POLYNOMIAL EXTENSIONS OF A DIMINNIE DELIGHT REVISITED: PART II

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Abstract. Recently, we investigated the Fibonacci polynomial recurrences $a_{n+1} = a_n(\Delta^2 a_n^2 + 3)$, where $a_n = a_n(x)$, $a_0 = f_e$, e is an even positive integer, $\Delta = \sqrt{x^2+4}$, and $n \geq 0$; and $a_{n+2} = a_{n+1}(\Delta^2 a_n^2 + 2)$, where $a_1 = f_{2k}$; k is an odd positive integer; and $n \geq 1$ [10]. We also studied their Lucas counterparts: $a_{n+1} = a_n(a_n^2 - 3)$, where $a_0 = l_e$; e is an even positive integer; and $n \geq 0$; and $a_{n+2} = a_{n+1}(a_n^2 - 2) - 2$, where $a_1 = l_{2k}$; $a_2 = l_{4k}$; k is an odd positive integer; and $n \geq 1$ [10]. This article focuses on the Jacobsthal, Vieta, and Chebyshev extensions of these charming recurrences and their implications.

1. Introduction

As in [11], our discourse hinges on the links among the gibonacci subfamilies of Fibonacci, Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev polynomials [5, 9, 13]; see Table 1.

$$
\begin{aligned}
&J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x}) && j_n(x) = x^{n/2} l_n(1/\sqrt{x})\\
&V_n(x) = i^{n-1} f_n(-ix) && v_n(x) = i^n l_n(-ix)\\
&V_n(x) = U_{n-1}(x/2) && v_n(x) = 2T_n(x/2)\\
&xV_n(x^2+2) = f_{2n} && xv_n(x^2+2) = l_{2n}\\
&J_{2n}(x) = x^{n-1} V_n\left(\frac{2x+1}{x}\right) && j_{2n}(x) = x^n v_n\left(\frac{2x+1}{x}\right).
\end{aligned}
$$

Table 1: Links Among the Gibonacci Subfamilies

These powerful tools pave the way for our explorations.

In the interest of brevity and clarity, again we omit the argument in the functional notation when there is *no* ambiguity; so g_n, for example, will mean $g_n(x)$. Also, we omit a lot of basic, but messy algebra.

We begin our pursuit with recurrence A from [10].

2. Polynomial Extensions of Recurrence A

Consider the first-order recurrence

$$a_{n+1} = a_n(\Delta^2 a_n^2 + 3), \tag{2.1}$$

where $a_0 = f_e$, e is an even positive integer, and $n \geq 0$ [10]. Its solution is $a_n = f_{e\cdot 3^n}$.

It follows by the relationships in Table 1 that recurrence (2.1) has Jacobsthal, Vieta, and Chebyshev implications.

2.1. **Jacobsthal Extensions.** Let $b_n = b_n(x) = x^{(e\cdot 3^n-1)/2}a_n(1/\sqrt{x})$. Replacing x with $1/\sqrt{x}$ in (2.1), and then multiplying the resulting equation by $x^{(e\cdot 3^{n+1}-1)/2}$, we get the recurrence

$$b_{n+1} = b_n\left[(4x+1)b_n^2 + 3x^{e\cdot 3^n}\right], \tag{2.2}$$

where $b_0 = x^{(e-1)/2}a_0(1/\sqrt{x}) = x^{(e-1)/2}f_e(1/\sqrt{x}) = J_e(x)$ and $n \geq 0$. Its solution is given by $b_n = x^{(e\cdot 3^n-1)/2}a_n(1/\sqrt{x}) = x^{(e\cdot 3^n-1)/2}f_{e\cdot 3^n}(1/\sqrt{x}) = J_{e\cdot 3^n}(x)$.

Letting $b_n(2) = B_n$, equation (2.2) yields the recurrence

$$B_{n+1} = B_n\left(9B_n^2 + 3\cdot 2^{e\cdot 3^n}\right), \tag{2.3}$$

where $B_0 = J_e$. Clearly, $B_n = J_{e\cdot 3^n}$, where $n \geq 0$.

