# THE SUMS OF THE CONSECUTIVE FIBONACCI NUMBERS

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Abstract. In this paper, we study integer numbers $d$ with the following property: the sum of any $d$ consecutive Fibonacci numbers is divisible by $d$. We call these $d$-numbers. We demonstrate a relation between $d$-numbers and the Pisano period, specifically, we prove that the original problem is equivalent to finding all integer numbers $d > 1$ that are divisible by their own Pisano period. We derive a general expression for all $d$-numbers and obtain convenient recurrent relations that significantly simplify practical calculation. Finally, we establish an equivalence between $d$-numbers and the OEIS sequence A072378.

## 1. Introduction

In this paper, we solve the following problem.

**Problem**. Find and investigate all integer numbers $d > 1$ such that the sum of any $d$ consecutive Fibonacci numbers is divisible by $d$, i.e., that satisfy the following relation for any integer $k$:

$$\sum_{i=k}^{d+k-1} F_i \equiv 0 \pmod{d}. \tag{1.1}$$

We refer to these numbers as $d$-numbers hereafter (not to be confused with $D$ numbers [4]).

We prove that condition (1.1) is equivalent to $\pi(d)|d$ where $\pi(d)$ denotes the Pisano period (see the definition below) of the Fibonacci sequence modulo $d$. Thus, we demonstrate that the original problem is equivalent to finding all integer numbers $d > 1$ that are divisible by their own Pisano period.

To solve the problem, we first derive a general expression (3.4) for the minimal $d$-number $d_k$ which is divisible by a given integer $k > 1$. Then, we prove that the set of all $d_k$ coincides with an infinite set of all $d$-numbers.

The direct use of (3.4), however, may be technically complicated. Hence, we prove theorems that allow us to significantly simplify the calculations. In particular, we derive the recurrent expression (3.8) to easily obtain $d$-numbers $d_p$ for all prime numbers $p$. We further prove that, once all such $d_p$ are known, all other $d$-numbers can be easily found using (3.11).

We show that all the results, formulas, and theorems, which have been obtained for the Fibonacci numbers, are also applicable for generalized Fibonacci numbers with arbitrary starting values $a$ and $b$, and with the usual recurrent formula for Fibonacci numbers.

Finally, we prove that all $d$-numbers are divisible by 24, and the sequence of quotients coincides with the sequence of numbers $n$ such that $12n$ divides $F_{12n}$ [6, A072378].

## 2. Known Results

To proceed, we will use some known results and definitions.

The sequence of the Fibonacci numbers $F_n$ satisfies the recurrent relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$ is periodic modulo $m$ for every integer $m > 1$ [8]. The minimal period of the Fibonacci sequence modulo $m$ is called the Pisano period of $m$ [7]. More rigorously, the Pisano period is defined as follows:

**Definition 2.1.** *For any integer $m > 1$, the least integer $n$ such that $(F_n, F_{n+1}) \equiv (0, 1) \pmod m$ is denoted by $\pi(m)$ and is called the Pisano period of $m$* [2].

**Theorem 2.2.** *(Iteration Theorem)*[2]*: For each integer $m > 1$, there exists a least integer $\omega$ such that $\pi^{\omega+1}(m) = \pi^{\omega}(m)$.*

*Here $\omega$ is called the Fibonacci frequency, and the following notations are used: $\pi^2(k) = \pi(\pi(k))$ and $\pi^{n+1}(k) = \pi(\pi^n(k))$.*

**Theorem 2.3.** *For each $m > 1$ and $n > 1$,*

$$\pi([n, m]) = [\pi(n), \pi(m)],$$

*where $[n, m]$ denotes the least common multiple* [2].

**Theorem 2.4.** *If $m > 2$, then $\pi(m)$ is an even number* [8].

**Theorem 2.5.** *If $p$ is prime and $a$ is a positive integer and $\pi(p^2) \neq \pi(p)$, then $\pi(p^a) = p^{a-1}\pi(p)$. Also, if $t$ is the largest integer with $\pi(p^t) = \pi(p)$, then $\pi(p^a) = p^{a-t}\pi(p)$ for $a > t$ (Theorem 5 in* [8]*).*

Also, we will use the next two corollaries, which follow from Theorem 2.5.

