# CONGRUENCES MODULO 5 FOR PARTITIONS INTO AT MOST FOUR PARTS

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Abstract. We give quick proofs of two congruences modulo 5 for $p(n,4)$, the number of partitions of $n$ into at most four parts, discovered and proved by Ali H. Al-Saedi, as well as a number of other congruences and related identities.

## 1. Introduction

Let $p(n,k)$ denote the number of partitions of $n$ into at most $k$ parts, or, what is the same thing, the number of partitions of a number into parts at most $k$.

The generating function of the $p(n,k)$ is

$$\sum_{n\geq 0} p(n,k)q^n = 1\Big/\prod_{l=1}^{k}(1-q^l).$$

Al-Saedi [1, Theorem 3.6], has discovered the following congruences for $p(n,4)$, namely

$$p(10n+2,4)+p(10n+3,4)+p(10n+4,4)\equiv 0 \pmod 5 \tag{1.1}$$

and

$$p(10n+6,4)+p(10n+7,4)+p(10n+8,4)\equiv 0 \pmod 5. \tag{1.2}$$

We give a quick proof of (1.1) and (1.2) as well as the following congruences,

$$p(20n+11,4)+p(20n+12,4)+p(20n+13,4)\equiv 0 \pmod 5 \tag{1.3}$$

and

$$p(20n+17,4)+p(20n+18,4)+p(20n+19,4)\equiv 0 \pmod 5. \tag{1.4}$$

Indeed, we will go rather further, and show that

$$p(10n+2,4)+p(10n+3,4)+p(10n+4,4)=(5n+3)(5n+4)(5n+5)/6, \tag{1.5}$$

$$p(10n+6,4)+p(10n+7,4)+p(10n+8,4)=(5n+5)(5n+6)(5n+7)/6, \tag{1.6}$$

$$p(20n+11,4)+p(20n+12,4)+p(20n+13,4)=25(n+1)(4n+3)(5n+4)$$
$$\equiv 0 \text{ (mod 50)} \tag{1.7}$$

and

$$p(20n+17,4)+p(20n+18,4)+p(20n+19,4)=25(n+1)(4n+5)(5n+6)/3$$
$$\equiv 0 \quad \text{(mod 50)}). \tag{1.8}$$

## 2. Proofs of (1.1)–(1.4)

Modulo 5,

$$\begin{aligned}
\sum_{n\geq 0} p(n,4)q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\
&= \frac{(1+q+q^2+q^3+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^4+q^8+q^{12}+q^{16})}{(1-q^5)(1-q^{10})(1-q^{15})(1-q^{20})} \\
&\equiv \frac{1}{(1-q^5)(1-q^{10})(1-q^{15})(1-q^{20})}\Big(1+q+2q^2-2q^3-2q^6-q^7+2q^8 \\
&-2q^9+q^{10}+2q^{11}+q^{12}+q^{13}-q^{14}-2q^{16}+2q^{17}-q^{19}+q^{20}-q^{21}+2q^{23} \\
&-2q^{24}-q^{26}+q^{27}+q^{28}+2q^{29}+q^{30}-2q^{31}+2q^{32}-q^{33}-2q^{34}-2q^{37} \\
&+2q^{38}+q^{39}+q^{40}\Big). \qquad (2.1)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0} p(5n,4)q^n &\equiv \frac{1+q^2+q^4+q^6+q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)}, \\
\sum_{n\geq 0} p(5n+1,4)q^n &\equiv \frac{1-2q+2q^2-2q^3-q^4-q^5-2q^6}{(1-q)(1-q^2)(1-q^3)(1-q^4)}, \\
\sum_{n\geq 0} p(5n+2,4)q^n &\equiv \frac{2-q+q^2+2q^3+q^5+2q^6-2q^7}{(1-q)(1-q^2)(1-q^3)(1-q^4)}, \\
\sum_{n\geq 0} p(5n+3,4)q^n &\equiv \frac{-2+2q+q^2+2q^4+q^5-q^6+2q^7}{(1-q)(1-q^2)(1-q^3)(1-q^4)}, \\
\sum_{n\geq 0} p(5n+4,4)q^n &\equiv \frac{-2q-q^2-q^3-2q^4+2q^5-2q^6+q^7}{(1-q)(1-q^2)(1-q^3)(1-q^4)}. \qquad (2.2)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n\geq 0} \Big(p(5n+1,4)+p(5n+2,4)+p(5n+3,4)\Big)q^n \\
&\equiv \frac{1-q-q^2+q^4+q^5-q^6}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\
&= \frac{(1-q)(1-q^2)(1-q^3)}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\
&= \frac{1}{1-q^4} \qquad (2.3)
\end{aligned}$$

and

$$\sum_{n\geq 0}\Big(p(5n+2,4)+p(5n+3,4)+p(5n+4,4)\Big)q^n$$
$$\equiv \frac{-q+q^2+q^3-q^5-q^6+q^7}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$
$$= -\frac{q}{1-q^4}. \tag{2.4}$$

All of (1.1)–(1.4) follow.

