SOME FIBONACCI-LUCAS-TRIBONACCI-LUCAS IDENTITIES

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Abstract. We derive new convolution relations between Fibonacci, Lucas, Tribonacci and Tribonacci-Lucas numbers.

1. Introduction

Let F_n, L_n, T_n, and K_n denote the Fibonacci, Lucas, Tribonacci, and Tribonacci-Lucas numbers, respectively. The four sequences are defined by the recurrence equations

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0,\ F_1 = 1, \tag{1.1}$$
$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2,\ L_1 = 1, \tag{1.2}$$
$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0,\ T_1 = T_2 = 1, \tag{1.3}$$
$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3,\ K_1 = 1, K_2 = 3. \tag{1.4}$$

The ordinary generating functions for these numbers are given by

$$f(x) = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n, \tag{1.5}$$

$$g(x) = \frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^n, \tag{1.6}$$

$$u(x) = \frac{x}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} T_n x^n, \tag{1.7}$$

and

$$v(x) = \frac{3-2x-x^2}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} K_n x^n. \tag{1.8}$$

See for instance [5], [6], [7], [8], and [1]. The mathematical literature contains many convolution identities for a series of important numbers such as Bernoulli, Euler, Cauchy, Fibonacci, Lucas, and Tribonacci numbers (see the references herein and those given in [6]). For Fibonacci numbers, one classic example is the following identity:

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}\left((n+1)L_n - 2F_{n+1}\right), \qquad n \geq 1. \tag{1.9}$$

This identity can be found in [4]. We will use this identity later in the proof of Theorem 3.1. More identities of this kind can by found in [4], [5], and [8]. Convolution identities for

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Tribonacci numbers have been derived recently in [6] and [7]. One example from [7] is

$$\sum_{k=0}^{n-3} T_k(T_{n-k} + T_{n-2-k} + 2T_{n-3-k}) = (n-2)T_{n-1} - T_{n-2}, \qquad n \geq 3. \tag{1.10}$$

In this paper, we continue the search for convolution identities. We present new relations between Fibonacci, Lucas, Tribonacci, and Tribonacci-Lucas numbers, respectively. More precisely, we derive new convolution identities for the pairs (F_n, T_n), (F_n, K_n), (L_n, T_n), and (L_n, K_n). To prove our results, we use some functional relations between the generating functions for these numbers. At the end of the article, we also propose an open problem.

2. First Results

Throughout the paper, we use the convention that $\sum_{k=0}^{n} a_k = 0$ for $n < k$. The first theorem is an identity that relates Fibonacci numbers to Tribonacci numbers.

Theorem 2.1. *Let $n \geq 1$ be an integer. Then,*

$$T_n = F_n + \sum_{k=0}^{n-2} F_k T_{n-2-k}. \tag{2.1}$$

Proof. Let $f(x)$ and $u(x)$ denote the generating functions for F_n and T_n, respectively. We have

$$\frac{x}{f(x)} = 1 - x - x^2.$$

Thus,

$$1 - x - x^2 - x^3 = \frac{x - x^3 f(x)}{f(x)},$$

and

$$u(x) = \frac{f(x)}{1 - x^2 f(x)},$$

or equivalently

$$u(x) - f(x) = x^2 f(x) u(x). \tag{2.2}$$

From the last equation, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}(T_n - F_n)x^n &= x^2\Big(\sum_{n=0}^{\infty} F_n x^n\Big)\Big(\sum_{n=0}^{\infty} T_n x^n\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} F_k T_{n-k}\Big)x^{n+2} \\ &= \sum_{n=2}^{\infty}\Big(\sum_{k=0}^{n-2} F_k T_{n-2-k}\Big)x^{n}. \end{aligned}$$