**Corollary 2.6.** $\pi(p^a) | p^{a-1}\pi(p)$.

**Corollary 2.7.** $\pi(p^a) | \pi(p^b)$, *if $a < b$.*

**Theorem 2.8.** *If $q$ and $p$ are prime and $q|\pi(p)$, where $p > 5$, then $q < p$ (Corollary 3.1 from Theorem 2.2 in* [2]*).*

The Fibonacci sequence can be extended to negative indices $n$ using the rearranged recurrence formula [5]

$$F_{n-2} = F_n - F_{n-1},$$

which yields the sequence of numbers satisfying the following relation:

$$F_n = (-1)^{n+1} F_{-n}. \tag{2.1}$$

## 3. The Main Results

Now, we solve the problem under consideration.

First, we prove that condition (1.1) is equivalent to $\pi(d)|d$, where $\pi(d)$ denotes the Pisano period of the Fibonacci sequence modulo $d$.

Using the definition of the Pisano period, and periodicities properties, we have

$$\pi(d)|d \Leftrightarrow F_{n+d} \equiv F_n \pmod d \tag{3.1}$$

for any integer $n$. Using mathematical induction, it is easy to show that last condition is equivalent to the following:

$$\sum_{i=k+1}^{k+d} F_i \equiv \sum_{j=1}^{d} F_j \pmod d \tag{3.2}$$

for any integer $k$.

Using (3.1) we obtain (see also Identity I. from [1], p. 66):

$$\sum_{i=1}^{d} F_i = F_{d+2} - 1 \equiv F_2 - 1 \equiv 0 \pmod{d} \tag{3.3}$$

Consequently, we proved that the original problem is equivalent to finding all integer numbers $d > 1$ that are divisible by their own Pisano period.

Next, we find the minimal $d$-number, $d_k$, which is divisible by a given integer number $k > 1$. Note, that the set of all $d_k$ numbers coincides with the set of all $d$-numbers. Indeed, the set of $d_k$ numbers is a subset of the set of $d$-numbers, and every $d$-number is a $d_k$-number (e.g., for $k = d$).

Before proceeding, we prove the following lemma.

**Lemma 3.1.** *If $d$-number is divisible by $k$, then $d$ is divisible by $[k, \pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^\omega(k)]$, where $\omega$ is the Fibonacci frequency defined in Theorem 2.2.*

*Proof.* Indeed, using Theorem 2.3 we find that $\pi(d) = \pi([k, d]) = [\pi(k), \pi(d)]$. Consequently, $d$ is divisible by $\pi(k)$. Similarly, we prove the divisibility of $d$ by $\pi^2(k)$, by $\pi^3(k)$, etc. □

The following theorem gives us a closed formula for the numbers $d_k$.

**Theorem 3.2.** *For any given integer $k > 1$, there exists the minimal $d$-number $d_k$, which is divisible by $k$, satisfies* (3.1)*, and may be calculated using*

$$d_k = [k, \pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^\omega(k)]. \tag{3.4}$$

*Proof.* We prove that the number $m = [k, \pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^\omega(k)]$ is a $d$-number. For this, we show that $\pi(m)|m$. Indeed, according to Theorem 2.3 and Iteration Theorem 2.2, we obtain

$$\pi(m) = [\pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^{\omega+1}(k)] = [\pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^\omega(k)], \tag{3.5}$$

so, $m$ is a $d$-number. And according to Lemma 3.1, we prove that $d_k$, given by (3.4), is indeed the minimal $d$-number divisible by $k$. □

Since condition (3.1) is equivalent to (1.1) and, as mentioned above, the set of all $d_k$ coincides with the set of all $d$-numbers, Theorem 3.2 solves the original problem, allowing us to find all $d$-numbers.

Even though (3.4) formally solves the problem, the use of this expression may be technically complicated.

