WEIGHTED SUMS OF SOME SECOND-ORDER SEQUENCES

KUNLE ADEGOKE

Abstract. We derive weighted summation identities involving the second-order recurrence sequence $\{w_n\} = \{w_n(a,b;p,q)\}$ defined by $w_0 = a$, $w_1 = b$; $w_n = pw_{n-1} - qw_{n-2}\,(n \ge 2)$, where a, b, p, and q are arbitrary complex numbers, with $p \ne 0$ and $q \ne 0$.

1. Introduction

Horadam [2] wrote a paper in which he established the basic arithmetical properties of his generalized Fibonacci sequence $\{w_n\} = \{w_n(a,b;p,q)\}$ defined by

$$w_0 = a,\ w_1 = b;\ w_n = pw_{n-1} - qw_{n-2}\,(n \ge 2)\,, \tag{1.1}$$

where a, b, p, and q are arbitrary complex numbers, with $p \ne 0$ and $q \ne 0$. Some well studied particular cases of $\{w_n\}$ are the sequences $\{u_n\}$, $\{v_n\}$, $\{G_n\}$, $\{P_n\}$, and $\{J_n\}$ given by:

$$w_n(1,p;p,q) = u_n(p,q)\,, \tag{1.2}$$

$$w_n(2,p;p,q) = v_n(p,q)\,, \tag{1.3}$$

$$w_n(a,b;1,-1) = G_n(a,b)\,, \tag{1.4}$$

$$w_n(0,1;2,-1) = P_n\,, \tag{1.5}$$

and

$$w_n(0,1;1,-2) = J_n\,. \tag{1.6}$$

Note that $u_n(1,-1) = F_{n+1}$ and $v_n(1,-1) = L_n$, where $F_n = G_n(0,1)$ and $L_n = G_n(2,1)$ are the classic Fibonacci numbers and Lucas numbers, respectively. P_n and J_n are the Pell numbers and Jacobsthal numbers, respectively. Note also that $u_n(2,-1) = P_{n+1}$ and $u_n(1,-2) = J_{n+1}$. The sequence $\{G_n\}$ was introduced by Horadam [1] in 1961, (under the notation $\{H_n\}$).

Extension of the definition of w_n to negative subscripts is provided by writing the recurrence relation as $w_{-n} = (pw_{-n+1} - w_{-n+2})/q$. Horadam [2] showed that:

$$u_{-n} = -q^{-n+1}u_{n-2}\,, \tag{1.7}$$

$$v_{-n} = q^n v_n\,, \tag{1.8}$$

and

$$w_{-n} = \frac{au_n - bu_{n-1}}{au_n + (b-pa)u_{n-1}}w_n\,. \tag{1.9}$$

Our main goal in this paper is to derive weighted summation identities involving the numbers w_n. For example, we shall derive (Theorem 5) the following weighted binomial sum:

$$(-qu_{r-1})^k \sum_{j=0}^{k} \binom{k}{j} \left(-\frac{u_r}{qu_{r-1}}\right)^j w_{m-k(r+1)+j} = w_m\,,$$

which generalizes Horadam's result [2, equation 3.19]:

$$(-q)^n \sum_{j=0}^{n} \binom{n}{j} \left(-\frac{p}{q}\right)^j w_j = w_{2n}\,,$$

the latter identity being an evaluation of the former at $m = 2n$, $k = n$, $r = 1$.

As another example, it is known (first identity of Cor. 15, [3]) that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} F_j = -F_n \,,$$

but this can be generalized to:

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{G_{rj}}{F_{r+1}^j} = \left(\frac{F_r}{F_{r+1}}\right)^k \frac{G_0 F_{k+1} - G_1 F_k}{G_0 F_{k-1} + G_1 F_k} G_k \,,$$

which is itself a special case of a more general result (see Theorem 6):

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{w_{rj}}{u_r^j} = \left(\frac{u_{r-1}}{u_r}\right)^k \frac{a u_k - b u_{k-1}}{a u_k + (b - pa) u_{k-1}} w_k \,.$$

As an example of non-binomial sums derived in this paper, we mention (see Theorem 3):

$$u_{r-1} \sum_{j=0}^{k} \frac{w_{rj}}{(-q u_{r-2})^j} = \frac{w_{kr+r-1}}{(-q u_{r-2})^k} + (ap - b) u_{r-2} \,,$$

of which a special case is

$$F_r \sum_{j=0}^{k} \frac{G_{rj}}{F_{r-1}^j} = \frac{G_{kr+r-1}}{F_{r-1}^k} - F_{r-1}(G_1 - G_0) \,.$$

