac{\overline{\alpha}^k}{\sqrt{5}}.$$

First, we suppose that $\beta_n^{2j}\gamma_n^{(\pm)} \leq \frac{\alpha^k}{\sqrt{5}}$. Thus, we see that

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(\pm)}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(-)}},$$

and

$$\begin{aligned} \frac{1}{4F_{2n+4}} \quad &< \quad \frac{1}{8F_{2n+4}} + \frac{1}{2F_{2n+6}} = \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} \\ &< \quad \beta_n^{-2j}\gamma_n^{(\mp)} + \frac{\overline{\alpha}^k}{\sqrt{5}} \leq \beta_n^{-2j}\gamma_n^{(+)} + \frac{1}{\sqrt{5} \cdot \alpha^k} \end{aligned}$$

imply

$$\frac{1}{4F_{2n+4}} < \beta_n^{-2j}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right). \tag{4.4}$$

Inequality (4.4) and Lemma 4.1 give

$$4^j F_{2n+6}^j F_{2n+4}^j < \beta_n^{2j} < 4F_{2n+4}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right) < 4F_{2n+4}(2.85\alpha^{-2n-4} + 10\alpha^{2n+4}),$$

so

$$4^{j-1}F_{2n+6}^j F_{2n+4}^{j-1} < 2.85\alpha^{-2n-4} + 10\alpha^{2n+4}. \tag{4.5}$$

Inequality (4.5) easily implies that $j < 2$, which contradicts the assumption.

So, we have $\beta_n^{2j}\gamma_n^{(\pm)} > \frac{\alpha^k}{\sqrt{5}}$. Therefore, $\Lambda > 0$. Moreover, as

$$\left|\alpha^k 5^{-1/2}\beta_n^{-2j}(\gamma_n^{(\pm)})^{-1} - 1\right| < \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} \cdot \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}} < \frac{1}{F_{2n+6}} \cdot \frac{1}{\beta_n^{2j}\gamma_n^{(-)}} < 50\beta_n^{-2j}$$

and the rightmost quantity above is $< 1/2$, we deduce that $\Lambda < 100\beta_n^{-2j}$. Here, we have used

$$|\Lambda| < 2|e^\Lambda - 1| \text{ whenever } |e^\Lambda - 1| < 1/2. \tag{4.6}$$

□

Now, we will prove the next proposition.

Proposition 4.3. *If equation* (3.4) *has a positive integer solution,* (j,k) *with* $j>1$*, then*

$$j < 2.3\cdot 10^{12}(n+3)\log(156j(n+3)). \tag{4.7}$$

Proof. We will apply Lemma 2.1 to the linear form in three logarithms.

$$\Lambda := 2j\log\beta_n - k\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}),$$

and take

$$N=3,\quad D=4,\quad b_1=2j,\quad b_2=-k,\quad b_3=1,$$

and

$$\alpha_1=\beta_n,\quad \alpha_2=\alpha,\quad \alpha_3=\sqrt{5}\cdot\gamma_n^{(\pm)}.$$

We will prove that $\alpha_1,\alpha_2,\alpha_3$ are multiplicatively independent. We know that $\alpha_2\in\mathbb{Q}(\sqrt{5})$ and $\alpha_1,\alpha_3^2\in\mathbb{Q}\left(\sqrt{F_{2n+4}\,F_{2n+6}}\right)$. Let us show that $F_{2n+4}F_{2n+6}$ is neither a square nor 5 times a square. Indeed, otherwise, since $\gcd(F_{2n+4},F_{2n+6})=F_{\gcd(2n+4,2n+6)}=F_2=1$, one of F_{2n+4} or F_{2n+6} would be a square. It is well-known that the only squares in the Fibonacci sequence are 1 and 144, which implies that $n=3,4$, but none of $F_{10}F_{12}$, $F_{12}F_{14}$ is either a square or 5 times square. Thus, if we write $F_{2n+4}F_{2n+6}=du^2$ for an integer u and a square-free integer d, then $d>1$ and $d\neq 5$. So, if $\alpha_1,\alpha_2,\alpha_3$ are multiplicatively dependent, then α_1 and α_3^2 are multiplicatively dependent (because no power of α_2 of a non-zero integer exponent is in $\mathbb{Q}(\sqrt{d})$). Since α_1 is a unit in $\mathbb{Q}(\sqrt{d})$, we deduce that $\alpha_3^2=5\left(\gamma_n^{(\pm)}\right)^2$ is a unit, which is false since the norm of $5\left(\gamma_n^{(\pm)}\right)^2$ is