**Example 3.3.** *As an example, we calculate $d_k$ for $k = 13^{10}$ using* (3.4)*. Since $\pi(13) = 28$ and $\pi(13^2) = 364 \neq \pi(13)$, using Theorem 2.5, we find: $\pi(13^a) = 13^{a-1}\pi(13)$.*

$$\pi(13^{10}) = 13^9\pi(13) = 13^9 \cdot 28;$$
$$\pi^2(13^{10}) = \pi(13^9 \cdot 28) = [\pi(13^9), \pi(28)] = [13^8 \cdot 28, 48] = 13^8 \cdot 7 \cdot 16 \cdot 3;$$
$$\pi^3(13^{10}) = \pi(13^8 \cdot 7 \cdot 16 \cdot 3) = [13^7 \cdot 7\pi(13), \pi(16), \pi(7), \pi(3)] = [13^7 \cdot 28, 24, 16, 8] = 13^7 \cdot 7 \cdot 16 \cdot 3;$$
$$\pi^4(13^{10}) = \pi(13^7 \cdot 7 \cdot 16 \cdot 3) = [13^6 \cdot 28, 24, 16, 8] = 13^6 \cdot 7 \cdot 16 \cdot 3;$$
$$\dots$$
$$\pi^{10}(13^{10}) = 7 \cdot 16 \cdot 3;$$
$$\pi^{11}(13^{10}) = \pi(7 \cdot 16 \cdot 3) = [16, 24, 8] = 48;$$
$$\pi^2(48) = \pi(16 \cdot 3) = [\pi(16), \pi(3)] = 24;$$
$$\pi(24) = \pi(8 \cdot 3) = [\pi(8), \pi(3)] = 24;$$
$$d_{13^{10}} = [13^{10}, \pi(13^{10}), \pi^2(13^{10}), \pi^3(13^{10}), \dots, \pi^{12}(13^{10})] = 13^{10} \cdot 336.$$

*We see that, in this case, $\omega = 12$. Consequently, we must calculate 13 Pisano periods to obtain $d_{13^{10}}$. Usually, for integers of the form $k = p^n$, where $p$ is a prime number, one needs to do more than $n + 1$ such operations, thus making the calculation even longer.*

Hence, we prove several theorems to simplify the calculation of $d$-numbers.

**Theorem 3.4.** *For any given integer $k > 1$, the minimal $d$-number $d_k$, which is divisible by $k$, may be calculated as*

$$d_k = [k, d_{\pi(k)}]. \tag{3.6}$$

*Proof.* Substituting $k = \pi(k)$ in (3.4), we find

$$d_{\pi(k)} = [\pi(k), \pi^2(k), \pi^3(k), \dots, \pi^\omega(k)]. \tag{3.7}$$

Then, using Theorem 2.3 with (3.4) and (3.7), we obtain the proof of the theorem. □

As will be shown below, for calculating $d_k$, it is convenient to first calculate $d_p$ for some prime numbers $p \le k$. Therefore, (3.6) formally allows us to obtain $d$-numbers $d_p$:

$$d_p = [p, d_{\pi(p)}]. \tag{3.8}$$

However, $\pi(p)$ is not necessarily a prime number. For example, $\pi(7) = 16$. Moreover, we know from Theorem 2.4 that if $m > 2$, then $\pi(m)$ is an even number. The following theorems allow us to find $d_k$ for a composite number, too. First, we prove Theorems 3.5 and 3.6.

**Theorem 3.5.** *Suppose the $d$-number $d_m$ is divisible by an integer $m > 1$ and $d$-number $d_n$ is divisible by an integer $n > 1$. Then, $[d_m, d_n]$ is the minimal $d$-number that is divisible by $[m, n]$.*

*Proof.* Since $\pi(d_m)|d_m$ and $\pi(d_n)|d_n$ then $\pi(d_m)|[d_m, d_n]$ and $\pi(d_n)|[d_m, d_n]$. Hence, one has $[\pi(d_m), \pi(d_n)]|[d_m, d_n]$ and, using Theorem 2.3, we find that $\pi([d_m, d_n])|[d_m, d_n]$, i.e., $[d_m, d_n]$ is a $d$-number.

