# LUCAS SEQUENCES AND TRACES OF MATRIX PRODUCTS

JOHN GREENE

Abstract. Given two noncommuting matrices, $A$ and $B$, it is well-known that $AB$ and $BA$ have the same trace. This extends to cyclic permutations of products of $A$'s and $B$'s. Thus if $A$ and $B$ are fixed matrices, then products of two $A$'s and four $B$'s can have three possible traces. For $2 \times 2$ matrices $A$ and $B$, we show that there are restrictions on the relative sizes of these traces. For example, if $M_1 = AB^2AB^2$, $M_2 = ABAB^3$, and $M_3 = A^2B^4$, then it is never the case that $\mathrm{Tr}(M_2) > \mathrm{Tr}(M_3) > \mathrm{Tr}(M_1)$, but the other five orderings of the traces can occur. By utilizing the connection between Lucas sequences and powers of a $2 \times 2$ matrix, a formula is given for the number of orderings of the traces that can occur in products of two $A$'s and $n$ $B$'s.

## 1. Introduction and Main Results

Given two square matrices $A$ and $B$, it is well-known [5, 7] that

$$Tr(AB) = Tr(BA), \tag{1.1}$$

where $Tr(A)$ is the trace of the matrix $A$. Consequently, for longer matrix products [7, p. 110]:

$$Tr(A_1A_2 \cdots A_k) = Tr(A_kA_1A_2 \cdots A_{k-1}). \tag{1.2}$$

Given a matrix written as the product of a collection of matrices, define the necklace of that matrix to be the set of all products of cyclic permutations of the collection. Thus, the necklace of $ABC$ is $\{ABC, CAB, BCA\}$, the necklace of $ABAB$ is $\{ABAB, BABA\}$, and the necklace of $A^2B^2$ is $\{A^2B^2, BA^2B, B^2A^2, AB^2A\}$. By (1.2), all products in a necklace have the same trace.

One might ask how traces of different necklaces compare. The author finds it somewhat surprising that in general, the trace of $ABAB$ tends to be larger than the trace of $A^2B^2$. To be more rigorous, if $A$ and $B$ are square matrices with independent random variables as entries, then Table 1 from [2] shows how often $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ in a simulation with 1,000,000 trials.

The first row in Table 1 suggests that for $2 \times 2$ matrices with independent random normal variables, $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ with probability $\frac{1}{\sqrt{2}}$. This was proved in [2]. The exact probability for larger matrices is unknown.

Some of the results in [2] apply to other necklaces. If $A$ and $B$ are $2 \times 2$ matrices, then $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(A^2B^4)$ with probability $\frac{1}{\sqrt{2}}$ as well. However, with two $A$'s and four $B$'s, there are three necklaces to consider, denoted by $AB^2AB^2$, $ABAB^3$, and $A^2B^4$. In simulations, whereas $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(A^2B^4)$ in 706,206 of 1,000,000 trials (as expected if the probability is $\frac{1}{\sqrt{2}}$), $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(ABAB^3)$ in 642,122 trials, and $\mathrm{Tr}(ABAB^3) > \mathrm{Tr}(A^2B^4)$ in 582,660 trials. Presumably, the exact probabilities for which these inequalities hold could be

TABLE 1. The frequency for which $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ in 1,000,000 trials for $m \times m$ matrices $A$ and $B$.

| m | Normal variables | Uniform variables |
|---|---|---|
| 2 | 707,456 | 720,660 |
| 3 | 703,004 | 703,320 |
| 4 | 701,885 | 700,959 |
| 5 | 702,375 | 700,259 |
| 10 | 706,124 | 704,561 |
| 20 | 709,715 | 710,189 |
| 50 | 714,473 | 714,627 |
| 100 | 716,805 | 717,009 |

calculated as in [2] provided the proper eightfold integrals could be evaluated.

One could also ask about the six possible total orderings of the traces of these necklaces. Again using independent random normal variables as entries for $A$ and $B$, in 1,000,000 trials, and letting $M_1 = AB^2AB^2$, $M_2 = ABAB^3$, and $M_3 = A^2B^4$, Table 2 resulted.