$$25\left(\gamma_n^{(+)}\gamma_n^{(-)}\right)^2 = 25\cdot\left(\frac{4F_{2n+4}-F_{2n+6}}{4F_{2n+4}F_{2n+6}}\right)^4<1,$$

so the above fraction is not an integer, for any $n\geq 1$ as in the reduced form that rational number has a denominator divisible by 2.

We consider

$$h(\alpha_1)=h(\beta_n)=\frac{1}{2}\log\beta_n \text{ and } h(\alpha_2)=h(\alpha)=\frac{1}{2}\log\alpha.$$

As $\gamma_n^{(+)}$, $\gamma_n^{(-)}$ are conjugate and roots of the quadratic polynomial

$$16F_{2n+4}^2F_{2n+6}^2X^2-8(F_{2n+6}^2F_{2n+4}+4F_{2n+6}F_{2n+4}^2)X+(4F_{2n+4}-F_{2n+6})^2,$$

and

$$\left|\gamma_n^{(\pm)}\right|\leq\left|\gamma_n^{(+)}\right|=\left(\frac{1}{\sqrt{F_{2n+6}}}+\frac{1}{2\sqrt{F_{2n+4}}}\right)^2<1,$$

we see that

$$h(\gamma_n^{(\pm)})=\frac{1}{2}\log(16F_{2n+4}^2F_{2n+6}^2)=\log(4F_{2n+4}F_{2n+6})<(4n+10)\log\alpha+\log(4/5),$$

where we have used $F_l<\alpha^l/\sqrt{5}$ for $l\in\{2n+4,2n+6\}$. We deduce that

$$\begin{aligned} h(\alpha_3) &= h(\sqrt{5}\cdot\gamma_n^{(\pm)})\leq h(\sqrt{5})+h(\gamma_n^{(\pm)})\\ &< \tfrac{1}{2}\log(5)+(4n+10)\log\alpha+\log(4/5)\\ &= \log(4/\sqrt{5})+(4n+10)\log\alpha<4(n+3)\log\alpha, \end{aligned}$$

where we used $4/\sqrt{5}<\alpha^2$. In conclusion, we take

$$A_1=2\log\beta_n,\quad A_2=2\log\alpha,\quad A_3=16(n+3)\log\alpha.$$

As $\alpha^{l-2} \le F_l \le \alpha^{l-1}$, for all $l \ge 1$, we deduce that

$$\beta_n < 2F_{2n+5} < 2\alpha^{2n+4} < \alpha^{2(n+3)}.$$

Moreover, using $\alpha^2 > 2$, we have

$$\begin{aligned}\alpha^{k-1} &< 2\alpha^{k-2} < 2F_k \le 8U_jV_j + 2(F_{2n+6} + 4F_{2n+4})U_j^2 \\ &< (V_j + U_j\sqrt{F_{2n+6}F_{2n+4}})^2 = (F_{2n+5} + \sqrt{F_{2n+6}F_{2n+4}})^{2j} \\ &< (2F_{2n+5})^{2j} < (2\alpha^{2n+4})^{2j} < \alpha^{4j(n+3)}.\end{aligned}$$

Therefore, we consider

$$E = \max\left\{\frac{2j\log\beta_n}{\log\alpha}, 8(n+3), k\right\} \le 4j(n+3).$$