Another example in this category is (see Theorem 2):

$$q^{n-r} e u_{r-1} \sum_{j=0}^{k} \frac{u_{rj}}{(w_n / w_{n-r})^j} = \frac{w_{n+kr+1} w_{n-r}}{(w_n / w_{n-r})^k} - w_n w_{n-r+1} \,,$$

giving rise to the following results for the $\{G_m\}$, $\{P_m\}$, and $\{J_m\}$ sequences:

$$(-1)^{n-r} (G_0 G_1 + G_0^2 - G_1^2) F_r \sum_{j=0}^{k} \frac{F_{rj+1}}{(G_n / G_{n-r})^j} = \frac{G_{n+kr+1} G_{n-r}}{(G_n / G_{n-r})^k} - G_n G_{n-r+1} \,,$$

$$(-1)^{n-r-1} P_r \sum_{j=0}^{k} \frac{P_{rj+1}}{(P_n / P_{n-r})^j} = \frac{P_{n+kr+1} P_{n-r}}{(P_n / P_{n-r})^k} - P_n P_{n-r+1} \,,$$

and

$$(-1)^{n-r-1} 2^{n-r} J_r \sum_{j=0}^{k} \frac{J_{rj+1}}{(J_n / J_{n-r})^j} = \frac{J_{n+kr+1} J_{n-r}}{(J_n / J_{n-r})^k} - J_n J_{n-r+1} \,.$$

We require the following identities, derived in [2]:

$$w_{m+r} = u_r w_m - q u_{r-1} w_{m-1} \,, \tag{1.10}$$

$$v_r w_m = w_{m+r} + q^r w_{m-r} \,, \tag{1.11}$$

and

$$w_{n-r} w_{m+n+r} = w_n w_{m+n} + q^{n-r} e u_{r-1} u_{m+r-1} \,, \tag{1.12}$$

where $e = pab - qa^2 - b^2$.

2. Weighted Sums

Lemma 1. *Let $\{X_m\}$ and $\{Y_m\}$ be any two sequences such that X_m and Y_m, $m \in \mathbb{Z}$, are connected by a second-order recurrence relation $X_m = xX_{m-\alpha} + yY_{m-\beta}$, where x and y are arbitrary non-vanishing complex functions, not dependent on m, and α and β are integers. Then,*

$$y\sum_{j=0}^{k}\frac{Y_{m-k\alpha-\beta+\alpha j}}{x^j} = \frac{X_m}{x^k} - xX_{m-(k+1)\alpha}\,,$$

for k a non-negative integer.

In particular,

$$y\sum_{j=0}^{k}\frac{Y_{\alpha j}}{x^j} = \frac{X_{k\alpha+\beta}}{x^k} - xX_{\beta-\alpha}\,. \tag{2.1}$$

Proof. The proof shall be by induction on k. Consider the proposition $\mathcal{P}_k$,

$$\mathcal{P}_k : \left(y\sum_{j=0}^{k}\frac{Y_{m-k\alpha-\beta+\alpha j}}{x^j} = \frac{X_m}{x^k} - xX_{m-(k+1)\alpha}\right)\,;$$

with respect to the relation $X_m = xX_{m-\alpha} + yY_{m-\beta}$. Clearly, $\mathcal{P}_0$ is true. Assume that $\mathcal{P}_n$ is true for a certain positive integer n. We want to prove that $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$. Now,