Comparing the coefficients of both sides of the equation gives the identity. □

Theorem 2.2. *Let $n \geq 3$ be an integer. Then,*

$$K_{n-1} = L_{n-1} + \sum_{k=0}^{n-3} F_k K_{n-3-k}. \tag{2.3}$$

Proof. Let $f(x)$ and $v(x)$ denote the generating functions for F_n and K_n, respectively. Then

$$v(x) = \frac{(3-2x-x^2)f(x)}{x - x^3 f(x)},$$

or equivalently

$$xv(x) - (3-2x-x^2)f(x) = x^3 f(x)v(x). \tag{2.4}$$

The left side of the last equation is

$$\sum_{n=1}^{\infty} K_{n-1}x^n - 3\sum_{n=0}^{\infty} F_n x^n + 2\sum_{n=1}^{\infty} F_{n-1}x^n + \sum_{n=2}^{\infty} F_{n-2}x^n, \tag{2.5}$$

whereas the right side equals

$$x^3 f(x)v(x) = \sum_{n=3}^{\infty}\Big(\sum_{k=0}^{n-3} F_k K_{n-3-k}\Big)x^n. \tag{2.6}$$

Comparing the coefficients of both power series and using that

$$-3F_n + 2F_{n-1} + F_{n-2} = -(F_{n-1} + 2F_{n-2}) = -L_{n-1}$$

completes the proof of the identity. □

Theorem 2.3. *Let $n \geq 3$ be an integer. Then,*

$$2T_n = T_{n-1} + L_{n-1} + \sum_{k=0}^{n-3} L_k T_{n-3-k}. \tag{2.7}$$

Proof. The formula is a consequence of

$$2u(x) - x(u(x) + g(x)) = x^3 g(x)u(x). \tag{2.8}$$

Writing this equation in terms of power series and comparing the coefficients gives the desired identity. □

We conclude this section with the following theorem.

Theorem 2.4. *Let $n \geq 3$ be an integer. Then,*

$$2K_n = K_{n-1} + L_{n-1} + 2L_{n-2} + \sum_{k=0}^{n-3} L_k K_{n-3-k}. \tag{2.9}$$

Proof. The identity follows essentially from

$$(2-x)v(x) - (3-2x-x^2)g(x) = x^3 g(x)v(x). \tag{2.10}$$

We omit the details. □

3. Higher-Order Identities with Three Factors

The functional relations between the generating functions for F_n, L_n, T_n, and K_n make it possible to derive identities for sums of products of three factors.

Theorem 3.1. *Let $n \geq 5$ and $k_1, k_2, k_3 \geq 1$ be integers. Then,*

$$\sum_{k_1+k_2+k_3=n-2} T_{k_1}F_{k_2}F_{k_3} = T_{n+2} - F_{n+2} - \frac{1}{5}\left((n+1)L_n - 2F_{n+1}\right). \tag{3.1}$$

Proof. From (2.2), we have

$$u(x)f(x) - f^2(x) = x^2 u(x) f^2(x). \tag{3.2}$$

In terms of power series, the relation becomes

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(T_k F_{n-k} - F_k F_{n-k}\Big)x^n &= x^2 \sum_{n=0}^{\infty}\Big(\sum_{k_1+k_2+k_3=n} T_{k_1}F_{k_2}F_{k_3}\Big)x^n \\ &= \sum_{n=2}^{\infty}\Big(\sum_{k_1+k_2+k_3=n-2} T_{k_1}F_{k_2}F_{k_3}\Big)x^n.\end{aligned}$$

Since $F_0 = T_0 = 0$, we can restrict all indices to be strictly positive. Comparing the coefficients of both sides gives

$$\sum_{k_1+k_2+k_3=n-2} T_{k_1}F_{k_2}F_{k_3} = \sum_{k=0}^{n}(T_k - F_k)F_{n-k}, \qquad n \geq 2. \tag{3.3}$$

From (1.9), it is known that

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}((n+1)L_n - 2F_{n+1}). \tag{3.4}$$

Finally, from (2.1), we also know that

$$\sum_{k=0}^{n} T_k F_{n-k} = T_{n+2} - F_{n+2}. \tag{3.5}$$

□

Corollary 3.2. *Let $N \geq 5$ be an integer. Then,*

$$\begin{aligned}\sum_{n=5}^{N} \sum_{\substack{k_1+k_2+k_3=n-2 \\ k_1,k_2,k_3\geq 1}} T_{k_1}F_{k_2}F_{k_3} &= \frac{1}{2}(T_{N+4} + T_{N+2} - 1) - F_{N+4} \\ &-\frac{1}{5}(4(N-1)F_N + (3N-4)F_{N-1} - 11F_{N-2} - 6F_{N-3}).\end{aligned} \tag{3.6}$$