By Lemmas 2.1 and 4.2, we get

$$\begin{aligned}C(3) &= \frac{8}{2!}(3+2)(6+3)(4^2e)^4 < 6.45\cdot 10^8, \\ C_0 &= \log(e^{4.4\cdot 3+7}3^{5.5}4^2\log(4e)) < 30, \\ W_0 &= \log(1.5eE4\log(4e)) < \log(156j(n+3)), \\ \Omega &= (2\log\beta_n)(2\log\alpha)(16(n+3)\log\alpha),\end{aligned}$$

so

$$2j\log\beta_n - \log 100 < -\log|\Lambda| < 198144\cdot 10^8\cdot\log(156j(n+3))(\log\beta_n)(\log\alpha)^2(n+3),$$

which leads to

$$j < 2.3\cdot 10^{12}(n+3)\log(156j(n+3)).$$

This completes the proof of the proposition. □

5. The Use of a Linear Form In Two Logarithms

In this section, we will introduce a linear form in two logarithms that will help us improve the result obtain in Proposition 4.3. From (3.7), when $j = 1$, we see that equation (3.4) has the solution

$$k = 2n, \quad \text{for } C = C_1^{(-)}. \tag{5.1}$$

We define the linear form in logarithms:

$$\Lambda_0 = 2\log\beta_n - 2n\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}). \tag{5.2}$$

First, we will determine an upper bound for Λ_0.

Lemma 5.1. *We have* $|\Lambda_0| < 3050\beta_n^{-2}$.

Proof. For $n = 1$, this can be checked directly. Assume that $n \ge 2$. We substitute (5.2) into (3.12) to obtain

$$\beta_n^2\gamma_n^{(\pm)} - \frac{\alpha^{2n}}{\sqrt{5}} = \frac{F_{2n+6} + 4F_{2n+4}}{2(F_{2n+5}^2 - 1)} - \beta_n^{-2}\gamma_n^{(\mp)} - \frac{\alpha^{-2n}}{\sqrt{5}}.$$

If $\beta_n^2\gamma_n^{(\pm)} \le \dfrac{\alpha^{2n}}{\sqrt{5}}$, then $\alpha^{-2n}/\sqrt{5} \le 1/(5\beta_n^2\gamma_n^{(\pm)})$ and

$$\begin{aligned}\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1\right| &< \frac{\beta_n^{-2}\gamma_n^{(\mp)} + \alpha^{-2n}/\sqrt{5}}{\beta_n^2\gamma_n^{(\pm)}} \\ &< \frac{\gamma_n^{(\mp)} + 1/(5\gamma^{(\pm)})}{\beta_n^4\gamma_n^{(\pm)}} < \frac{142.5 + 500\alpha^{4n+8}}{\beta_n^4}.\end{aligned}$$

This inequality together with $\beta_n \ge 13 + 2\sqrt{42}$ and $\beta_n \ge \alpha^{2n+4}$ gives

$$\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1\right| < 501\beta_n^{-2}.$$

On the other hand, if $\beta^2\gamma_n^{(\pm)} > \alpha^{2n}/(\sqrt{5})$, then

$$\begin{aligned}\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1\right| &< \frac{1/(2F_{2n+6}) + 1/(2F_{2n+6})}{\beta_n^2\gamma_n^{(\pm)}} \\ &< \frac{1}{F_{2n+6}\beta^2\gamma_n^{(\pm)}} < 50\beta_n^{-2}.\end{aligned}$$

In both cases,

$$\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1\right| < 501\beta_n^{-2}. \tag{5.3}$$

Since $n \ge 2$, we have $\beta_n \ge 34 + \sqrt{1155}$, so $501\beta_n^{-2} < 1/2$, and inequalities (4.6) and (5.3) imply $|\Lambda_0| < 2 \cdot 501\beta_n^{-2} < 3050\beta_n^{-2}$. □

Let $K := (2j-1)(2n+5) - k - 5$ and

$$\Lambda_1 := K\log\alpha - (j-1)\log(5/4). \tag{5.4}$$

Now, we determine an upper bound for Λ_1.

Lemma 5.2. *We have* $|\Lambda_1| < (8j + 4192)\alpha^{-4n-10}$.