Further, it follows from Lemma 3.1, that if a $d$-number is divisible by an integer $k > 1$, then this $d$ is also divisible by $d_k$, where $d_k$ is defined in Theorem 3.2. Consequently, the minimal $d$-number, which is divisible by $[m, n]$, must be divisible by $d_m$ and $d_n$, which is $[d_m, d_n]$. □

**Theorem 3.6.** *For a prime number* $p$,

$$d_{p^n} = [p^n, d_p]. \tag{3.9}$$

*Proof.* From Lemma 3.1, we find that if $k|d$, then $d_k|d$. Then, because $p|d_{p^n}$, we obtain $d_p|d_{p^n}$. Since, by definition, $p^n|d_{p^n}$. Then,

$$d_{p^n} \geq [p^n, d_p]. \tag{3.10}$$

Now, we prove that the number $[p^n, d_p]$ is a $d$-number. Indeed, using Theorem 2.3 we obtain: $\pi([p^n, d_p]) = [\pi(p^n), \pi(d_p)] = [p^{n-a}\pi(p), \pi(d_p)]$ where, according to Theorem 2.5, integer $a$ satisfies an inequality $0 < a \leq n$. Next, for $p$ equal to 2, 3, or 5 we have: $\pi(2) = 3$, $\pi(3) = 8$, and $\pi(5) = 20$. For $p > 5$ we shall use Theorem 2.8: If $q$ and $p$ are prime, and $q|\pi(p)$, where $p > 5$, then $q < p$, i.e., $p^{n-a}\pi(p) = [p^{n-a}, \pi(p)]$ for $p \neq 5$, and $p^{n-a}\pi(p) = [p^{n-a+1}, \pi(p)]$ for $p = 5$. Consequently, $\pi([p^n, d_p]) = [p^{n-b}, \pi(p), \pi(d_p)]$, where $0 \leq b \leq n$. Then, since $\pi(d_p)|d_p$ by the definition of $d$-numbers, $\pi(p)|d_p$ according to Theorem 3.2 and $p^{n-b}|p^n$, we have $\pi([p^n, d_p])|[p^n, d_p]$. Then, using (3.10), we obtain (3.9). □

Now, using Theorems 3.4, 3.5, and 3.6, we can prove Theorems 3.7 and 3.8. The expressions derived in these theorems are recurrent, hence, convenient for computer calculations.

**Theorem 3.7.** *The* $d$*-number* $d_k$ *for arbitrary integer* $k > 2$ *with the prime factorization* $k = \Pi_{i=1}^m q_i^{a_i}$ *(hereafter all* $a_i > 0$*) can be calculated according to the following recurrent expression:*

$$d_k = [k, d_{q_1}, d_{q_2}, \ldots, d_{q_m}]. \tag{3.11}$$

*Proof.* From Theorems 3.5 and 3.6, we obtain:

$$d_k = [d_{q_1^{a_1}}, d_{q_2^{a_2}}, \ldots, d_{q_m^{a_m}}] = [k, d_{q_1}, d_{q_2}, \ldots, d_{q_m}] \tag{3.12}$$

□

**Theorem 3.8.** *All* $d_{p_n}$*, for the nth prime number* $p_n$ *(*$n > 3$*), can be calculated using the following recurrent expression:*

$$d_{p_n} = [p_n, \pi(p_n), d_{q_1}, d_{q_2}, \ldots, d_{q_m}], \tag{3.13}$$

*where* $\pi(p_n) = \Pi_{i=1}^m q_i^{a_i}$ *is the prime factorization of* $\pi(p_n)$*.*

*Proof.* Substituting $k = \pi(p_n)$ in (3.12), we find that

$$d_{\pi(p_n)} = [d_{q_1^{a_1}}, d_{q_2^{a_2}}, \ldots, d_{q_m^{a_m}}] = [\pi(p_n), d_{q_1}, d_{q_2}, \ldots, d_{q_m}], \tag{3.14}$$

where $\pi(p_n) = \Pi_{i=1}^m q_i^{a_i}$.

Next, using Theorem 3.4 and (3.14), we obtain (3.13). □

According to Theorem 2.8, $\pi(p)$ is not divisible by a prime $q \geq p$ for any prime $p > 5$. Consequently, expression (3.13) is recurrent indeed, since, for calculating $d_{p_n}$, where $p_n$ is the $n$th prime number and $n > 3$, one only needs to know $d_{q_i}$, where $q_i < p_n$. Therefore, after calculating $d_2 = 24, d_3 = 24$, and $d_5 = 120$, all other $d_p$ can be found using recurrent expression (3.13).