## 3. Proofs of (1.5)–(1.8)

We have

$$\sum_{n\geq 0} p(n,4)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$
$$= \frac{(1+q+q^2+\ \cdots\ +q^{11})(1+q^2+\ \cdots\ +q^{10})(1+q^3+\ \cdots\ +q^9)(1+q^4+q^8)}{(1-q^{12})^4}$$
$$= \frac{1}{(1-q^{12})^4}\Big(1+q+2q^2++3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}$$
$$+27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19}+48q^{20}$$
$$+48q^{21}+22q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28}+18q^{29}+15q^{30}$$
$$+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}\Big). \tag{3.1}$$

It follows that

$$p(12n,4) = \binom{n+3}{3}+30\binom{n+2}{3}+39\binom{n+1}{3}+2\binom{n}{3}$$
$$= 12n^3+15n^2+6n+1, \tag{3.2}$$
$$p(12n+1,4) = \binom{n+3}{3}+35\binom{n+2}{3}+35\binom{n+1}{3}+\binom{n}{3}$$
$$= 12n^3+18n^2+8n+1, \tag{3.3}$$
$$p(12n+2,4) = 2\binom{n+3}{3}+39\binom{n+2}{3}+30\binom{n+1}{3}+\binom{n}{3}$$
$$= 12n^3+21n^2+12n+2, \tag{3.4}$$
$$p(12n+3,4) = 3\binom{n+3}{3}+42\binom{n+2}{3}+27\binom{n+1}{3}$$
$$= 12n^3+24n^2+15n+3, \tag{3.5}$$
$$p(12n+4,4) = 5\binom{n+3}{3}+44\binom{n+2}{3}+23\binom{n+1}{3}$$
$$= 12n^3+27n^2+20n+5, \tag{3.6}$$
$$p(12n+5,4) = 6\binom{n+3}{3}+48\binom{n+2}{3}+18\binom{n+1}{3}$$

$$= 12n^3 + 30n^2 + 24n + 6, \tag{3.7}$$

$$p(12n+6,4) = 9\binom{n+3}{3} + 48\binom{n+2}{3} + 15\binom{n+1}{3}$$
$$= 12n^3 + 33n^2 + 30n + 9, \tag{3.8}$$

$$p(12n+7,4) = 11\binom{n+3}{3} + +50\binom{n+2}{3} + 11\binom{n+1}{3}$$
$$= 12n^3 + 36n^2 + 35n + 11, \tag{3.9}$$

$$p(12n+8,4) = 15\binom{n+3}{3} + +48\binom{n+2}{3} + 9\binom{n+1}{3}$$
$$= 12n^3 + 39n^2 + 42n + 15, \tag{3.10}$$

$$p(12n+9,4) = 18\binom{n+3}{3} + 48\binom{n+2}{3} + 6\binom{n+1}{3}$$
$$= 12n^3 + 42n^2 + 48n + 18, \tag{3.11}$$

$$p(12n+10,4) = 23\binom{n+3}{3} + 44\binom{n+2}{3} + 5\binom{n+1}{3}$$
$$= 12n^3 + 45n^2 + 56n + 23, \tag{3.12}$$

$$p(12n+11,4) = 27\binom{n+3}{3} + 42\binom{n+2}{3} + 3\binom{n+1}{3}$$
$$= 12n^3 + 48n^2 + 63n + 27. \tag{3.13}$$

If in (3.2)–(3.13) we substitute $5n$, $5n+1$, $5n+2$, $5n+3$ and $5n+4$ for $n$, we obtain formulas for $p(60n+r,4)$ for $r = 0, 1, \ldots, 59$.

For example,

$$\begin{aligned} p(60n,4) &= p(12(5n),4) \\ &= 12(5n)^3 + 15(5n)^2 + 6(5n) + 1 \\ &= 1500n^3 + 375n^2 + 30n + 1, \end{aligned} \tag{3.14}$$

$$\begin{aligned} p(60n+59,4) &= p(12(5n+4)+11,4) \\ &= 12(5n+4)^3 + 48(5n+4)^2 + 63(5n+4) + 27 \\ &= 1500n^3 + 4800n^2 + 5115n + 1815. \end{aligned} \tag{3.15}$$

We are now in a position to prove (1.5)–(1.8). Let us start with (1.5).