Proof. We know that

$$\begin{aligned}\beta_n &= F_{2n+5} + \sqrt{F_{2n+5}^2 - 1} = 2F_{2n+5} - \frac{1}{F_{2n+5} + \sqrt{F_{2n+5}^2 - 1}} \\ &= 2F_{2n+5}\left(1 - \frac{1}{2F_{2n+5}(F_{2n+5} + \sqrt{F_{2n+5}^2 - 1})}\right)\end{aligned} \tag{5.5}$$

and

$$2F_{2n+5} = \frac{2}{\sqrt{5}}\left(\alpha^{2n+5} - \overline{\alpha}^{2n+5}\right) = \frac{2}{\sqrt{5}}\alpha^{2n+5}\left(1 + \frac{1}{\alpha^{4n+10}}\right).$$

We define

$$\delta_n = \left(1 - \frac{1}{2F_{2n+5}(F_{2n+5} + \sqrt{F_{2n+5}^2 - 1})}\right)\left(1 + \frac{1}{\alpha^{4n+10}}\right).$$

From the above, we deduce that

$$\log\beta_n = \log(2/\sqrt{5}) + (2n+5)\log\alpha + \log\delta_n.$$

We use (4.3) and (5.2) to obtain

$$\begin{aligned}\Lambda - \Lambda_0 &= (2j-2)\log\beta_n - (k-2n)\log\alpha\\ &= (2j-2)\log(2/\sqrt{5}) + (2j-2)(2n+5)\log\alpha\\ &\quad +(2j-2)\log\delta_n - (k-2n)\log\alpha\\ &= (2j-2)\log\delta_n + K\log\alpha - (j-1)\log(5/4).\end{aligned}$$

The above calculation and the definition of Λ_1 give

$$\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log\delta_n.$$

One can see that Lemmas 4.2, 5.1, and the inequalities

$$\begin{aligned}|\log\delta_n| &\leq \left|\log\left(1 - \frac{1}{2F_{2n+5}(F_{2n+5}+\sqrt{F_{2n+5}^2-1})}\right)\right| + \left|\log\left(1+\frac{1}{\alpha^{4n+10}}\right)\right|\\ &< \frac{1}{F_{2n+5}(F_{2n+5}+\sqrt{F_{2n+5}^2-1})} + \frac{1}{\alpha^{4n+10}} < \frac{4}{\alpha^{4n+2}}\end{aligned}$$

imply that

$$|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + |(2j-2)||\log\delta_n| < \frac{3150}{\beta_n^2} + \frac{8(j-1)}{\alpha^{4n+10}}. \tag{5.6}$$

Clearly, we get

$$\begin{aligned}\beta_n &= F_{2n+5}\left(1+\sqrt{1-\frac{1}{F_{2n+5}^2}}\right) \geq F_{2n+5}(1+2\sqrt{42}/13)\\ &> \frac{\alpha^{2n+5}}{\sqrt{5}}(1+2\sqrt{42}/13),\end{aligned}$$

and then

$$\beta_n^2 > \alpha^{4n+10}\cdot\frac{(1+2\sqrt{42}/13)^2}{5} > \frac{3\alpha^{4n+10}}{4}. \tag{5.7}$$

From (5.6) and (5.7), we obtain the desired result. □

We are ready now to reach the aim of this section by proving the next result.

Lemma 5.3. *If equation* (3.4) *has a positive integer solution* (j,k) *with* $j > 1$*, then*

$$j < 3.7\cdot 10^{19} \quad \textit{and} \quad n < 207062.$$

Proof. To obtain a lower bound for Λ_1, we will apply Lemma 2.2. So, we put

$$D = 2, \quad \gamma_1 = \frac{5}{4}, \quad \gamma_1 = \alpha, \quad b_1 = (j-1), \quad b_2 = K.$$

The condition of the lemma are fulfilled for our choices of parameters. Furthermore, we can take $h_1 = \log 5$, $h_2 = \frac{1}{2}$. By Lemma 5.2 , we have

$$\begin{aligned}K &< \frac{(j-1)\log(5/4) + (8j+4192)\alpha^{-4n-10}}{\log\alpha}\\ &< 0.47(j-1) + 0.02j + 10.34 < 0.5j + 9.9.\end{aligned}$$