Suppose we let $e = 4$. Then

$$B_{n+1} = B_n\left(9B_n^2 + 3\cdot 16^{3^n}\right),$$

where $B_0 = J_4 = 5$. Then $B_1 = 5(9\cdot 5^2 + 3\cdot 16) = 1,365 = J_{4\cdot 3}$, and hence, $B_2 = 1365(9\cdot 1365^2 + 3\cdot 16^3) = 22,906,492,245 = J_{4\cdot 3^2}$.

2.2. **Vieta Extensions.** Let $b_n = b_n(x) = i^{e\cdot 3^n-1}a_n(-ix)$. Replacing x with $-ix$ in (2.1), and then multiplying the resulting equation by $i^{e\cdot 3^{n+1}-1}$, we get the recurrence

$$b_{n+1} = b_n\left[(x^2-1)b_n^2 + 3\right], \tag{2.4}$$

where $b_0 = i^{e-1}a_0(-ix) = i^{e-1}f_e(-ix) = V_e(x)$ and $n \geq 0$. Its solution is given by $b_n = i^{e\cdot 3^n-1}a_n(1/\sqrt{x}) = i^{e\cdot 3^n-1}f_{e\cdot 3^n}(-ix) = V_{e\cdot 3^n}(x)$.

It follows from Table 1 that recurrence (2.4) has Fibonacci, Jacobsthal, and Chebyshev consequences.

2.2.1. *Fibonacci Byproducts.* Let $z_n = z_n(x) = xb_n(x^2+2)$. Then recurrence (2.4) implies that

$$z_{n+1} = z_n[(x^2+4)z_n^2 + 3], \tag{2.5}$$

where $z_0 = xb_0(x^2+2) = xV_e(x^2+2) = f_{2e}$. Then $z_n = xb_n(x^2+2) = xV_{e\cdot 3^n}(x^2+2) = f_{2e\cdot 3^n}$.

When $z_n(1) = Z_n$, equation (2.5) gives the recurrence

$$z_{n+1} = z_n(5z_n^2 + 3),$$

where $Z_0 = F_{2e}$. Clearly, $Z_n = F_{2e\cdot 3^n}$, where $n \geq 0$.

In particular, let $e = 6$. Then $Z_0 = F_{12} = 144$, and hence, $Z_1 = Z_0(5Z_0^2+3) = 14,930,352 = F_{12\cdot 3}$. Consequently, $Z_2 = 14930352(5\cdot 14930352^2 + 3) = 16,641,027,750,620,563,662,096 = F_{12\cdot 3^2}$.

2.2.2. *Jacobsthal Byproducts.* Letting $t = \dfrac{2x+1}{x}$ and $z_n = z_n(x) = x^{e\cdot 3^n-1}b_n(t)$. Then equation (2.4) yields

$$z_{n+1} = z_n\left[(4x+1)z_n^2 + 3x^{2e\cdot 3^n}\right], \tag{2.6}$$

where $z_0 = x^{e-1}b_0(t) = x^{e-1}V_e(t) = J_{2e}(x)$. Then $z_n = x^{e\cdot 3^n-1}b_n(t) = x^{e\cdot 3^n-1}V_{e\cdot 3^n}(t) = J_{2e\cdot 3^n}(x)$, where $n \geq 0$.

Suppose we let $z_n(2) = Z_n$. Then equation (2.6) yields the Jacobsthal recurrence

$$Z_{n+1} = Z_n\left(9Z_n^2 + 3\cdot 4^{e\cdot 3^n}\right),$$

where $Z_0 = J_{2e}$. Clearly, $Z_n = J_{2e\cdot 3^n}$, where $n \geq 0$.

In particular, let $e = 4$. Then $Z_0 = J_8 = 85$ and $Z_1 = 85(9\cdot 85^2 + 3\cdot 4^4) = 5,592,405 = J_{8\cdot 3}$. Consequently, $Z_2 = 5592405(9\cdot 5592405^2 + 3\cdot 4^{12}) = 1,574,122,160,956,548,404,565 = J_{8\cdot 3^2}$.

Next we pursue Chebyshev byproducts.

2.2.3. *Chebyshev Byproducts.* When $z_n = z_n(x) = b_n(2x)$, it follows from equation (2.4) that

$$z_{n+1} = z_n \left[4(x^2-1)z_n^2 + 3\right], \tag{2.7}$$

where $z_0 = b_0(2x) = V_e(2x) = U_{e-1}(x)$. Then $z_n = b_n(2x) = V_{e\cdot 3^n}(2x) = U_{e\cdot 3^n-1}(x)$, where $n \geq 0$.