$$\mathcal{P}_n : \left(f(n) = \frac{X_m}{x^n} - xX_{m-(n+1)\alpha}\right)\,;$$

where

$$f(n) = y\sum_{j=0}^{n}\frac{Y_{m-n\alpha-\beta+j\alpha}}{x^j}\,.$$

We have

$$\begin{aligned}
f(n+1) &= y\sum_{j=0}^{n+1}\frac{Y_{m-n\alpha-\alpha-\beta+j\alpha}}{x^j} = y\sum_{j=-1}^{n}\frac{Y_{m-n\alpha-\alpha-\beta+j\alpha+\alpha}}{x^{j+1}}\\
&= \frac{y}{x}\sum_{j=-1}^{n}\frac{Y_{m-n\alpha-\beta+j\alpha}}{x^j} = \frac{y}{x}\left(xY_{m-n\alpha-\beta-\alpha} + \sum_{j=0}^{n}\frac{Y_{m-n\alpha-\beta+j\alpha}}{x^j}\right)\\
&= yY_{m-n\alpha-\alpha-\beta} + \frac{1}{x}\left(y\sum_{j=0}^{n}\frac{Y_{m-n\alpha-\beta+j\alpha}}{x^j}\right)\\
&\quad\text{(invoking the induction hypothesis } \mathcal{P}_n)\\
&= yY_{m-n\alpha-\alpha-\beta} + \frac{1}{x}\left(\frac{X_m}{x^n} - xX_{m-n\alpha-\alpha}\right)\\
&= \frac{X_m}{x^{n+1}} - \left(X_{m-n\alpha-\alpha} - yY_{m-n\alpha-\alpha-\beta}\right).
\end{aligned}$$

Since $X_{m-n\alpha-\alpha} - yY_{m-n\alpha-\alpha-\beta} = xX_{m-n\alpha-\alpha-\alpha}$, we finally have

$$f(n+1) = \frac{X_m}{x^{n+1}} - xX_{m-(n+2)\alpha}\,.$$

Thus,

$$\mathcal{P}_{n+1} : \left(f(n+1) = \frac{X_m}{x^{n+1}} - xX_{m-(n+1+1)\alpha} \right) ;$$

i.e. $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$ and the induction is complete. □

Note that the identity of Lemma 1 can also be written in the equivalent form:

$$y \sum_{j=0}^{k} x^j Y_{m-\beta-j\alpha} = X_m - x^{k+1} X_{m-(k+1)\alpha} \,. \tag{2.2}$$

In particular,

$$y \sum_{j=0}^{k} x^j Y_{-j\alpha} = X_\beta - x^{k+1} X_{\beta-(k+1)\alpha} \,. \tag{2.3}$$

Theorem 1. *For integer m, non-negative integer k and any integer r for which $w_{r-1} \neq 0$, the following identity holds:*

$$\sum_{j=0}^{k} \left(\frac{w_r}{qw_{r-1}} \right)^j w_{m+r-k+j} = \left(\frac{w_r}{qw_{r-1}} \right)^k u_m w_r - qu_{m-k-1} w_{r-1} \,.$$

In particular,

$$q^{r-1} \sum_{j=0}^{k} \left(\frac{w_r}{qw_{r-1}} \right)^j w_j = \left(\frac{w_r}{qw_{r-1}} \right)^k q^{r-1} u_{k-r} w_r + u_{r-1} w_{r-1} \,. \tag{2.4}$$

Proof. Interchange m and r in identity (1.10) and write the resulting identity as

$$u_m = \frac{qw_{r-1}}{w_r} u_{m-1} + \frac{1}{w_r} w_{m+r} \,.$$

Identify $X = u$, $Y = w$, $x = qw_{r-1}/w_r$, $y = 1/w_r$, $\alpha = 1$, and $\beta = -r$, and use these in Lemma 1. □

The Fibonacci, Lucas, and Pell versions of Theorem 1 are, respectively,

$$\sum_{j=0}^{k} (-1)^j \left(\frac{F_r}{F_{r-1}} \right)^j F_{m+r-k+j} = (-1)^k \left(\frac{F_r}{F_{r-1}} \right)^k F_{m+1} F_r + F_{m-k} F_{r-1} \,, \tag{2.5}$$

$$\sum_{j=0}^{k} (-1)^j \left(\frac{L_r}{L_{r-1}} \right)^j L_{m+r-k+j} = (-1)^k \left(\frac{L_r}{L_{r-1}} \right)^k F_{m+1} L_r + F_{m-k} L_{r-1} \,, \tag{2.6}$$

and

$$\sum_{j=0}^{k} (-1)^j \left(\frac{P_r}{P_{r-1}} \right)^j P_{m+r-k+j} = (-1)^k \left(\frac{P_r}{P_{r-1}} \right)^k P_{m+1} P_r + P_{m-k} P_{r-1} \,. \tag{2.7}$$