So, we can take

$$b' = 1.16j + 2.1 > (j-1) + \frac{K}{2\log 5} = \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Therefore, Lemma 2.2 yields

$$\log|\Lambda_1| > -17.9 \cdot 8 \log 5 \cdot (\max\{\log(1.16j + 2.1) + 0.38, 15\})^2. \tag{5.8}$$

From Lemma 5.2, we get

$$\log|\Lambda_1| < -(4n + 10)\log\alpha + \log(8j + 4192). \tag{5.9}$$

Combining the two bounds (5.8) and (5.9) on $\log|\Lambda_1|$, we have

$$n < 120(\max\{\log(1.16j + 2.1) + 0.38, 15\})^2 + 0.6\log(8j + 4192). \tag{5.10}$$

If

$$\log(1.16j + 2.1) + 0.38 \leq 15,$$

then

$$j < 1927201 \quad \text{and} \quad n < 120 \cdot 15^2 + 0.6\log(8 \cdot 1927201 + 4192) < 27010.$$

Otherwise,

$$n < 120(\log(1.16j + 2.1) + 0.38)^2 + 0.6\log(8j + 4192). \tag{5.11}$$

Substituting inequality (5.11) into Proposition 4.3, we have

$$\begin{aligned} j \quad < \quad & 2.3 \cdot 10^{12}(120(\log(1.16j + 2.1) + 0.38)^2 + 0.6\log(8j + 4192) + 3) \\ & \times \log(156j(120(\log(1.16j + 2.1) + 0.38)^2 + 0.6\log(8j + 4192) + 3)). \end{aligned} \tag{5.12}$$

A straightforward calculation gives $j < 3.7 \cdot 10^{19}$, which together with (5.11) implies $n < 207062$. □

6. Better Bounds on j and n

The goal of this section is to obtain better bounds on j and n. We use Lemma 5.9 to obtain

$$|K\log\alpha - (j-1)\log(5/4)| < (8j + 4192)\alpha^{-4n-10}.$$

Hence, we have

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\right| < \frac{8j + 4192}{(j-1)\alpha^{4n+10}\log\alpha}. \tag{6.1}$$

First, we assume that

$$\frac{8j + 4192}{(j-1)\alpha^{4n+10}\log\alpha} < \frac{1}{2(j-1)^2}. \tag{6.2}$$

Then, we get

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\right| < \frac{1}{2(j-1)^2}.$$

Using a criterion of Legendre, we see that $K/(j-1)$ is a convergent in the simple continued fraction expansion of $\log(5/4)/\log\alpha$. We know that

$$\begin{aligned} \frac{\log 2}{\log\alpha} \quad = \quad & [0, 2, 6, 2, 1, 1, 3, 7, 1, 3, 1, 1, 22, 2, 1, 1, 4, 3, 1, 2, 1, 1, 4, 1, 1, \\ & 1, 12, 6, 1, 1, 4, 1, 8, 2, 1, 49, 1, 10, 6, 1, 1, 3, 1, 1, 1, 5, 22, 1, 1, \ldots]. \end{aligned}$$

The denominator of the 46th convergent

$$\frac{25158053660121411107}{54253653513327093513}$$

is greater than the upper bound $3.7 \cdot 10^{19}$ on j. The 45th convergent

$$\frac{4460457560349832575}{9619031832089360168}$$

gives the lower bound

$$\left|\frac{\log(5/4)}{\log\alpha}-\frac{K}{j-1}\right|>1.9\cdot 10^{-39}. \tag{6.3}$$

The combination of (6.1) and (6.3) gives

$$1.9\cdot 10^{-39}<\frac{8j+4192}{(j-1)\alpha^{4n+10}\log\alpha}<4208\alpha^{-4n-10}(\log\alpha)^{-1},$$

which implies that $n<49$. We know (see [6]) that if p_r/q_r is the rth such convergent of $\log(5/4)/\log\alpha$, then

$$\left|\frac{\log(5/4)}{\log\alpha}-\frac{p_r}{q_r}\right|>\frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$st partial quotient of $\log(5/4)/\log\alpha$. We thus have

$$\min\left\{\frac{1}{(a_{r+1}+2)(j-1)^2}\right\}<\frac{8j+4192}{(j-1)\alpha^{4n+10}\log\alpha},\quad \text{for } 2\le r\le 45. \tag{6.4}$$