Next we investigate the polynomial extensions of recurrence B in [10]. The steps and technique involved are quite similar; so we will highlight the key steps only.

3. Polynomial Extensions of Recurrence B

Consider the second-order recurrence

$$a_{n+2} = a_{n+1}(\Delta^2 a_n^2 + 2), \tag{3.1}$$

where $a_n = a_n(x)$; $a_1 = f_{2k}$; $a_2 = f_{4k}$; k is an odd positive integer; and $n \geq 1$ [10]. The solution of this recurrence is $a_n = f_{k\cdot 2^n}$.

3.1. Jacobsthal Extensions.

3.1. Jacobsthal Extensions. Letting $b_n = b_n(x) = x^{(k\cdot 2^n-1)/2}a_n(1/\sqrt{x})$, it follows from equation (3.1) that

$$b_{n+2} = b_{n+1}\left[(4x+1)b_n^2 + 2x^{k\cdot 2^n}\right], \tag{3.2}$$

where $b_1 = x^{(2k-1)/2}a_1(1/\sqrt{x}) = x^{(2k-1)/2}f_{2k}(1/\sqrt{x}) = J_{2k}(x); b_2 = x^{(4k-1)/2}a_2(1/\sqrt{x}) = x^{(4k-1)/2}f_{4k}(1/\sqrt{x}) = J_{4k}(x)$, where $n \geq 1$. Clearly, $b_n = x^{(k\cdot 2^n-1)/2}a_n(1/\sqrt{x}) = x^{(k\cdot 2^n-1)/2}f_{k\cdot 2^n} = J_{k\cdot 2^n}(x)$.

Suppose we let $b_n(2) = B_n$. Then equation (3.2) implies

$$B_{n+2} = B_{n+1}\left(9B_n^2 + 2\cdot 2^{k\cdot 2^n}\right), \tag{3.3}$$

where $B_1 = J_{2k}$ and $B_2 = J_{4k}$. We then have $B_n = J_{k\cdot 2^n}$, where $n \geq 1$.

In particular, let $k = 7$. Then

$$B_{n+2} = B_{n+1}\left(9B_n^2 + 2\cdot 128^{2^n}\right),$$

where $B_1 = J_{14} = 5461$ and $B_2 = J_{28} = 89,478,485$. Consequently, $B_3 = 89478485\left(9\cdot 5461^2 + 2\cdot 128^2\right) = 24,019,198,012,642,645 = J_{7\cdot 2^3}$.

3.2. Vieta Extensions. Letting $b_n = b_n(x) = i^{k\cdot 2^n-1}a_n(-ix)$, equation (3.1) yields

$$b_{n+2} = b_{n+1}\left[(x^2-4)b_n^2 + 2\right], \tag{3.4}$$

where $b_1 = i^{2k-1}a_1(-ix) = i^{2k-1}f_{2k}(-ix) = V_{2k}(x); b_2 = i^{4k-1}a_2(-ix) = i^{4k-1}f_{4k}(-ix) = V_{4k}(x)$, where $n \geq 1$. Since $b_n = i^{k\cdot 2^n-1}a_n(-ix)$, it follows that $b_n = i^{k\cdot 2^n-1}f_{k\cdot 2^n}(-ix) = V_{k\cdot 2^n}(x)$.

We now pursue the Fibonacci, Jacobsthal, and Chebyshev consequences of recurrence (3.4).

3.2.1. *Fibonacci Consequences.* Suppose we let $z_n = z_n(x) = xb_n(x^2+2)$. Recurrence (3.4) then yields

$$z_{n+2} = z_{n+1}\left[(x^2+4)z_n^2 + 2\right], \tag{3.5}$$

where $z_1 = xV_{2k}(x^2+2) = f_{4k}$ and $z_2 = xV_{4k}(x^2+2) = f_{8k}$. Then $z_n = xV_{k\cdot 2^n}(x^2+2) = f_{2k\cdot 2^n}$, where $n \geq 1$.