In particular, we have

$$\sum_{j=0}^{k} (-1)^j \left(\frac{F_r}{F_{r-1}} \right)^j F_{r+j} = (-1)^k \left(\frac{F_r}{F_{r-1}} \right)^k F_{k+1} F_r \,, \tag{2.8}$$

$$\sum_{j=0}^{k} (-1)^j \left(\frac{L_r}{L_{r-1}} \right)^j L_{r+j} = (-1)^k \left(\frac{L_r}{L_{r-1}} \right)^k F_{k+1} L_r \tag{2.9}$$

and

$$\sum_{j=0}^{k}(-1)^j\left(\frac{P_r}{P_{r-1}}\right)^j P_{r+j}=(-1)^k\left(\frac{P_r}{P_{r-1}}\right)^k P_{k+1}P_r\,. \tag{2.10}$$

Theorem 2. *For non-negative integer k, integers m and r and any integer n for which $w_n\neq 0$, the following identity holds:*

$$q^{n-r}eu_{r-1}\sum_{j=0}^{k}\frac{u_{m-(n+1)-kr+rj}}{(w_n/w_{n-r})^j}=\frac{w_m w_{n-r}}{(w_n/w_{n-r})^k}-w_n w_{m-(k+1)r}\,.$$

In particular,

$$q^{n-r}eu_{r-1}\sum_{j=0}^{k}\frac{u_{rj}}{(w_n/w_{n-r})^j}=\frac{w_{n+kr+1}w_{n-r}}{(w_n/w_{n-r})^k}-w_n w_{n-r+1}\,. \tag{2.11}$$

Proof. Write identity (1.12) as

$$w_m=\frac{w_n}{w_{n-r}}w_{m-r}+q^{n-r}\frac{eu_{r-1}}{w_{n-r}}u_{m-n-1}\,.$$

Identify $x=w_n/w_{n-r}$, $y=q^{n-r}eu_{r-1}/w_{n-r}$, $\alpha=r$, and $\beta=n+1$, and use these in Lemma 1. □

Results for the $\{G_m\}$ and $\{P_m\}$ sequences emanating from identity (2.11) are the following:

$$(-1)^{n-r}(G_0G_1+G_0^2-G_1^2)F_r\sum_{j=0}^{k}\frac{F_{rj+1}}{(G_n/G_{n-r})^j}=\frac{G_{n+kr+1}G_{n-r}}{(G_n/G_{n-r})^k}-G_nG_{n-r+1} \tag{2.12}$$

and

$$(-1)^{n-r-1}P_r\sum_{j=0}^{k}\frac{P_{rj+1}}{(P_n/P_{n-r})^j}=\frac{P_{n+kr+1}P_{n-r}}{(P_n/P_{n-r})^k}-P_nP_{n-r+1}\,. \tag{2.13}$$

Lemma 2. *Let $\{X_m\}$ be any arbitrary sequence, where X_m, $m\in\mathbb{Z}$, satisfies a second order recurrence relation $X_m=xX_{m-\alpha}+yX_{m-\beta}$, where x and y are arbitrary non-vanishing complex functions, not dependent on m, and α and β are integers. Then,*

$$y\sum_{j=0}^{k}\frac{X_{m-k\alpha-\beta+\alpha j}}{x^j}=\frac{X_m}{x^k}-xX_{m-(k+1)\alpha} \tag{2.14}$$

and

$$x\sum_{j=0}^{k}\frac{X_{m-k\beta-\alpha+\beta j}}{y^j}=\frac{X_m}{y^k}-yX_{m-(k+1)\beta}\,, \tag{2.15}$$

for k a non-negative integer.

In particular,

$$y\sum_{j=0}^{k}\frac{X_{\alpha j}}{x^j}=\frac{X_{k\alpha+\beta}}{x^k}-xX_{\beta-\alpha} \tag{2.16}$$

and

$$x\sum_{j=0}^{k}\frac{X_{\beta j}}{y^j}=\frac{X_{k\beta+\alpha}}{y^k}-yX_{\alpha-\beta}\,. \tag{2.17}$$

Proof. Identity (2.14) is a direct consequence of Lemma 1 with $Y_m = X_m$. Identity (2.15) is obtained from the symmetry of the recurrence relation by interchanging x and y and α and β. □

Note that the identities (2.14) and (2.15) can be written in the following equivalent forms:

$$y\sum_{j=0}^{k} x^j X_{m-\beta-\alpha j} = X_m - x^{k+1}X_{m-(k+1)\alpha} \tag{2.18}$$

and

$$x\sum_{j=0}^{k} y^j X_{m-\alpha-\beta j} = X_m - y^{k+1}X_{m-(k+1)\beta}\,. \tag{2.19}$$