Proof. First, we note that from $L_n = F_{n+1} + F_{n-1}$, we easily deduce that $(n+1)L_n - 2F_{n+1} = (n-1)F_n + 2nF_{n-1}$. Hence, we have

$$\begin{aligned}\sum_{n=5}^{N} \sum_{\substack{k_1+k_2+k_3=n-2 \\ k_1,k_2,k_3\geq 1}} T_{k_1}F_{k_2}F_{k_3} &= \sum_{n=1}^{N} \sum_{\substack{k_1+k_2+k_3=n-2 \\ k_1,k_2,k_3\geq 1}} T_{k_1}F_{k_2}F_{k_3} \\ &= \sum_{n=1}^{N}(T_{n+2} - F_{n+2}) - \frac{1}{5}\Big(\sum_{n=0}^{N-1} nF_{n+1} + 2\sum_{n=1}^{N} nF_{n-1}\Big) \\ &= \sum_{n=1}^{N}(T_{n+2} - F_{n+2}) - \frac{1}{5}\Big(\sum_{n=1}^{N-1} nF_n + 3\sum_{n=1}^{N-1} nF_{n-1} + 2NF_{N-1}\Big).\end{aligned}$$

To finish the proof, use the identities

$$\sum_{n=1}^{N} F_n = F_{N+2} - 1, \tag{3.7}$$

$$\sum_{n=1}^{N} nF_n = NF_{N+2} - 3F_N - 2F_{N-1} + 2, \tag{3.8}$$

$$\sum_{n=1}^{N} nF_{n-1} = NF_{N+1} - 3F_{N-1} - 2F_{N-2} + 1, \tag{3.9}$$

and

$$\sum_{n=1}^{N} T_n = \frac{1}{2}(T_{N+2} + T_N - 1). \tag{3.10}$$

The three Fibonacci sums are discussed in [8] (see also [9]). The last sum for Tribonacci numbers appears in [2] and [3]. □

Theorem 3.3. *Let $n \geq 4$, $k_1 \geq 1$, and $k_2, k_3 \geq 0$ be integers. Then,*

$$\sum_{k_1+k_2+k_3=n-3} T_{k_1} L_{k_2} L_{k_3} = 5T_{n+1} + 4T_n - nL_{n-1} - 2F_n - 5F_{n+1}. \tag{3.11}$$

Proof. Using (2.8), we start with

$$2u(x)g(x) - xu(x)g(x) - xg^2(x) = x^3 u(x) g^2(x). \tag{3.12}$$

In terms of power series, the left side equals

$$\sum_{n=1}^{\infty} \Big(2\sum_{k=0}^{n} T_k L_{n-k} - \sum_{k=0}^{n-1} T_k L_{n-1-k} - \sum_{k=0}^{n-1} L_k L_{n-1-k} \Big) x^n, \tag{3.13}$$

whereas the right side is given by

$$x^3 u(x) g^2(x) = \sum_{n=3}^{\infty} \Big(\sum_{k_1+k_2+k_3=n-3} T_{k_1} L_{k_2} L_{k_3} \Big) x^n, \tag{3.14}$$

with $k_1 \geq 1$ and $k_2, k_3 \geq 0$. To simplify the left side further, use

$$2\sum_{k=0}^{n} T_k L_{n-k} = 2\sum_{k=1}^{n-1} T_k L_{n-k} + 4T_n,$$

and

$$\sum_{k=0}^{n-1} L_k L_{n-1-k} = 2L_{n-1} + \sum_{k=1}^{n-1} L_k L_{n-1-k}.$$

Next, note that

$$2L_{n-k} - L_{n-1-k} = L_{n-2-k} + L_{n-k} = 5F_{n-1-k}.$$

Comparing the coefficients of both sides shows that

$$\sum_{k_1+k_2+k_3=n-3} T_{k_1} L_{k_2} L_{k_3} = 4T_n - 2L_{n-1} + \sum_{k=1}^{n-1} (5F_{n-1-k} T_k - L_k L_{n-1-k}). \tag{3.15}$$