When $z_n(1) = Z_n$, this recurrence implies

$$Z_{n+2} = Z_{n+1}\left(5Z_n^2 + 2\right),$$

where $Z_1 = F_{4k}$ and $Z_2 = F_{8k}$. It also follows that $Z_n = F_{2k\cdot 2^n}$, where $n \geq 1$.

Suppose, for example, $k = 5$. Then $Z_1 = F_{20} = 6765$ and $Z_2 = F_{40} = 102,334,155$. Consequently, $Z_3 = 102334155\left(5 \cdot 6765^2 + 2\right) = 23,416,728,348,467,685 = F_{10\cdot 2^3}$.

3.2.2. *Jacobsthal Consequences.* Let $t = \dfrac{2x+1}{x}$ and $z_n = z_n(x) = x^{k\cdot 2^n - 1} b_n(t)$. Equation (3.4) yields

$$z_{n+2} = z_{n+1}\left[(4x+1)z_n^2 + 2x^{2k\cdot 2^n}\right], \tag{3.6}$$

where $z_1 = x^{2k-1}b_1(t) = x^{2k-1}V_{2k}(t) = J_{4k}(x)$ and $z_2 = x^{4k-1}b_2(t) = x^{4k-1}V_{4k}(t) = J_{8k}(x)$. Since $z_n = x^{k\cdot 2^n - 1}b_n(t)$, it follows that $z_n = J_{2k\cdot 2^n}(x)$, where $n \geq 1$.

Letting $z_n(2) = Z_n$, recurrence (3.6) implies that

$$Z_{n+2} = Z_{n+1}\left(9Z_n^2 + 2\cdot 4^{k\cdot 2^n}\right), \tag{3.7}$$

where $Z_1 = J_{4k}$ and $Z_2 = J_{8k}$. The solution of this recurrence is $Z_n = J_{2k\cdot 2^n}$, where $n \geq 1$.

Suppose, for example, we let $k = 5$. Then $Z_1 = J_{20} = 349,525$ and $Z_2 = J_{40} = 366,503,875,925$. Recurrence (3.7), coupled with these two initial conditions, implies that $Z_3 = 366503875925(9 \cdot 349525^2 + 2 \cdot 4^{10} = 402,975,273,204,876,391,568,725 = J_{10\cdot 2^3}$.

Finally, we present the Chebyshev ramifications.

3.2.3. *Chebyshev Consequences.* Letting $z_n = z_n(x) = b_n(2x)$, it follows from recurrence (3.4) that

$$z_{n+2} = z_{n+1}\left[4(x^2-1)z_n^2 + 2\right], \tag{3.8}$$

where $z_1 = U_{2k-1}(x)$ and $z_2 = U_{4k-1}(x)$. Clearly, $z_n = U_{k\cdot 2^n - 1}(x)$, where $n \geq 1$.

Interestingly, Fibonacci recurrences (2.1) and (3.4) have delightful Lucas counterparts [10]. We now investigate their Jacobsthal-Lucas, Vieta, and Chebyshev extensions and implications.

4. Lucas Extensions of Recurrence A

Consider the recurrence

$$a_{n+1} = a_n(a_n^2 - 3), \tag{4.1}$$

where $a_0 = l_e$, e is an even positive integer, and $n \geq 0$. Its solution is $a_n = l_{e\cdot 3^n}$ [10].

We now pursue its Jacobsthal-Lucas Extensions.

4.1. Jacobsthal-Lucas Extensions.

Let $b_n = b_n(x) = x^{(e\cdot 3^n)/2}a_n(1/\sqrt{x})$. Replacing x with $1/\sqrt{x}$ in (4.1), and then multiplying the resulting equation by $x^{e\cdot 3^{n+1}}$, we get the recurrence

$$b_{n+1} = b_n\left(b_n^2 - 3x^{e\cdot 3^n}\right), \tag{4.2}$$

where $b_0 = x^{e/2}a_0(1/\sqrt{x}) = x^{e/2}l_e(1/\sqrt{x}) = j_e(x)$ and $n \geq 0$. Its solution is
$b_n = x^{(e\cdot 3^n)/2}a_n(1/\sqrt{x}) = x^{(e\cdot 3^n)/2}l_{e\cdot 3^n}(1/\sqrt{x}) = j_{e\cdot 3^n}(x)$.