Since $\max\{a_{r+1}:2\le r\le 45\}=a_{36}=49$, from (6.4) we get

$$\alpha^{4n+10}<51(j-1)(8j+4192)(\log\alpha)^{-1}.$$

All this was when inequality (6.2) holds. On the other hand, if (6.2) does not hold, then

$$\alpha^{4n+10}\le 2(j-1)(8j+4192)(\log\alpha)^{-1}.$$

Both possibilities give

$$\alpha^{4n+10}<51(j-1)(8j+4192)(\log\alpha)^{-1}<848j(j+524)<445200j^2.$$

Therefore, we deduce the following result.

Proposition 6.1. *If equation* (3.4) *has a positive integer solution* (j,k) *with* $j>1$*, then*

$$n<1.04\log j+4.3. \tag{6.5}$$

This bound is better than that in (5.11). Combining Propositions 4.3 and 6.1, we get

$$j<120\cdot 10^{12}(1.04\log j+7.3)\log(156\cdot j\cdot(1.04\log j+7.3)),$$

which implies that $j<4.7\cdot 10^{15}$. Using Proposition 6.1, we get the following result.

Lemma 6.2. *If equation* (3.4) *has a positive integer solution* (j,k) *with* $j>1$*, then*

$$j<4.7\cdot 10^{15}\quad \textit{and}\quad n<42.$$

7. Proof of Theorem 1.1

In this section, we will use all the above results to complete the proof of Theorem 1.1. So, to address the remaining cases, for $1\le n\le 41$, first we will use the Baker-Davenport reduction method to reduce the bounds of n and j. Then, we will address the small values of n and j.

We know that

$$0<2j\log\beta_n-k\log\alpha+\log(\sqrt{5}\cdot\gamma_n^{(\pm)})<100\beta_n^{-2j}.$$

To apply Lemma 2.3, we will consider

$$\kappa=\frac{2\log\beta_n}{\log\alpha},\quad \mu=\frac{\log(\sqrt{5}\cdot\gamma_n^{(\pm)})}{\log\alpha},\quad A=\frac{100}{\log\alpha},\quad B=\beta_n^2,\quad M=4.7\cdot 10^{15}.$$

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q>6M$ does not satisfy the condition $\eta>0$, then we use the next convergent until we find the one that satisfies the conditions. In one minute, all the computations were done. In all cases, we obtained $j \leq 8$. From Proposition 6.1, we deduce that $1 \leq n \leq 6$. So we have the following result.

Lemma 7.1. *If equation* (3.5) *has a positive integer solution* (j, k) *with* $j > 1$*, then*

$$j \leq 8 \quad \text{and} \quad n \leq 6.$$

We are now ready to prove Theorem 1.1. For $1 \leq n \leq 6$, $2 \leq j \leq 8$, we compute all $C_j^{(\pm)}$. None of them is a Fibonacci number. This means that equation (3.5) has no positive integer solution (j, k) with $j \geq 2$. When $j = 1$, we have

$$C_1^{(+)} = 4V_1U_1 + (F_{2n+6} + 4F_{2n+4})\,U_1^2 = 4F_{2n+5} + F_{2n+6} + 4F_{2n+4} = 5F_{2n+6}, \quad \text{for } n \geq 1$$

and

$$C_1^{(-)} = -4V_1U_1 + (F_{2n+6} + 4F_{2n+4})\,U_1^2 = -4F_{2n+5} + F_{2n+6} + 4F_{2n+4} = F_{2n}, \quad \text{for } n \geq 1.$$

But $F_{2n+9} < 5F_{2n+6} < F_{2n+10}$. Since $F_1 = F_2$, the additional solutions come from the triple $\{F_1, F_8, 4F_6\} = \{F_2, F_8, 4F_6\} = \{1, 21, 32\}$.

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