By (2.1), the first convolution equals

$$\sum_{k=1}^{n-1} F_{n-1-k} T_k = \sum_{k=0}^{n-1} F_k T_{n-1-k} = T_{n+1} - F_{n+1}. \tag{3.16}$$

Finally, the convolution (see [4])

$$\sum_{k=0}^{n} L_k L_{n-k} = (n+1)L_n + 2F_{n+1}, \qquad n \geq 1, \tag{3.17}$$

shows that

$$\sum_{k=1}^{n-1} L_k L_{n-1-k} = nL_{n-1} + 2F_n - 2L_{n-1}. \tag{3.18}$$

□

Corollary 3.4. *Let $N \geq 4$ be an integer. Then,*

$$\begin{aligned}\sum_{n=4}^{N} \sum_{\substack{k_1+k_2+k_3=n-3\\ k_1,\geq 1,k_2,k_3\geq 0}} T_{k_1}L_{k_2}L_{k_3} &= \frac{9}{2}(T_{N+2}+T_N-1)+5T_{N+1}+4-(N+7)F_{N+2}\\ &\quad -5F_{N+1}-(N-3)F_N+4F_{N-1}+F_{N-2}.\end{aligned} \tag{3.19}$$

Proof. We have

$$\begin{aligned}\sum_{n=4}^{N} \sum_{\substack{k_1+k_2+k_3=n-3\\ k_1\geq 1,k_2,k_3\geq 0}} T_{k_1}L_{k_2}L_{k_3} &= \sum_{n=1}^{N} \sum_{\substack{k_1+k_2+k_3=n-3\\ k_1\geq 1,k_2,k_3\geq 0}} T_{k_1}L_{k_2}L_{k_3}\\ &= \sum_{n=1}^{N}(5T_{n+1}+4T_n) - \sum_{n=1}^{N} nL_{n-1} - \sum_{n=1}^{N}(2F_n+5F_{n+1}).\end{aligned}$$

The evaluation of the sums is straightforward but lengthy and is left as an exercise. □

Theorem 3.5. *Let $n \geq 5$, $k_1 \geq 0$, and $k_2, k_3 \geq 1$ be integers. Then*

$$\sum_{k_1+k_2+k_3=n-3} K_{k_1}F_{k_2}F_{k_3} = K_{n+1} - L_{n+1} - (n+1)F_{n-1}. \tag{3.20}$$

Proof. Using (2.4), our starting point is the relation

$$xv(x)f(x) - 3f^2(x) + 2xf^2(x) + x^2f^2(x) = x^3v(x)f^2(x). \tag{3.21}$$

The power series on the left side equals

$$\sum_{n=2}^{\infty}\Big(\sum_{k=0}^{n-1} K_kF_{n-1-k} - 3\sum_{k=0}^{n} F_kF_{n-k} + 2\sum_{k=0}^{n-1} F_kF_{n-1-k} + \sum_{k=0}^{n-2} F_kF_{n-2-k}\Big)x^n, \tag{3.22}$$

whereas the right side is given by

$$x^3v(x)f^2(x) = \sum_{n=3}^{\infty}\Big(\sum_{k_1+k_2+k_3=n-3} K_{k_1}F_{k_2}F_{k_3}\Big)x^n, \tag{3.23}$$

with $k_1 \geq 0$ and $k_2, k_3 \geq 1$. In the next step, we use

$$-3F_{n-k} + 2F_{n-1-k} + F_{n-2-k} = -(F_{n-1-k} + 2F_{n-2-k}) = -L_{n-1-k}.$$

This produces

$$\sum_{k_1+k_2+k_3=n-3} K_{k_1}F_{k_2}F_{k_3} = \sum_{k=1}^{n-2}(K_kF_{n-1-k} - F_kL_{n-1-k}). \tag{3.24}$$

From (2.3), we see that

$$\sum_{k=1}^{n-2} K_k F_{n-1-k} = \sum_{k=0}^{n-1} K_k F_{n-1-k} - 3F_{n-1} = K_{n+1} - L_{n+1} - 3F_{n-1}. \tag{3.25}$$