TABLE 2. The frequency for orders of necklace traces, 1,000,000 trials total.

| Trace combination | Number of cases |
|---|---|
| $Tr(M_1) > Tr(M_2) > Tr(M_3)$ | 300,092 |
| $Tr(M_1) > Tr(M_3) > Tr(M_2)$ | 123,546 |
| $Tr(M_2) > Tr(M_1) > Tr(M_3)$ | 282,568 |
| $Tr(M_2) > Tr(M_3) > Tr(M_1)$ | 0 |
| $Tr(M_3) > Tr(M_1) > Tr(M_2)$ | 218,484 |
| $Tr(M_3) > Tr(M_2) > Tr(M_1)$ | 75,310 |

Of interest to us here is that the order $\mathrm{Tr}(M_2) > \mathrm{Tr}(M_3) > \mathrm{Tr}(M_1)$ did not occur in the 1,000,000 trials. Exploring further, it was discovered that this is common. As the number of $B$'s grew, a smaller and smaller portion of orders occurred in simulations, as shown in Table 3 from [10].

Table 3. The number of necklace orderings in numerical simulations.

| # of $B$'s | Necklaces | Possible orders | Orders occurring |
|---|---|---|---|
| 2 | 2 | 2 | 2 |
| 3 | 2 | 2 | 2 |
| 4 | 3 | 6 | 5 |
| 5 | 3 | 6 | 6 |
| 6 | 4 | 24 | 8 |
| 7 | 4 | 24 | 12 |
| 8 | 5 | 120 | 12 |
| 9 | 5 | 120 | 22 |
| 10 | 6 | 720 | 17 |
| 11 | 6 | 720 | 32 |

Our main theorem is the following.

**Theorem 1.1.** *Consider products of two $A$'s and $n$ $B$'s, where $A$ and $B$ are $2\times 2$ matrices and $n \geq 3$. If $\phi$ is Euler's totient function, then among those matrices for which no two distinct necklaces have the same trace, there are*

$$3 + \frac{1}{2}\sum_{k=1}^{n-1} \phi(k) \tag{1.3}$$

*possible arrangements for the orders of the traces when $n$ is even, and*

$$\sum_{k=1}^{n-1} \phi(k) \tag{1.4}$$

*possible arrangements when $n$ is odd.*

For example, when $n = 10$, the number of allowable orders is $3 + \frac{1}{2}(1+1+2+2+4+2+6+4+6) = 17$. Now

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2}{\pi^2} + O(n \ln n), \tag{1.5}$$

an estimate from [4, Theorem 330], and the number of necklace orderings is the factorial of $\left\lceil \frac{n+1}{2} \right\rceil$. Thus, the frequency of possible orders rapidly goes to 0 as $n$ increases. Two distinct necklaces have the same trace with probability 0 if their entries are selected independently at random from a normal distribution. One can easily construct $A$ and $B$ for which different necklaces have the same trace, even when $A$ and $B$ do not commute. For example, if

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

then $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(ABAB^3) = \mathrm{Tr}(A^2B^4)$. In this paper, we only consider strict inequalities, so in what follows, we restrict ourselves to matrices whose necklaces have distinct traces.

The proof of Theorem 1.1 follows from properties of Lucas sequences. In the next section, the required information on Lucas sequences is provided. These properties are related to traces of necklaces in Section 3. Theorem 1.1 is proved in Section 4, and we give some concluding remarks in Section 5.

## 2. Properties of Lucas Sequences

As usual [8, pp. 41-61], [9, pp. 107-108], Lucas sequences $U_n = U_k(P,Q)$ may be defined by the recurrence $U_0 = 0$, $U_1 = 1$, and $U_k = PU_{k-1} - QU_{k-2}$ for $k \geq 2$. Lucas sequences naturally enter into this study as follows. Let $B$ be a $2 \times 2$ matrix with trace $P$ and determinant $Q$. Then, by the $2 \times 2$ version of the Cayley-Hamilton theorem,

$$B^2 = PB - QI,$$

and by an easy induction,

$$B^k = U_k B - QU_{k-1} I. \tag{2.1}$$

We use the following properties of Lucas sequences.