In particular,

$$y\sum_{j=0}^{k} x^j X_{-\alpha j} = X_\beta - x^{k+1}X_{\beta-(k+1)\alpha} \tag{2.20}$$

and

$$x\sum_{j=0}^{k} y^j X_{-\beta j} = X_\alpha - y^{k+1}X_{\alpha-(k+1)\beta}\,. \tag{2.21}$$

Theorem 3. *For non-negative integer k and any integer m, the following identities hold:*

$$qu_r^k u_{r-1}\sum_{j=0}^{k}\frac{w_{m-kr-r-1+rj}}{u_r^j} = u_r^{k+1}w_{m-kr-r} - w_m,\quad r\in\mathbb{Z},\quad r\neq -1\,, \tag{2.22}$$

$$u_{r-1}\sum_{j=0}^{k}\frac{w_{m-kr-r+1+rj}}{(-qu_{r-2})^j} = \frac{w_m}{(-qu_{r-2})^k} + qu_{r-2}w_{m-(k+1)r},\quad r\in\mathbb{Z},\quad r\neq 1\,, \tag{2.23}$$

and

$$\sum_{j=0}^{k}\frac{w_{m-k+r+j}}{(qu_{r-1}/u_r)^j} = \frac{u_r w_m}{(qu_{r-1}/u_r)^k} - qu_{r-1}w_{m-k-1},\quad r\in\mathbb{Z},\quad r\neq 0\,. \tag{2.24}$$

In particular,

$$qu_r^k u_{r-1}\sum_{j=0}^{k}\frac{w_{rj}}{u_r^j} = bu_r^{k+1} - w_{kr+r+1}\,, \tag{2.25}$$

$$u_{r-1}\sum_{j=0}^{k}\frac{w_{rj}}{(-qu_{r-2})^j} = \frac{w_{kr+r-1}}{(-qu_{r-2})^k} + (ap-b)u_{r-2}\,, \tag{2.26}$$

and

$$\sum_{j=0}^{k}\frac{w_j}{(qu_{r-1}/u_r)^j} = \frac{u_r w_{k-r}}{(qu_{r-1}/u_r)^k} - \frac{1}{q^r}\frac{au_{r+1}-bu_r}{au_{r+1}+(b-pa)u_r}u_{r-1}w_{r+1}\,. \tag{2.27}$$

Proof. To prove identities (2.22) and (2.23), write the relation (1.10) as $w_m = u_r w_{m-r} - qu_{r-1}w_{m-r-1}$, identify $X = w$, $x = u_r$, $y = -qu_{r-1}$, $\alpha = r$, and $\beta = r+1$, and use these in Lemma 2. Similarly, identity (2.24) is proved by writing the relation (1.10) as $w_m = (1/u_r)w_{m+r} + (qu_{r-1}/u_r)w_{m-1}$, identifying $X = w$, $x = 1/u_r$, $y = qu_{r-1}/u_r$, $\alpha = -r$, and $\beta = 1$ and using these in Lemma 2. □

Explicit examples from identity (2.27) include

$$\sum_{j=0}^{k}(-1)^j\frac{G_j}{(F_r/F_{r+1})^j}=(-1)^k\frac{F_{r+1}}{(F_r/F_{r+1})^k}G_{k-r}-(-1)^r\frac{F_{r+2}G_0-F_{r+1}G_1}{F_{r+2}G_0+F_{r+1}(G_1-G_0)}F_rG_{r+1}\,, \tag{2.28}$$

$$\sum_{j=0}^{k}(-1)^j\frac{P_j}{(P_r/P_{r+1})^j}=(-1)^k\frac{P_{r+1}P_{k-r}}{(P_r/P_{r+1})^k}+(-1)^rP_rP_{r+1}\,, \tag{2.29}$$

and

$$\sum_{j=0}^{k}\frac{(-1)^j}{2^j}\frac{J_j}{(J_r/J_{r+1})^j}=\frac{(-1)^k}{2^k}\frac{J_{r+1}J_{k-r}}{(J_r/J_{r+1})^k}+\frac{(-1)^r}{2^r}J_rJ_{r+1}\,. \tag{2.30}$$