Letting $B_n = b_n(2)$, recurrence (4.2) yields its Jacobsthal-Lucas numeric counterpart:

$$B_{n+1} = B_n\left(B_n^2 - 3\cdot 2^{e\cdot 3^n}\right), \tag{4.3}$$

where $B_0 = j_e$. Clearly, $B_n = j_{e\cdot 3^n}$, where $n \geq 0$.

When $e = 4$, this recurrence yields $B_1 = 17\left(17^2 - 3\cdot 2^4\right) = 4,097 = j_{4\cdot 3}$. Consequently, $B_2 = 4097\left(4097^2 - 3\cdot 2^{12}\right) = 68,719,476,737 = j_{4\cdot 3^2}$.

Next we explore Vieta extensions.

4.2. **Vieta Extensions.** This time, we let $b_n = b_n(x) = i^{e\cdot 3^n} a_n(-ix)$. Replacing x with $-ix$ in equation (4.1), and then multiplying the resulting equation by $i^{e\cdot 3^{n+1}}$, yields

$$b_{n+1} = b_n(b_n^2 - 3), \tag{4.4}$$

where $b_0 = i^e a_n(-ix) = i^e l_e(-ix) = v_e(x)$. The corresponding solution is given by $b_n = i^{e\cdot 3^n} a_n(-ix) = i^{e\cdot 3^n}(-ix) = v_{e\cdot 3^n}(x)$, where $n \geq 0$.

It follows from Table 1 that recurrence (4.4) has Lucas, Jacobsthal-Lucas, and Chebyshev implications.

4.2.1. *Lucas Implications.* Suppose we let $z_n = z_n(x) = xb_n(x^2+2)$ in equation (4.4). Then

$$z_{n+1} = z_n\left(\frac{z_n^2}{x^2} - 3\right), \tag{4.5}$$

where $z_0 = xb_0(x^2+2) = xv_e(x^2+2) = l_{2e}$. Clearly, $z_n = xb_n(x^2+2) = xv_{e\cdot 3^n}(x^2+2) = l_{2e\cdot 3^n}$, where $n \geq 0$.

In particular, letting $z_n(1) = Z_n$, equation (4.5) yields the recurrence

$$Z_{n+1} = Z_n\left(Z_n^2 - 3\right),$$

where $Z_0 = L_{2e}$. Then $Z_n = L_{2e\cdot 3^n}$, where $n \geq 0$.

For example, let $e = 6$. Then $Z_0 = L_{12} = 322$ and hence, $Z_1 = 322(322^2 - 3) = 33{,}385{,}282$. Consequently, $Z_2 = 33385282(33385282^2 - 3) = 37{,}210{,}469{,}265{,}847{,}998{,}489{,}922 = L_{12\cdot 3^2}$.

Next we pursue Jacobsthal-Lucas consequences.

4.2.2. *Jacobsthal-Lucas Implications.* Let $t = \dfrac{2x+1}{x}$ and $z_n = z_n(x) = x^{e\cdot 3^n} b_n(t)$. It then follows from recurrence (4.4) that

$$z_{n+1} = z_n\left(z_n^2 - 3x^{2e\cdot 3^n}\right), \tag{4.6}$$

where $z_0 = x^e b_0(t) = x^e v_e(t) = j_{2e}(x)$ and $n \geq 0$. The solution of this recurrence is $z_n = x^{e\cdot 3^n} b_n(t) = x^{e\cdot 3^n} v_{e\cdot 3^n}(t) = j_{2e\cdot 3^n}(x)$.

Letting $z_n(2) = Z_n$, we then have the Jacobsthal-Lucas counterpart:

$$Z_{n+1} = Z_n\left(Z_n^2 - 3\cdot 4^{e\cdot 3^n}\right),$$

where $Z_0 = j_{2e}$. Correspondingly, $Z_n = j_{2e\cdot 3^n}$, where $n \geq 0$.

For example, let $e = 4$. Then $Z_0 = j_8 = 257$ and $Z_1 = 257(257^2 - 3\cdot 4^4) = 16{,}777{,}217 = j_{8\cdot 3}$. Consequently, $Z_2 = 16777217\left(16777217^2 - 3\cdot 4^{12}\right) = 4{,}722{,}366{,}482{,}869{,}645{,}213{,}697 = j_{8\cdot 3^2}$.