Finally, the convolution (see [4])

$$\sum_{k=0}^{n} L_k F_{n-k} = (n+1)F_n, \tag{3.26}$$

shows that

$$\sum_{k=1}^{n-2} F_k L_{n-1-k} = (n-2)F_{n-1}. \tag{3.27}$$

□

Corollary 3.6. *Let $N \geq 5$ be an integer. Then,*

$$\sum_{n=5}^{N} \sum_{\substack{k_1+k_2+k_3=n-3\\ k_1,\geq 0, k_2,k_3\geq 1}} K_{k_1} F_{k_2} F_{k_3} = \frac{1}{2}(K_{N+3} + K_{N+1}) - L_{N+3} - (N+1)F_{N+1} + 3F_{N-1} + 2F_{N-2}. \tag{3.28}$$

Proof. The statement follows from

$$\sum_{n=5}^{N} \sum_{\substack{k_1+k_2+k_3=n-3\\ k_1\geq 0, k_2,k_3\geq 1}} K_{k_1} F_{k_2} F_{k_3} = \sum_{n=1}^{N} K_{n+1} - \sum_{n=1}^{N} L_{n+1} - \sum_{n=1}^{N} (n+1)F_{n-1}, \tag{3.29}$$

combined with (see [8])

$$\sum_{n=1}^{N} L_n = L_{N+2} - 3, \tag{3.30}$$

and (see [2])

$$\sum_{n=1}^{N} K_n = \frac{1}{2}(K_{N+2} + K_N - 6). \tag{3.31}$$

□

Theorem 3.7. *Let $n \geq 3$ and $k_1, k_2, k_3 \geq 0$ be integers. Then,*

$$\sum_{k_1+k_2+k_3=n-3} K_{k_1} L_{k_2} L_{k_3} = 5K_{n+1} + 4K_n - 11L_n - 4L_{n-1} - 5nF_{n-1}. \tag{3.32}$$

Proof. Using (2.10), we start with

$$2v(x)g(x) - xv(x)g(x) - 3g^2(x) + 2xg^2(x) + x^2g^2(x) = x^3v(x)g^2(x). \tag{3.33}$$

The power series on the left side equals

$$\sum_{n=2}^{\infty} \Big(2\sum_{k=0}^{n} K_k L_{n-k} - \sum_{k=0}^{n-1} K_k L_{n-1-k} - 3\sum_{k=0}^{n} L_k L_{n-k} + 2\sum_{k=0}^{n-1} L_k L_{n-1-k} + \sum_{k=0}^{n-2} L_k L_{n-2-k} \Big) x^n, \tag{3.34}$$

whereas the right side is given by

$$x^3 v(x) g^2(x) = \sum_{n=3}^{\infty} \Big(\sum_{k_1+k_2+k_3=n-3} K_{k_1} L_{k_2} L_{k_3} \Big) x^n, \tag{3.35}$$

with $k_1, k_2, k_3 \geq 0$. The coefficient of x^n on the left side can be written as

$$\sum_{k=0}^{n-1} K_k (2L_{n-k} - L_{n-1-k}) + 4K_n + L_{n-1} - 6L_n + \sum_{k=0}^{n-2} L_k (-3L_{n-k} + 2L_{n-1-k} + L_{n-2-k}).$$

Simplifying further and making use of the formula

$$2L_{n-k} - L_{n-1-k} = L_{n-1-k} + 2L_{n-2-k} = 5F_{n-1-k}.$$

allows us to write

$$\sum_{k_1+k_2+k_3=n-3} K_{k_1} L_{k_2} L_{k_3} = 4K_n + L_{n-1} - 6L_n + 5 \sum_{k=0}^{n-2} F_{n-1-k} (K_k - L_k). \tag{3.36}$$