Theorem 4. *For non-negative integer k and all integers r and m, the following identities hold:*

$$\sum_{j=0}^{k}\frac{w_{m-kr+r+rj}}{(q^r/v_r)^j}=\frac{v_rw_m}{(q^r/v_r)^k}-q^rw_{m-(k+1)r}\,, \tag{2.31}$$

$$v_r^kq^r\sum_{j=0}^{k}\frac{w_{m-r+rj}}{v_r^j}=v_r^{k+1}w_m-w_{m+(k+1)r}\,, \tag{2.32}$$

$$v_r\sum_{j=0}^{k}\frac{w_{m-2kr-r+2rj}}{(-q^r)^j}=\frac{w_m}{(-q^r)^k}+q^rw_{m-(k+1)2r}\,, \tag{2.33}$$

$$v_r\sum_{j=0}^{k}\frac{w_{m+r+2rj}}{q^{rj}}=\frac{w_{m+2r(k+1)}}{q^{kr}}-q^rw_m\,, \tag{2.34}$$

and

$$\sum_{j=0}^{k}\left(-\frac{v_r}{q^r}\right)^jw_{m+2r+rj}=q^rw_m+v_r\left(-\frac{v_r}{q^r}\right)^kw_{m+(k+1)r}\,. \tag{2.35}$$

In particular,

$$\sum_{j=0}^{k}\frac{w_{rj}}{(q^r/v_r)^j}=\frac{v_rw_{kr-r}}{(q^r/v_r)^k}-\frac{1}{q^r}\frac{au_{2r}-bu_{2r-1}}{au_{2r}+(b-pa)u_{2r-1}}w_{2r}\,, \tag{2.36}$$

$$v_r^kq^r\sum_{j=0}^{k}\frac{w_{rj}}{v_r^j}=v_r^{k+1}w_r-w_{(k+2)r}\,, \tag{2.37}$$

$$v_r\sum_{j=0}^{k}\frac{w_{2rj}}{(-q^r)^j}=\frac{w_{(2k+1)r}}{(-q^r)^k}+\frac{au_r-bu_{r-1}}{au_r+(b-pa)u_{r-1}}w_r\,, \tag{2.38}$$

$$v_r\sum_{j=0}^{k}\frac{w_{2rj}}{q^{rj}}=\frac{w_{2rk+r}}{q^{kr}}-\frac{au_r-bu_{r-1}}{au_r+(b-pa)u_{r-1}}w_r\,, \tag{2.39}$$

and

$$q^r\sum_{j=0}^{k}\left(-\frac{v_r}{q^r}\right)^jw_{rj}=\frac{au_{2r}-bu_{2r-1}}{au_{2r}+(b-pa)u_{2r-1}}w_{2r}+q^rv_r\left(-\frac{v_r}{q^r}\right)^kw_{kr-r}\,. \tag{2.40}$$

Proof. To prove identities (2.31) and (2.32), write identity (1.11) as $w_m = (1/v_r)w_{m+r} + (q^r/v_r)w_{m-r}$. Identify $X = w$, $x = 1/v_r$, $y = q^r/v_r$, $\alpha = -r$, and $\beta = r$, and use these in Lemma 2, identities (2.15) and (2.18). Likewise, to prove identity (2.34), write identity (1.11) as $q^r w_m = w_{m+2r} - v_r w_{m+r}$. Identify $X = w$, $x = 1/q^r$, $y = -v_r/q^r$, $\alpha = -2r$, and $\beta = -r$, and use these in Lemma 2, identity (2.18). □

Lemma 3. *Let $\{X_m\}$ be any arbitrary sequence. Let X_m, $m \in \mathbb{Z}$, satisfy a second order recurrence relation $X_m = xX_{m-\alpha} + yX_{m-\beta}$, where x and y are non-vanishing complex functions, not dependent on m, and α and β are integers. Then,*

$$\sum_{j=0}^{k} \binom{k}{j} \left(\frac{x}{y}\right)^j X_{m-k\beta+(\beta-\alpha)j} = \frac{X_m}{y^k}\,,$$

for k a non-negative integer.