Next we present Chebyshev consequences of recurrence (4.4).

4.2.3. *Chebyshev Implications.* Let $z_n = z_n(x) = \dfrac{1}{2}b_n(2x)$. Then equation (4.4) yields the recurrence

$$z_{n+1} = z_n\left(4z_n^2 - 3\right), \tag{4.7}$$

where $z_0 = \dfrac{1}{2}b_0(2x) = \dfrac{1}{2}v_e(2x) = T_{2e}(x)$ and $n \geq 0$. Its solution is $z_n = \dfrac{1}{2}b_n(2x) = \dfrac{1}{2}v_{e\cdot 3^n}(2x) = T_{2e\cdot 3^n}(x)$.

Next we focus on Recurrence B from [10].

5. Lucas Extensions of Recurrence B

Consider the second-order recurrence

$$a_{n+2} = a_{n+1}(a_n^2 - 2) - 2, \tag{5.1}$$

where $a_1 = l_{2k}$; $a_2 = l_{4k}$; and k is an odd positive integer [10]. Then $a_n = l_{k\cdot 2^n}$ [10].

As can be predicted, recurrence (5.1) also has Jacobsthal-Lucas, Vieta, and Chebyshev extensions.

5.1. **Jacobsthal-Lucas Extensions.**

Let $b_n = b_n(x) = x^{(k\cdot 2^n)/2} a_n(1/\sqrt{x})$. It then follows from equation (5.1) that

$$b_{n+2} = b_{n+1}\left(b_n^2 - 2x^{k\cdot 2^n}\right) - 2x^{2k\cdot 2^n}, \tag{5.2}$$

where $b_1 = x^k a_1(1/\sqrt{x}) = x^k l_{2k}(1/\sqrt{x}) = j_{2k}(x); b_2 = x^k a_2(1/\sqrt{x}) = x^{2k} l_{4k}(1/\sqrt{x}) = j_{4k}(x)$; and $n \ge 1$. The solution of this recurrence is $b_n = x^{(k\cdot 2^n)/2} a_n(1/\sqrt{x}) = x^{(k\cdot 2^n)/2} l_{k\cdot 2^n}(1/\sqrt{x}) = j_{k\cdot 2^n}$.

Letting $b_n(2) = B_n$, this yields the numeric counterpart:

$$B_{n+2} = B_{n+1}\left(B_n^2 - 2\cdot 2^{k\cdot 2^n}\right) - 2\cdot 4^{k\cdot 2^n}, \tag{5.3}$$

where $B_1 = j_{2k}, B_2 = j_{4k}$, and $n \ge 1$.

When $k = 5$, for example, this yields the recurrence

$$B_{n+2} = B_{n+1}\left(B_n^2 - 2\cdot 2^{5\cdot 2^n}\right) - 2\cdot 1024^{2^n},$$

where $B_1 = j_{10} = 1025$ and $B_2 = j_{20} = 1,048,577$. Consequently,
$B_3 = 1048577\left(1025^2 - 2^{11}\right) - 2\cdot 1024^2 = 1,099,511,627,777 = j_{5\cdot 2^3}$.

Next we pursue Vieta-Lucas polynomial extensions.

5.2. **Vieta-Lucas Extensions.**

Let $b_n = b_n(x) = i^{k\cdot 2^n} a_n(-ix)$. Replacing x with $-ix$ in (5.1) and multiplying the resulting equation by $i^{k\cdot 2^{n+2}}$, we get the recurrence

$$b_{n+2} = b_{n+1}\left(b_n^2 - 2\right) - 2, \tag{5.4}$$

where $b_1 = i^{2k} a_1(-ix) = i^{2k} l_{2k}(-ix) = v_{2k}(x); b_2 = i^{4k} a_2(-ix) = i^{4k} l_{4k}(-ix) = v_{4k}(x$, and $n \ge 1$. Its solution is $b_n = i^{k\cdot 2^n} a_n(-ix) = i^{k\cdot 2^n} l_{k\cdot 2^n}(-ix) = v_{k\cdot 2^n}(x)$.