**Lemma 2.1.** *Viewing $U_k(P,Q)$ as a polynomial in $P$ and $Q$ we have the following.*

(a) *As a polynomial in $P$, $U_k$ has degree $k-1$. If $k$ is even, then $U_k$ is an odd function in $P$; if $k$ is odd, then it is an even function in $P$.*

(b) *As a polynomial in $Q$, $U_k$ has degree $\left\lfloor \dfrac{k-1}{2} \right\rfloor$. Also, $U_k$ has exactly $\left\lfloor \dfrac{k+1}{2} \right\rfloor$ terms, one for each allowable power of $Q$ and the coefficient of $Q^l$ has the form $P^{k-1-2l}(-1)^l c_l$ for some integer $c_l > 0$.*

(c) *If $P^2 \geq 4Q$, then $U_k > 0$ when $k$ is odd; and $PU_k > 0$ when $k > 0$ is even.*

*Proof.* The proofs of (a) and (b) are easy inductions. Part (c) follows from the representation [8, p. 44]

$$U_k(P,Q) = \frac{1}{2^{k-1}} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} P^{k-2i-1}(P^2 - 4Q)^i.$$

$\square$

Of special interest are the cases where $Q = \pm 1$. With $Q = -1$, $U_k(x,-1)$ are usually referred to as Fibonacci polynomials. When $Q = 1$, $U_k(x,1)$ are a scaled version of Chebyshev polynomials of the second kind, with the actual Chebyshev polynomials being $U_{k+1}(2x,1)$. We require the following facts about $U_k(x,1)$.

**Lemma 2.2.** *The zeros of $U_k(x,1)$ have the form $x = 2\cos\frac{l\pi}{k}$ where $1 \leq l \leq k-1$. In particular, for all $k \geq 3$, $U_k(x,1)$ has exactly $\lfloor \frac{k-1}{2} \rfloor$ simple positive zeros and the zeros of $U_k(x,1)$ and $U_{k+1}(x,1)$ separate each other. That is, between each pair of successive positive zeros of one polynomial there is exactly one zero of the other.*

*Proof.* That the zeros are simple and separate each other follows from $\{U_k(2x,1)\}$ being a set of orthogonal polynomials. See [1, Theorem 5.4.1, Theorem 5.4.2], for example. A standard representation for Chebyshev polynomials [1, p 101] is $U_k(2\cos\theta, 1) = \frac{\sin k\theta}{\sin\theta}$, giving the formula for the zeros. Since $U_k$ has degree $k-1$ and is an even or odd function, depending on whether $n$ is odd or even, the count for the number of positive zeros follows. $\square$

We end this section with one more root separation result.

**Lemma 2.3.** *Suppose that $a < b$, that $a$ and $b$ are not zeros of any Lucas polynomial $U_i(x,1)$ for $2 \leq i \leq N$, and that there is a zero of $U_m(x,1)$ between $a$ and $b$ for some $m \leq N$. Then there is an index $k \leq N$ for which there is exactly one zero of $U_k(x,1)$ between $a$ and $b$.*

*Proof.* Suppose $a$ and $b$ are separated by a zero of the $U_m$. Let $k$ be minimal with the property that there is a zero of $U_k$ separating $a$ and $b$. If there were two or more zeros between $a$ and $b$, then by the interlacing property, $U_{k-1}$ would also have a zero between $a$ and $b$, contradicting the minimality of $k$. □

## 3. Lucas Sequences and Necklace Traces

In this section, $A$ and $B$ will always denote $2\times 2$ matrices. Moreover, we let $P = \mathrm{Tr}(B)$ and $Q = \det(B)$, and define the Lucas sequence $\{U_k(P,Q)\}$ as in the previous section. The main result of this section is the following.

**Theorem 3.1.** *Let $A$ and $B$ be $2\times 2$ matrices and let $T = Tr\ (ABAB - A^2B^2)$. If $j \geq i \geq k$ then*

$$\mathrm{Tr}\ (AB^iAB^j) - \mathrm{Tr}\ (AB^{i-k}AB^{j+k}) = Q^{i-k}U_k\,U_{j-i+k}\,T. \tag{3.1}$$

This theorem allows us to convert a question about trace orders to the positivity of a collection of products on the right side of (3.1).