In particular,

$$\sum_{j=0}^{k} \binom{k}{j} \left(\frac{x}{y}\right)^j X_{(\beta-\alpha)j} = \frac{X_{k\beta}}{y^k}\,. \tag{2.41}$$

Proof. We apply mathematical induction on k. Obviously, the lemma is true for $k = 0$. We assume that it is true for $k = n$ a positive integer. The induction hypothesis is

$$\mathcal{P}_n : \left(f(n) = \frac{X_m}{y^n}\right)\,;$$

where

$$f(n) = \sum_{j=0}^{n} \binom{k}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta+(\beta-\alpha)j}.$$

We want to prove that $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$. We proceed,

$$\begin{aligned}
f(n+1) &= \sum_{j=0}^{n+1} \binom{n+1}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j}\\
&\qquad \left(\text{since } \binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}\right)\\
&= \sum_{j=0}^{n+1} \binom{n}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j} + \sum_{j=0}^{n+1} \binom{n}{j-1} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j}\\
&= \sum_{j=0}^{n+1} \binom{n}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j} + \sum_{j=1}^{n+1} \binom{n}{j-1} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j}\\
&= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)j} + \frac{x}{y}\sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{y}\right)^j X_{m-n\beta-\beta+(\beta-\alpha)(j+1)}\\
&= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{y}\right)^j \left(X_{m-n\beta-\beta+(\beta-\alpha)j} + \frac{x}{y} X_{m-n\beta-\beta+(\beta-\alpha)(j+1)}\right)\\
&= \frac{1}{y}\sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{y}\right)^j \left(xX_{m-n\beta-\beta+(\beta-\alpha)(j+1)} + yX_{m-n\beta-\beta+(\beta-\alpha)j}\right)
\end{aligned}$$

$$\begin{aligned}&(\text{since } xX_{m-n\beta-\beta+(\beta-\alpha)(j+1)} + yX_{m-n\beta-\beta+(\beta-\alpha)j} = X_{m-n\beta+(\beta-\alpha)j})\\ &= \frac{1}{y}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{x}{y}\right)^j X_{m-n\beta+(\beta-\alpha)j}\\ &= \frac{1}{y}\frac{X_m}{y^n} \quad (\text{by the induction hypothesis})\,.\end{aligned}$$

Thus,

$$f(n+1) = \frac{X_m}{y^{n+1}}\,,$$

so that

$$\mathcal{P}_{n+1} : \left(f(n+1) = \frac{X_m}{y^{n+1}}\right)\,;$$

i.e. $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$ and the induction is complete. □

Note that the identity of Lemma 3 can also be written as

$$\sum_{j=0}^{k}\binom{k}{j}\left(\frac{y}{x}\right)^j X_{m-k\alpha+(\alpha-\beta)j} = \frac{X_m}{x^k}\,, \tag{2.42}$$

with the particular case

$$\sum_{j=0}^{k}\binom{k}{j}\left(\frac{y}{x}\right)^j X_{(\alpha-\beta)j} = \frac{X_{k\alpha}}{x^k}\,. \tag{2.43}$$

Theorem 5. *For non-negative integer k and any integer m, the following identities hold:*

$$(-qu_{r-1})^k\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{u_r}{qu_{r-1}}\right)^j w_{m-k(r+1)+j} = w_m, \quad r\in\mathbb{Z}, \quad r\neq 0\,, \tag{2.44}$$

$$\sum_{j=0}^{k}\binom{k}{j}\frac{w_{m-k+rj}}{(qu_{r-2})^j} = \left(\frac{u_{r-1}}{qu_{r-2}}\right)^k w_m, \quad r\in\mathbb{Z}, \quad r\neq 1, \tag{2.45}$$

and

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{w_{m+k+rj}}{u_r^j} = \left(\frac{qu_{r-1}}{u_r}\right)^k w_m, \quad r\in\mathbb{Z}, \quad r\neq -1\,. \tag{2.46}$$

In particular,

$$(-qu_{r-1})^k\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{u_r}{qu_{r-1}}\right)^j w_j = w_{k(r+1)}\,, \tag{2.47}$$

$$\sum_{j=0}^{k}\binom{k}{j}\frac{w_{rj}}{(qu_{r-2})^j} = \left(\frac{u_{r-1}}{qu_{r-2}}\right)^k w_k\,, \tag{2.48}$$

and

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{w_{rj}}{u_r^j} = \left(\frac{u_{r-1}}{u_r}\right)^k \frac{au_k - bu_{k-1}}{au_k + (b-pa)u_{k-1}} w_k\,. \tag{2.49}$$

Proof. To prove identity (2.44), use, in Lemma 3, the x, y, α, and β found in the proof of identities (2.22) and (2.23) of Theorem 3. To prove identity (2.45), use in Lemma 3, the x, y, α, and β found in the proof of identity (2.24) of Theorem 3. To prove identity (2.46), write the relation (1.10) as $w_m = -(1/(qu_{r-1}))w_{m+r+1} + (u_r/(qu_{r-1}))w_{m+1}$. Identify $X = w$, $x = -(1/(qu_{r-1}))$, $y = (u_r/(qu_{r-1}))$, $\alpha = -1-r$, and $\beta = -1$, and use these in Lemma 3. □