As an example, we calculate $d_2$ from (3.4): since Fibonacci frequency $\omega$ in this case equals 3, and $\pi(2) = 3$, $\pi^2(2) = 8$, $\pi^3(2) = 24$, we find that $d_2 = [2, \pi(2), \pi^2(2), \pi^3(2)] = 24$. Analogously, one easily obtains $d_3 = 24$ and $d_5 = 120$. Then, to obtain $d_7$, we use expression(3.13): $d_7 = [7, \pi(7), d_2] = [7, 16, 24] = 336$, where we used that $\pi(7) = 16$ and $d_2 = 24$.

**Example 3.9.** *Next, we use Theorems 3.7 and 3.8 to calculate* $d_k$ *for* $k = 13^{10}$.
*Since* $\pi(13) = 28 = 2^2 \cdot 7$, *using* (3.11), *we find:*

$$d_{13^{10}} = [13^{10}, d_{13}] = 13^{10} \cdot 7 \cdot 16 \cdot 3 = 13^{10} \cdot 336.$$

*We calculated* $d_{13}$ *using* (3.13)*:*

$$d_{13} = [13, \pi(13), d_2, d_7] = [13, 4 \cdot 7, 8 \cdot 3, 7 \cdot 16 \cdot 3] = 13 \cdot 7 \cdot 16 \cdot 3 = 13 \cdot 336 = 4368.$$

*We see that using Theorems 3.7 and 3.8 significantly simplifies the calculations compared with the direct use of* (3.4)*, see Example 3.3.*

Proposition 3.10 allows us to easily generate new $d$-numbers from known $d$-numbers. Its proof is straightforward from Theorem 3.7.

**Proposition 3.10.** *If the prime factorization of d-number* $m$ *has the form* $m = \Pi_{i=1}^{n} q_i^{a_i}$, *then the number* $M$ *with prime factorization* $M = \Pi_{i=1}^{n} q_i^{b_i}$, *where* $b_i \geq a_i$, *is a d-number.*

To summarize, we showed that Theorem 3.2 allows us to calculate all minimal $d$-numbers $d_k$ that are divisible by a given integer $k$. Since the set of all $d_k$ coincides with the set of all $d$-numbers, the proof of Theorem 3.2 solves the original problem. However, even though Eq. (3.4) formally solves the problem, it can be technically complicated to use it. Hence, to simplify the calculation, we derived the recurrent expressions (3.11) and (3.13). Finally, Proposition 3.10 allows us to easily generate $d$-numbers from already known $d$-numbers.

The $d$-numbers have a number of interesting properties.

1. From the definition of $d_k$, we easily obtain:

$$d_{d_k} = d_k, \tag{3.15}$$

because the minimal $d$-number that is divisible by $d_k$ is $d_k$.

2. Note, that the right sides of expressions

$$\pi(d_k) = [\pi(k), \pi^2(k), \pi^3(k), \ldots, \pi^{\omega}(k)], \tag{3.16}$$

and (3.7) are equal. Hence, we obtain from (3.16) and (3.7)

$$\pi(d_k) = d_{\pi(k)}. \tag{3.17}$$

**Theorem 3.11.** *The number* $m$ *with the prime factorization* $m = \Pi_{i=1}^{n} q_i^{a_i}$, *where integers* $a_i > 0$, *is a d-number if and only if* $m$ *is divisible by* $[\pi(q_1); \pi(q_2); \ldots; \pi(q_n)]$.

*Proof.* First, for $q_i$ equal to 2, 3, or 5 we have: $\pi(2) = 3$, $\pi(3) = 8$, and $\pi(5) = 20$. In all these cases, $\pi(q_i)$ is not divisible by $q_i^2$. For $q_i > 5$ we use Theorem 2.8: If $q$ and $p$ are prime, and $q|\pi(p)$, where $p > 5$, then $q < p$, i.e., $\pi(q_i)$ is not divisible by $q_i$ for $q_i > 5$. Then, since $q_i^{a_i}|m$ we find that if $\pi(q_i)|m$ then $q_i^{a_i-1}\pi(q_i)|m$. Further, using Corollary 2.6 from Theorem 2.5, we find that $\pi(q_i^{a_i})|m$. The proof follows from Theorem 2.3. The inverse statement follows directly from Theorems 2.3 and 2.5. □