*Proof.* The result follows via a number of applications of (2.1). We have

$$\begin{aligned} AB^iAB^j - AB^{i-k}AB^{j+k} &= (AB^iAB^{i-k} - AB^{i-k}AB^i)B^{j-i+k} \\ &= U_{j-i+k}\,(AB^iAB^{i+1-k} - AB^{i-k}AB^{i+1}) \\ &\qquad - QU_{j-i+k-1}\,(AB^iAB^{i-k} - AB^{i-k}AB^i). \end{aligned}$$

Since $\mathrm{Tr}(AB^iAB^{i-k}) = \mathrm{Tr}(AB^{i-k}AB^i)$, we have

$$\mathrm{Tr}(AB^iAB^j - AB^{i-k}AB^{j+k}) = U_{j-i+k}\,\mathrm{Tr}(AB^iAB^{i+1-k} - AB^{i-k}AB^{i+1}).$$

We now use $B^k = U_kB - QU_{k-1}I$ to obtain

$$\begin{aligned} AB^iAB^{i+1-k} - AB^{i-k}AB^{i+1} &= AB^{i-k}B^kAB^{i+1-k} - AB^{i-k}AB^{i+1-k}B^k \\ &= U_k\,(AB^{i+1-k}AB^{i+1-k} - AB^{i-k}AB^{i+2-k}). \end{aligned}$$

Letting $i - k = l$, we are left to evaluate

$$AB^{l+1}AB^{l+1} - AB^lAB^{l+2}.$$

We have

$$\begin{aligned} AB^{l+1}AB^{l+1} - AB^lAB^{l+2} &= PAB^lAB^{l+1} - QAB^{l-1}AB^{l+1} \\ &\qquad - PAB^lAB^{l+1} + QAB^lAB^l \\ &= Q(AB^lAB^l - AB^{l-1}AB^{l+1}). \end{aligned}$$

A simple induction now gives

$$AB^{l+1}AB^{l+1} - AB^lAB^{l+2} = Q^l(ABAB - A^2B^2),$$

and the proof follows. □

For necklaces with two $A$'s and $n$ $B$'s; if $l = \lfloor n/2 \rfloor$, then there are $l+1$ necklaces. If we list the products in a necklace in lexicographic order, then the first element of the necklace will have the form $AB^kAB^{n-k}$ with $0 \leq k \leq l$. We will use such matrices to represent their necklaces in what follows. A natural way to order necklaces is by how far the $A$'s are apart in the product (viewed cyclically). With this ordering, for $n$ even, these necklaces are represented by $AB^lAB^l$, $AB^{l-1}AB^{l+1}$, …, $A^2B^n$. We associate the $k$th necklace, $AB^{l+1-k}AB^{l-1+k}$, with the number $k$. When $n$ is odd, the $k$th necklace is $AB^{l+1-k}AB^{l+k}$ for $1 \le k \le l+1$.

Given two matrices $A$ and $B$, define the permutation $\pi = \pi(A,B) = \langle \pi_1, \pi_2, \ldots, \pi_{l+1}\rangle$ so that necklaces $\pi_1, \ldots, \pi_{l+1}$ have their traces in decreasing order. For example, if $n = 6$ then there are four necklaces, represented by $AB^3AB^3$, $AB^2AB^4$, $ABAB^5$, $A^2B^6$. We associate 1 with the first necklace, 2 with the second, etc. Since $AB^3AB^3$, $AB^2AB^4$, $ABAB^5$, $A^2B^6$ have traces 32, 24, 28, 30, respectively, we have

$$\pi\left(\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & -2\\ 1 & 0\end{pmatrix}\right) = \langle 1,4,3,2\rangle.$$

Define a function on permutations, $\pi$, by

$$S_{i,j}(\pi) = \begin{cases} 1, & \text{if } i \text{ appears to the left of } j \text{ in } \pi;\\ -1, & \text{otherwise.}\end{cases}$$

For example, $S_{1,3}(\langle 1,4,3,2\rangle) = 1$ and $S_{2,3}(\langle 1,4,3,2\rangle) = -1$. Note that $S_{i,j}(\pi) = -S_{j,i}(\pi)$.