We have the following specific examples from identity (2.48):

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{G_{rj}}{F_{r-1}^j} = (-1)^k \left(\frac{F_r}{F_{r-1}}\right)^k G_k\,, \tag{2.50}$$

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{P_{rj}}{P_{r-1}^j} = (-1)^k \left(\frac{P_r}{P_{r-1}}\right)^k P_k\,, \tag{2.51}$$

and

$$\sum_{j=0}^{k} \frac{(-1)^j}{2^j} \binom{k}{j} \frac{J_{rj}}{J_{r-1}^j} = \frac{(-1)^k}{2^k} \left(\frac{J_r}{J_{r-1}}\right)^k J_k\,. \tag{2.52}$$

Note that identity (2.44) is a generalization of identity (48) of Vajda [4], the latter being the evaluation of the former at $r = 1$ and $q = -1$.

Theorem 6. *For non-negative integer k and all integers m and r, the following identities hold:*

$$\sum_{j=0}^{k} \binom{k}{j} \frac{w_{m-kr+2rj}}{q^{rj}} = \left(\frac{v_r}{q^r}\right)^k w_m\,, \tag{2.53}$$

$$\sum_{j=0}^{k} \binom{k}{j} \left(-\frac{v_r}{q^r}\right)^j w_{m-2kr+rj} = \frac{w_m}{(-q^r)^k}\,, \tag{2.54}$$

and

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{w_{m+kr+rj}}{v_r^j} = (-1)^k \frac{q^{rk} w_m}{v_r^k}\,. \tag{2.55}$$

In particular,

$$\sum_{j=0}^{k} \binom{k}{j} \frac{w_{2rj}}{q^{rj}} = \left(\frac{v_r}{q^r}\right)^k w_{rk}\,, \tag{2.56}$$

$$\sum_{j=0}^{k} \binom{k}{j} \left(-\frac{v_r}{q^r}\right)^j w_{rj} = \frac{w_{2kr}}{(-q^r)^k}\,, \tag{2.57}$$

and

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{w_{rj}}{v_r^j} = (-1)^k \left(\frac{au_{kr} - bu_{kr-1}}{au_{kr} + (b-pa)u_{kr-1}}\right) \frac{w_{kr}}{v_r^k}\,. \tag{2.58}$$

Proof. To prove identity (2.53), use, in Lemma 3, the x, y, α, and β found in the proof of identities (2.31) and (2.32) of Theorem 4. To prove identity (2.54), write identity (1.11) as $w_m = v_r w_{m-r} - q^r w_{m-2r}$. Identify $X = w$, $x = v_r$, $y = -q^r$, $\alpha = r$, and $\beta = 2r$, and use these in Lemma 3. To prove identity (2.55), use, in Lemma 3, the x, y, α, and β found in the proof of identity (2.34) of Theorem 4. □

Setting $p = 1 = -q$ in identity (2.53), we have

$$\sum_{j=0}^{k} (-1)^{rj} \binom{k}{j} G_{m-kr+2rj} = (-1)^{rk} L_r^k G_m \,. \tag{2.59}$$

Identity (2.58) at $p = 1 = -q$ gives

$$\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \frac{G_{rj}}{L_r^j} = (-1)^k \frac{F_{kr+1}G_0 - F_{kr}G_1}{F_{kr+1}G_0 + F_{kr}(G_1 - G_0)} \frac{G_{kr}}{L_r^k} \,. \tag{2.60}$$

3. Acknowledgement

I thank the anonymous referee whose comments helped to improve this paper.

References

[1] A. F. Horadam, *A generalized Fibonacci sequence*, The American Mathematical Monthly, **68.5** (1961), 455–459.

[2] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quarterly, **3.3** (1965), 161–176.

[3] P. Stanica, *Generating functions, weighted and non-weighted sums for powers of second order recurrence sequences*, The Fibonacci Quarterly, **41.4** (2003), 321–333.

[4] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Press, 2008.

MSC2010: 11B37, 11B39, 65B10

Department of Physics and Engineering Physics, Obafemi Awolowo University, 220005 Ile-Ife, Nigeria

E-mail address: adegoke00@gmail.com