**Theorem 3.12.** *All d-numbers are divisible by 24.*

*Proof.* First, we directly verify that $d \neq 2$. Then, using Theorem 2.4, we find that $\pi(d)$ is even. Next, it is clear that $2|\pi(d)|d$. Using Lemma 3.1, we obtain $[2, \pi(2), \pi^2(2), \pi^3(2)]|d$. Finally, taking into account that $\pi(2) = 3, \pi^2(2) = 8, \pi^3(2) = 24$, and that the Fibonacci frequency $\omega$ in this case equals 3, we find that $24|d$. □

Note that all $d$-numbers that can be calculated using theorems and propositions of this Section also satisfy (1.1) for the generalized Fibonacci numbers [3] (p.109), with two arbitrary starting integer values $a$ and $b$. Indeed, let $G_0 = a$, $G_1 = b$, and $G_{n+1} = G_n + G_{n-1}$. Then one can derive that $G_{n+1} = bF_n + aF_{n-1}$. Let $m$ be a $d$-number, i.e., it satisfies (1.1). Then, $\sum_{n=k}^{m+k-1} G_n = a\sum_{i=k-1}^{m+k-2} F_i + b\sum_{j=k}^{m+k-1} F_j$. Thus, $m$ divides $\sum_{n=k}^{m+k-1} G_n$.

## 4. The Connection Between $d$-numbers and Another Known Sequence

Now, we prove that the set of all $d$-numbers divided by 24 coincides with the sequence of numbers $n$ such that $12n$ divides $F_{12n}$ [6]. Namely, we prove the equivalency:

$$n = d/24 \Leftrightarrow 12n | F_{12n}.$$

*Proof.* The following lemma is useful to prove the necessity.

**Lemma 4.1.** *If $3n|F_{3n}$, then $6n|F_{3n}$.*

*Proof.* Let $n = 2^k z$, where $z$ is an odd number. Then, using Theorem 2.5 and $\pi(2) = 3$ and $\pi(4) = 6$, i.e., $\pi(2) \neq \pi(2^2)$, we find that $F_{2^k 3z} = F_{\pi(2^{k+1})z}$. Thus, using the definition of the Pisano period and its properties, we obtain $2^{k+1}|F_{\pi(2^{k+1})z} \Rightarrow 2^{k+1}|F_{2^k 3z}$. □

Necessity.

Since $12n|F_{12n}$, from Lemma 4.1 we have $24n|F_{12n}$. Further, using (2.1), we find that $24n|F_{-12n}$. Next, using the definition of the Fibonacci numbers, $24n|F_{12n}$, and (2.1), we obtain $F_{12n+1} = F_{12n-1}+F_{12n} \equiv F_{-12n+1} (\mathrm{mod} 24n)$. Finally, since $(F_{-12n}, F_{-12n+1}) \equiv (F_{12n}, F_{12n+1}) (\mathrm{mod} 24n)$, $24n$ is a period of the Fibonacci sequence modulo $24n$, i.e., $24n$ is a $d$-number.

Sufficiency.

Using (2.1), we obtain $F_{12n} = -F_{-12n}$, or, equivalently, $F_{12n} + F_{-12n} = 0$. Since $24n$ is a period, we find $F_{12n} - F_{-12n} \equiv 0 \pmod{24n}$. Consequently, $2F_{12n} \equiv 0 \pmod{24n}$, so $F_{12n} \equiv 0 \pmod{12n}$. □

## 5. Summary

In this paper, we calculated and investigated all integers $d > 1$, such that the sum of any $d$ consecutive Fibonacci numbers is divisible by $d$. We call these numbers $d$-numbers.

We demonstrated a relation between $d$-numbers and the Pisano period, namely, we proved that all $d$-numbers are multiple of their own Pisano period.

We obtained a closed formula (3.4) for calculating all minimal $d$-numbers, $d_k$, which are divisible by a given integer $k$, and proved that the set of all $d_k$ coincides with the set of all $d$-numbers. Further, we obtained convenient recurrent relations (3.11) and (3.13), which significantly simplify practical calculations. Proposition 3.10 allows us to easily generate new $d$-numbers from the already known $d$-numbers. We found some interesting properties of the $d$-numbers. We proved results in Section 3 that are applicable to the generalized Fibonacci numbers with two arbitrary starting integer values $a$ and $b$.

Finally, we proved that a set of all $d$-numbers divided by 24 coincides with the sequence of numbers $n$ such that $12n$ divides $F_{12n}$ [6].

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