**Corollary 3.2.** *If $i < j$ and $\pi = \pi(A,B)$, then with the notation above,*

$$S_{i,j}(\pi) = \begin{cases} \operatorname{sgn}(Q^{l-j+1}U_{j-i}\,U_{i+j-2}\,T), & \text{when } n \text{ is even;}\\ \operatorname{sgn}(Q^{l-j+1}U_{j-i}\,U_{i+j-1}\,T), & \text{when } n \text{ is odd.}\end{cases} \tag{3.2}$$

*Proof.* By (3.1), we have

$$\mathrm{Tr}\ (AB^{l+1-i}AB^{l-1+i}) - \mathrm{Tr}\ (AB^{l+1-j}AB^{l-1+j}) = Q^{l-j+1}U_{j-i}\,U_{i+j-2}\,T,$$

$$\mathrm{Tr}\ (AB^{l+1-i}AB^{l+i}) - \mathrm{Tr}\ (AB^{l+1-j}AB^{l+j}) = Q^{l-j+1}U_{j-i}\,U_{i+j-1}\,T.$$

□

**Corollary 3.3.** *If $n$ is even and $Q$ and $T$ are positive, then the necklace containing $AB^lAB^l$ has the largest trace.*

*Proof.* The matrix $AB^lAB^l$ is represented by 1 and

$$S_{1,j} = \operatorname{sgn}(Q^{l-j+1}U_{j-1}^2\,T) = 1,$$

when $Q$ and $T$ are positive. □

Finally for this section, we mention the following result.

**Lemma 3.4.** *If $A$ and $B$ are $2\times 2$ matrices and $\mathrm{Tr}(ABAB) < \mathrm{Tr}(A^2B^2)$, then $P^2 \ge 4Q$, where $P$ is the trace of $B$ and $Q$ is the determinant of $B$. Consequently, $U_k(P,Q) > 0$ for odd $k$ and $PU_k(P,Q) > 0$ for even $k$.*

*Proof.* In Lemma 3.6 of [2], it is shown that $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ when either $A$ or $B$ has complex eigenvalues. Thus, for $\mathrm{Tr}(ABAB) < \mathrm{Tr}(A^2B^2)$, $B$ must have real eigenvalues; call them $\lambda_1$ and $\lambda_2$. Now $P = \lambda_1 + \lambda_2$ and $Q = \lambda_1\lambda_2$, so $P^2 - 4Q = (\lambda_1 - \lambda_2)^2 \ge 0$. By Lemma 2.1 (c), the positivity of the $U_k$ follows. □

## 4. A Proof of Theorem 1.1

We show that the expressions in (1.3) and (1.4) give upper bounds for the numbers of possible trace orders, and that these bounds are achieved. For the upper bound, we use Corollary 3.2 and Lemma 3.4 to give information on permutations of necklace trace orders. We need information on the sign of $Q^{m-k}U_kU_{n-m+k}T$. We break up the investigation into three cases: $Q < 0$, $Q > 0$ but $T < 0$, and both $Q > 0$ and $T > 0$. We investigate these cases in order.

**Lemma 4.1.** *If $Q < 0$, then there are exactly two possible permutations of trace orders.*

*Proof.* If $Q < 0$, then $P^2 > 4Q$; so by Lemma 2.1(c), $U_k > 0$ for odd $k$ and $PU_k$ is positive for even $k$. Let $\pi = \pi(A, B)$. By Corollary 3.2, if $i < j$, then $S_{i,j}(\pi) = \operatorname{sgn}((-1)^{l-j+1}TU_{j-i}U_{j+i-\epsilon})$, where $\epsilon = 1$ if $n$ is odd and $\epsilon = 2$ otherwise. This means that

$$S_{i,j}(\pi) = \begin{cases} \operatorname{sgn}((-1)^{l-j+1}\,T), & \text{when } n \text{ is even;} \\ \operatorname{sgn}((-1)^{l-j+1}\,TP), & \text{when } n \text{ is odd.} \end{cases}$$