Recurrence (5.4) also has Lucas, Jacobsthal-Lucas, and Chebyshev ramifications.

5.2.1. *Lucas Byproducts.*

Let $z_n = z_n(x) = xb_n(x^2+2)$. Replacing x with x^2+2 in recurrence (5.4) and then multiplying the resulting equation by x yields

$$z_{n+2} = z_{n+1}\left(\frac{z_n^2}{x^2} - 2\right) - 2x, \tag{5.5}$$

where $z_1 = xb_1(x^2+2) = xv_{2k}(x^2+2) = l_{2k}; z_2 = xb_2(x^2+2) = xv_{4k}(x^2+2) = l_{4k}$; and $n \ge 1$. The solution of recurrence (5.5) $z_n = xb_n(x^2+2) = xv_{k\cdot 2^n}(x^2+2) = l_{2k\cdot 2^n}$.

Suppose we let $z_n(1) = Z_n$. Then equation (5.5) yields its Lucas counterpart:

$$Z_{n+2} = Z_{n+1}\left(Z_n^2 - 2\right) - 2, \tag{5.6}$$

where $Z_1 = L_{2k}$ and $Z_2 = L_{4k}$. Then $Z_n = L_{2k\cdot 2^n}$, where $n \ge 1$.

In particular, let $k = 5$. Then $Z_1 = L_{10} = 123$ and $Z_2 = L_{20} = 15,127$. Consequently, $Z_3 = 15127(123^2 - 2) - 2 = 228,826,127 = L_{5\cdot 2^3}$ and hence, $Z_4 = 228826127(15127^2 - 2) - 2 = 52,361,396,397,820,127 = L_{5\cdot 2^4}$.

Next we pursue implications to Jacobsthal-Lucas polynomials.

5.2.2. *A Jacobsthal-Lucas Byproducts.* Let $t = \dfrac{2x+1}{x}$ and $z_n = z_n(x) = x^{k\cdot 2^n} b_n(t)$. It then follows from equation (5.4) that

$$z_{n+2} = z_{n+1}\left(z_n^2 - 2x^{4k\cdot 2^n}\right), \tag{5.7}$$

where $z_1 = x^{2k}b_1(t) = x^{2k}v_{2k}(t) = j_{4k}(x); z_2 = x^{4k}b_2(t) = x^{4k}v_{4k}(t) = j_{8k}(x)$; and $n \geq 1$. It now follows that $z_n = x^{k\cdot 2^n}b_n(t) = x^{k\cdot 2^n}v_{k\cdot 2^n}(t) = j_{2k\cdot 2^n}(x)$.

Letting $z_n(2) = Z_n$, it follows from recurrence (5.7) that

$$Z_{n+2} = Z_{n+1}\left(Z_n^2 - 2\cdot 4^{k\cdot 2^n}\right) - 2\cdot 16^{k\cdot 2^n},$$

where $Z_1 = j_{4k}, Z_2 = j_{8k}$, and $n \geq 1$. Clearly, the solution of this numeric version is $Z_n = j_{2k\cdot 2^n}$.

In particular, let $k = 5$. Since $Z_1 = j_{20} = 1,048,577$ and $Z_2 = j_{40} = 1,099,511,627,777$, it follows that $Z_3 = 1099511627777\left(1048577^2 - 2\cdot 4^{10}\right) - 2\cdot 16^{10} =$
$1,208,925,819,614,629,174,706,177 = j_{10\cdot 2^3}$.

Finally, we study Chebyshev consequences.

5.2.3. *Chebyshev Byproducts.* Letting $z_n = z_n(x) = \dfrac{1}{2}b_n(2x)$, equation (5.4) yields the recurrence

$$z_{n+2} = z_{n+1}\left(4z_n^2 - 2\right) - 2,$$

where $z_1 = \dfrac{1}{2}b_1(2x) = \dfrac{1}{2}v_{2k}(2x) = T_{2k}(x)$; $z_2 = \dfrac{1}{2}b_1(2x) = \dfrac{1}{2}v_{4k}(2x) = T_{4k}(x)$, and $n \geq 1$. Consequently, $z_n = \dfrac{1}{2}b_n(2x) = \dfrac{1}{2}v_{k\cdot 2^n}(2x) = T_{k\cdot 2^n}(x)$.

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