We complete the proof by noting that

$$\sum_{k=0}^{n-2} F_{n-1-k} L_k = nF_{n-1}, \tag{3.37}$$

and

$$\sum_{k=0}^{n-2} K_k F_{n-1-k} = K_{n+1} - L_{n+1}. \tag{3.38}$$

□

Corollary 3.8. *Let $N \geq 3$ be an integer. Then,*

$$\begin{aligned} \sum_{n=3}^{N} \sum_{\substack{k_1+k_2+k_3=n-3 \\ k_1,k_2,k_3 \geq 0}} K_{k_1} L_{k_2} L_{k_3} &= \frac{5}{2}(K_{N+3} + K_{N+1}) + 2(K_{N+2} + K_N) - 11L_{N+2} - 4L_{N+1} \\ &\quad -5NF_{N+1} + 15F_{N-1} + 10F_{N-2}. \end{aligned} \tag{3.39}$$

Proof. The identity follows from similar arguments as in the previous corollaries. To evaluate the Tribonacci-Lucas sums, we again use (3.31). We have

$$\begin{aligned} \sum_{n=3}^{N} \sum_{\substack{k_1+k_2+k_3=n-3 \\ k_1,k_2,k_3 \geq 0}} K_{k_1} L_{k_2} L_{k_3} &= 5 \sum_{n=2}^{N+1} K_n + 4 \sum_{n=1}^{N} K_n - 11 \sum_{n=1}^{N} L_n - 4 \sum_{n=0}^{N-1} L_n - 5 \sum_{n=1}^{N} nF_{n-1} \\ &= 5\Big(\frac{1}{2}\big(K_{N+3} + K_{N+1} - 6\big) - 1\Big) + 4\Big(\frac{1}{2}\big(K_{N+2} + K_N - 6\big)\Big) \\ &\quad -11\big(L_{N+2} - 3\big) - 4\big(L_{N+1} - 1\big) \\ &\quad -5\big(NF_{N+1} - 3F_{N-1} - 2F_{N-2} + 1\big). \end{aligned}$$

Gathering like terms establishes the result. □

4. The General Case

In this section, we give some remarks on the general nature of the relations derived in this paper.

Theorem 4.1. *Let $m \geq 0$ and $n \geq m+4$ be integers. Then,*

$$\sum_{\substack{k_1+k_2+\cdots+k_{m+2}=n-2\\ k_1,k_2,\ldots,k_{m+2}\geq 1}} T_{k_1}F_{k_2}\cdots F_{k_{m+2}} = \sum_{\substack{k_1+k_2+\cdots+k_{m+1}=n\\ k_1,k_2,\ldots,k_{m+1}\geq 1}} T_{k_1}F_{k_2}\cdots F_{k_{m+1}} - H(n,m), \tag{4.1}$$

with $H(n,0) = F_n$ and for $m \geq 1$,

$$\begin{aligned} H(n,m) &= \sum_{\substack{k_1+k_2+\cdots+k_{m+1}=n\\ k_1,k_2,\ldots,k_{m+1}\geq 1}} F_{k_1}F_{k_2}\cdots F_{k_{m+1}} \\ &= \frac{C_{m-1}}{(2m-2)!2^{2m-2}} \sum_{j=1}^{n-m} \frac{(n+j+m-2)!!(n-j+m-2)!!}{(n+j-m)!!(n-j-m)!!} jF_j \cos\Big(\frac{(n-j-m)\pi}{2}\Big), \end{aligned} \tag{4.2}$$

where C_n is the nth Catalan number, and $n!! = n(n-2)(n-4)\cdots 1$ if n is odd and $n!! = n(n-2)(n-4)\cdots 2$ if n is even.

Proof. From (2.2) (or (3.2)), it is clear that if $m \geq 0$ is an arbitrary fixed integer, then

$$u(x)f^m(x) - f^{m+1}(x) = x^2u(x)f^{m+1}(x). \tag{4.3}$$

From this identity, it follows that

$$\begin{aligned} \sum_{\substack{k_1+k_2+\cdots+k_{m+2}=n-2\\ k_1,k_2,\ldots,k_{m+2}\geq 1}} T_{k_1}F_{k_2}\cdots F_{k_{m+2}} &= \sum_{\substack{k_1+k_2+\cdots+k_{m+1}=n\\ k_1,k_2,\ldots,k_{m+1}\geq 1}} T_{k_1}F_{k_2}\cdots F_{k_{m+1}} \\ &\quad - \sum_{\substack{k_1+k_2+\cdots+k_{m+1}=n\\ k_1,k_2,\ldots,k_{m+1}\geq 1}} F_{k_1}F_{k_2}\cdots F_{k_{m+1}}. \end{aligned} \tag{4.4}$$