For any given $B$, the sign of $T$ and the sign of $PT$ are fixed. That is, the relevant sign is the same for every permutation. Consequently, there are two possible permutations corresponding to whether $T$ (or $PT$) is positive or negative. □

We can state explicitly what these permutations are. Since $l - j + 1$ does not depend on $i$, but only on the parity of $j$, once we know $S_{1,2}(\pi)$, we know $\pi$. If $S_{1,2}(\pi) = 1$, then all even numbers follow all odd numbers. Also, $S_{i,i+2j}(\pi)$ will be 1 if $i$ is even and $-1$ if $i$ is odd. This means that $\pi$ starts with the largest odd number and descends through the odds to 1, followed by the even numbers in increasing order. If $n = 9$ for example, then there are five necklaces and this permutation would be $\langle 5, 3, 1, 2, 4\rangle$. On the other hand, if $S_{1,2}(\pi) = -1$, we have the reverse of this permutation, $\langle 4, 2, 1, 3, 5\rangle$. These two permutations must occur since they will be produced by the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

To see this, we note that

$$\operatorname{Tr}(A_1BA_1B - A_1^2B^2) = -9,$$
$$\operatorname{Tr}(A_2BA_2B - A_2^2B^2) = 9.$$

In these examples, $P = 1$ and $Q = -2 < 0$. Since $T = \operatorname{Tr}(ABAB - A^2B^2)$, by the proof of Lemma 4.1, $S_{1,2}(\pi) = \operatorname{sgn}((-1)^{l-1}T)$, and both sign patterns will occur. One thing is left to establish: that all necklaces have different traces. For this, noting that $T$ and $Q$ are nonzero, by Theorem 3.1, two traces can only be the same when $U_{j-i}U_{j+i-k} = 0$. In this case, $U_m = U_{m-1} + 2U_{m-2}$ implies that no $U_m$ is zero if $m > 0$.

**Lemma 4.2.** *If $Q > 0$ but $T < 0$, then there is one possible permutation when $n$ is even, and two permutations if $n$ is odd.*

*Proof.* By Lemma 3.4, we again have $P^2 > 4Q$, but now we know the sign of $T$. Thus, for $i < j$ we have

$$S_{i,j}(\pi) = \begin{cases} -\operatorname{sgn}(Q^{l-j+1}) = -1, & \text{when } n \text{ is even;} \\ -\operatorname{sgn}(Q^{l-j+1}\,P) = -\operatorname{sgn}(P), & \text{when } n \text{ is odd.} \end{cases}$$

For an even $n$, the permutation must be $\pi = \langle l+1, l, \ldots, 2, 1\rangle$. When $n$ is odd, there are two possible permutations, one for each sign of $P$. The first is the same as the above, the other is $\pi = \langle 1, 2, \ldots, l, l+1\rangle$. □

These cases are realized in the examples

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad B_2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

This leaves us with the following case.

**Theorem 4.3.** *Consider the set*

$$M = \{a > 0 \mid U_k(a, 1) = 0 \textit{ for some } k \textit{ with } 3 \le k \le n-1\},$$

*the set of distinct positive zeros of* $U_3$, …, $U_{n-1}$, *and suppose* $M$ *has size* $m$. *The number of permutations of trace orders in the case where* $Q > 0$ *and* $T > 0$ *is*

$$\begin{cases} 1+m, & \textit{when } n \textit{ is even;} \\ 2(1+m), & \textit{when } n \textit{ is odd.} \end{cases}$$