Consider the quantity

$$p(10n+2,4) + p(10n+3,4) + p(10n+4,4).$$

If $n \equiv 0 \pmod 6$, write $n = 6m$. Then,

$$\begin{aligned} &p(10n+2,4) + p(10n+3,4) + p(10n+4,4) \\ &= p(60m+2,4) + p(60m+3,4) + p(60m+4,4) \\ &= 4500m^3 + 1800m^2 + 235m + 10 \end{aligned}$$

$$= 4500\left(\frac{n}{6}\right)^3 + 1800\left(\frac{n}{6}\right)^2 + 235\left(\frac{n}{6}\right) + 10$$
$$= \frac{1}{6}(125n^3 + 300n^2 + 235n + 60)$$
$$= \frac{1}{6}(5n+3)(5n+4)(5n+5), \tag{3.16}$$

if $n \equiv 1 \pmod 6$, $n = 6m+1$,

$$p(10n+2,4) + p(10n+3,4) + p(10n+4,4)$$
$$= p(60n+12,4) + p(60n+13,4) + p(60n+14,4)$$
$$= 4500m^3 + 4050m^2 + 1210m + 120$$
$$= 4500\left(\frac{n-1}{6}\right)^3 + 4050\left(\frac{n-1}{6}\right)^2 + 1210\left(\frac{n-1}{6}\right) + 47$$
$$= \frac{1}{6}(125n^3 + 300n^2 + 235n + 60), \tag{3.17}$$

if $n \equiv 2 \pmod 6$, $n = 6m+2$,

$$p(10n+2,4) + p(10n+3,4) + p(10n+4,4)$$
$$= p(60m+22,4) + p(60m+23,4) + p(60m+24,4)$$
$$= 4500m^3 + 6300m^2 + 2935m + 455$$
$$= 4500\left(\frac{n-2}{6}\right)^3 + 6300\left(\frac{n-2}{6}\right)^2 + 2935\left(\frac{n-2}{6}\right) + 455$$
$$= \frac{1}{6}(125n^3 + 300n^2 + 235n + 60), \tag{3.18}$$

if $n \equiv 3 \pmod 6$, $n = 6m+3$,

$$p(10n+2,4) + p(10n+3,4) + p(10n+4,4)$$
$$= p(60m+32,4) + p(60m+33,4) + p(60m+34,4)$$
$$= 4500m^3 + 8550m^2 + 5410m + 1140$$
$$= 4500\left(\frac{n-3}{6}\right)^3 + 8550\left(\frac{n-3}{6}\right)^2 + 5410\left(\frac{n-3}{6}\right) + 1140$$
$$= \frac{1}{6}(125n^3 + 300n^2 + 235n + 60), \tag{3.19}$$

if $n \equiv 4 \pmod 6$, $n = 6m+4$,

$$p(10n+2,4) + p(10n+3,4) + p(10n+4,4)$$
$$= p(60m+42,4) + p(60m+43,4) + p(60m+44,4)$$
$$= 4500m^3 + 10800m^2 + 8635m + 2300$$
$$= 4500\left(\frac{n-4}{6}\right)^3 + 10800\left(\frac{n-4}{6}\right)^2 + 8635\left(\frac{n-4}{6}\right) + 2300$$
$$= \frac{1}{6}(125n^3 + 300n^2 + 235n + 60), \tag{3.20}$$

and finally, if $n \equiv 5 \pmod 6$, $n = 6m+5$,

$$\begin{aligned}
&p(10n+2,4)+p(10n+3,4)+p(10n+4,4)\\
&=p(60m+52,4)+p(60m+53,4)+p(60m+54,4)\\
&=4500m^3+13050m^2+12610m+4060\\
&=4500\left(\frac{n-5}{6}\right)^3+13050\left(\frac{n-5}{6}\right)^2+12610\left(\frac{n-5}{6}\right)+4060\\
&=\frac{1}{6}(125n^3+300n^2+235n+60). \qquad (3.21)
\end{aligned}$$

Thus, (1.5) is proved.

The proofs of (1.6)–(1.8) are similar, and so omitted.

## 4. Additional Comments

It follows from (1.5)–(1.8) that

$$\sum_{n\geq 0}(p(10n+2,4)+p(10n+3,4)+p(10n+4,4))q^n=\frac{10+80q+35q^2}{(1-q)^4}, \qquad (4.1)$$

$$\sum_{n\geq 0}(p(10n+6,4)+p(10n+7,4)+p(10n+8,4))q^n=\frac{35+80q+10q^2}{(1-q)^4}, \qquad (4.2)$$

$$\sum_{n\geq 0}(p(20n+11,4)+p(20n+12,4)+p(20n+13,4))q^n=\frac{100+650q+250q^2}{(1-q)^4}, \qquad (4.3)$$

$$\sum_{n\geq 0}(p(20n+17,4)+p(20n+18,4)+p(20n+19,4))q^n=\frac{250+650q+100q^2}{(1-q)^4}. \qquad (4.4)$$

## References

[1] A. Al-Saedi, *Using periodicity to obtain partition congruences*, Journal of Number Theory, **178** (2017), 158–178.

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