The second sum in (4.4) allows the stated closed-form expression as was shown by Komatsu, et al. (2014) ([5], Theorem 4.2). □

According to Theorem 4.1, the convolution of $T_{k_1}F_{k_2}\cdots F_{k_{m+2}}$ can be specified in an iterative manner, using the expression for the convolution for $F_{k_1}F_{k_2}\cdots F_{k_{m+1}}$. When $m=0$, Theorem 4.1 reduces to Theorem 2.1. When $m=1$, Theorem 4.1 reduces to Theorem 3.1, since (see [5], Proposition 6.1)

$$\sum_{j=1}^{n-1} jF_j \cos\Big(\frac{(n-j-1)\pi}{2}\Big) = \frac{(n-1)F_n + 2nF_{n-1}}{5}. \tag{4.5}$$

When $m=2$, we have the following identity.

Theorem 4.2. *Let $n \geq 6$ be an integer. Then,*

$$\sum_{\substack{k_1+k_2+k_3+k_4=n-2\\k_1,k_2,k_3,k_4\geq 1}} T_{k_1}F_{k_2}F_{k_3}F_{k_4} = T_{n+4} - F_{n+4} - \frac{(n+1)F_{n+2}+2(n+2)F_{n+1}}{5} - \sum_{j=1}^{n-2}\frac{(n+j)(n-j)jF_j}{8}\cos\Big(\frac{(n-j-2)\pi}{2}\Big). \tag{4.6}$$

An equivalent expression for the above four-factor sum was discovered by the author during the study. The expression is stated in the following theorem.

Theorem 4.3. *Let $n \geq 6$ be an integer. Then,*

$$\sum_{\substack{k_1+k_2+k_3+k_4=n-2\\k_1,k_2,k_3,k_4\geq 1}} T_{k_1}F_{k_2}F_{k_3}F_{k_4} = T_{n+4} - F_{n+4} - \frac{(n+1)F_{n+2}+2(n+2)F_{n+1}}{5} - \frac{(n-1)(n-2)}{50}F_n - \frac{(n-2)(2n+1)}{25}F_{n-1} - \frac{2(n-1)(n+1)}{25}F_{n-2}. \tag{4.7}$$

Proof. It remains to show that

$$\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 1}} F_{k_1}F_{k_2}F_{k_3} = \frac{(n-1)(n-2)}{50}F_n + \frac{(n-2)(2n+1)}{25}F_{n-1} + \frac{2(n-1)(n+1)}{25}F_{n-2}. \tag{4.8}$$

The equation holds for $n \geq 3$. The proof of the last identity can be done as follows:

$$\begin{aligned}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 1}} F_{k_1}F_{k_2}F_{k_3} &= \sum_{k_3=0}^{n}\sum_{k_2=0}^{k_3} F_{k_2}F_{k_3-k_2}F_{n-k_3}\\ &= \frac{1}{5}\sum_{k_3=0}^{n} F_{n-k_3}((k_3-1)F_{k_3}+2k_3F_{k_3-1}).\end{aligned}$$

Since,

$$\sum_{k_3=0}^{n} k_3F_{k_3}F_{n-k_3} = \sum_{k_3=0}^{n}(n-k_3)F_{k_3}F_{n-k_3},$$

it follows that

$$\sum_{k_3=0}^{n} k_3F_{k_3}F_{n-k_3} = \frac{n}{2}\Big(\frac{(n-1)F_n+2nF_{n-1}}{5}\Big).$$

Next,

$$\sum_{k_3=0}^{n} k_3F_{k_3-1}F_{n-k_3} = \sum_{k_3=0}^{n-1} k_3F_{k_3}F_{n-1-k_3} + \sum_{k_3=0}^{n-1} F_{k_3}F_{n-1-k_3}.$$

Gathering like terms, we obtain the following equation.

$$\sum_{k_1+k_2+k_3=n} F_{k_1}F_{k_2}F_{k_3} = \frac{n(n-1)}{50}F_n + \frac{n^2}{25}F_{n-1} - \frac{(n-1)F_n+2nF_{n-1}}{25} + \frac{2(n-1)(n-2)}{50}F_{n-1} + \frac{2(n-1)^2}{25}F_{n-2} + \frac{2((n-2)F_{n-1}+2(n-1)F_{n-2})}{25}.$$

Simplifying the equation completes the proof. □

For the pair (K_n, F_n), we also obtain an iterative relation in the next theorem.

Theorem 4.4. *Let $m \geq 0$ and $n \geq m+4$ be integers. Then,*

$$\sum_{\substack{k_1+k_2+\cdots+k_{m+2}=n-3 \\ k_1\geq 0, k_2,\ldots,k_{m+2}\geq 1}} K_{k_1}F_{k_2}\cdots F_{k_{m+2}} = \sum_{\substack{k_1+k_2+\cdots+k_{m+1}=n-1 \\ k_1\geq 0, k_2,\ldots,k_{m+1}\geq 1}} K_{k_1}F_{k_2}\cdots F_{k_{m+1}} - 3H(n,m) + 2H(n-1,m) + H(n-2,m), \tag{4.9}$$

where $H(n,m)$ is defined in (4.2).

Proof. The statement is a consequence of the general relation

$$xv(x)f^m(x) - (3-2x-x^2)f^{m+1}(x) = x^3v(x)f^{m+1}(x), \tag{4.10}$$

which follows from (2.4). □

When $m=0$, Theorem 4.4 reduces to Theorem 2.2. Also, when $m=1$, it is easily verified that $-3H(n,1)+2H(n-1,1)+H(n-2,1) = -(n+1)F_{n-1}$. This shows that, when $m=1$, Theorem 4.4 reduces to Theorem 3.5. When $m=2$, we have the following identity.

Theorem 4.5. *Let $n \geq 6$ be an integer. Then,*

$$\begin{aligned}\sum_{\substack{k_1+k_2+k_3+k_4=n-3 \\ k_1\geq 0, k_2,k_3,k_4\geq 1}} K_{k_1}F_{k_2}F_{k_3}F_{k_4} &= K_{n+3} - L_{n+3} - (n+3)F_{n+1} \\ &-3\sum_{j=1}^{n-2}\frac{(n+j)(n-j)jF_j}{8}\cos\Big(\frac{(n-j-2)\pi}{2}\Big) \\ &+2\sum_{j=1}^{n-3}\frac{(n-1+j)(n-1-j)jF_j}{8}\cos\Big(\frac{(n-j-3)\pi}{2}\Big) \\ &+\sum_{j=1}^{n-4}\frac{(n-2+j)(n-2-j)jF_j}{8}\cos\Big(\frac{(n-j-4)\pi}{2}\Big).\end{aligned} \tag{4.11}$$

This result can be stated equivalently as

$$\sum_{\substack{k_1+k_2+k_3+k_4=n-3 \\ k_1\geq 0, k_2,k_3,k_4\geq 1}} K_{k_1}F_{k_2}F_{k_3}F_{k_4} = K_{n+3} - L_{n+3} - (n+3)F_{n+1} - \frac{3(n-1)(n-2)}{50}F_n - \frac{2(n-2)(n+9)}{50}F_{n-1} - \frac{7n^2+29n-36}{50}F_{n-2}. \tag{4.12}$$

5. Final Remark

From

$$2u(x)g^m(x) - xu(x)g^m(x) - xg^{m+1}(x) = x^3u(x)g^{m+1}(x) \tag{5.1}$$

and

$$2v(x)g^m(x) - xv(x)g^m(x) - (3-2x-x^2)g^{m+1}(x) = x^3v(x)g^{m+1}(x), \tag{5.2}$$

it is clear that a general solution for the pairs (L_n, T_n) and (L_n, K_n) will preserve its iterative accessibility. However, a closed form requires an expression for the sum

$$\sum_{k_1+k_2+\cdots+k_{m+1}=n} L_{k_1}L_{k_2}\cdots L_{k_{m+1}}.$$

Such an expression is currently unknown. The expressions for two- and three-factor sums that have been derived here are special cases of a more general identity that is to be found. The author proposes this task as an open problem.

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