*Proof.* The $Q$-parameter in $U_n(P, Q)$ may be scaled away by multiplying $B$ by $\frac{1}{\sqrt{Q}}$. This will have no effect on the orders of the traces of the necklaces. Thus, we need only consider the Lucas sequence $U_k(x, 1)$. Since $T > 0$,

$$S_{i,j}(\pi) = \begin{cases} \operatorname{sgn}(U_{j-i}U_{i+j-2}), & \text{when } n \text{ is even;} \\ \operatorname{sgn}(U_{j-i}U_{i+j-1}), & \text{when } n \text{ is odd;} \end{cases} \tag{4.1}$$

for all $1 \le i < j \le l+1$. Note that the maximum value of the subscript $i + j - 2$ is $l + (l+1) - 2 = 2l - 1 = n - 1$ when $n$ is even, and the maximum value of $i + j - 1$ is $l + (l+1) - 1 = 2l = n - 1$ when $n$ is odd as well. Thus, when examining $S_{i,j}(\pi)$, only Lucas polynomials up to $U_{n-1}$ are required. We focus on $S_{i,i+1}$ and $S_{i,i+2}$. When $n$ is even, these have the form $\operatorname{sgn}(U_1(x,1)U_{2i-1}(x,1)) = \operatorname{sgn}(U_{2i-1})$, and $\operatorname{sgn}(U_2(x,1)U_{2i}(x,1)) = \operatorname{sgn}(xU_{2i})$, respectively. When $n$ is odd, the important quantities are $\operatorname{sgn}(U_{2i})$ and $\operatorname{sgn}(xU_{2i+1})$. Given an $x \notin M$, the conditions of (4.1) will determine a permutation, call it $\pi(x)$.

Suppose we order the set of positive zeros $0 < x_1 < x_2 < \cdots < x_m$. These zeros partition the half line $(0, \infty)$ into $m + 1$ regions. If $x$ and $y$ belong to the same region, say $x_i < x$ and $y < x_{i+1}$, then $\pi(x) = \pi(y)$ since signs of $U_k(x, 1)$ and $U_k(y, 1)$ will match for all $k$. Thus, there can be no more than $m+1$ permutations associated with the regions between the elements of $M$. When $n$ is even, all the products of the $U$'s will be even polynomials, leaving us with at most these $m+1$ permutations. When $n$ is odd, the products of the $U$'s will be odd polynomials. Thus $\pi(-x)$ will be the reverse of $\pi(x)$, doubling the possible number of permutations. If

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix},$$

then $P = x$, $Q = 1$, $T = x^2 > 0$, so for every region between the zeros of the $U_k$, there is a matrix $B$ with an $x$-value in that region, with its associated permutation.

For example, suppose that $n = 6$. Then,

$$\begin{aligned} M &= \{a > 0 \mid U_3(a,1) = 0,\ U_4(a,1) = 0 \text{ or } U_5(a,1) = 0\} \\ &= \{a > 0 \mid a^2 - 1 = 0,\ a^3 - 2a = 0 \text{ or } a^4 - 3a^2 + 1 = 0\} \\ &= \left\{\frac{1}{2}(\sqrt{5}-1),\ 1,\ \sqrt{2},\ \frac{1}{2}(\sqrt{5}+1)\right\}. \end{aligned}$$

If we were to select $x = 1.25$, then $U_1(x) > 0$, $U_2(x) > 0$, $U_3(x) > 0$, $U_4(x) < 0$, and $U_5(x) < 0$, meaning 1 is to the left of 2 and 3, 2 is left of 3 but right of 4, and 3 is right of 4. This information did not tell us how 1 and 4 relate, but it turns out $\pi(1.25) = \langle 1, 4, 2, 3\rangle$. Selecting an $x$ from each of the five possible regions, we have

$$\pi(.5) = \langle 1,3,4,2\rangle,\ \pi(.8) = \langle 1,4,3,2\rangle,\ \pi(1.25) = \langle 1,4,2,3\rangle,$$
$$\pi(1.5) = \langle 1,2,4,3\rangle,\ \text{and}\ \ \pi(2) = \langle 1,2,3,4\rangle.$$

Finally, we must show that all resulting permutations are distinct. Knowing $S_{i,i+1}$ and $S_{i,i+2}$ for all $i$ is not enough to specify $\pi(x)$, but it is enough information to show $\pi(x) \neq \pi(y)$, if $x$ and $y$ come from different regions. To see this, suppose that $x$ and $y$ are separated by a zero for one of the $U_i$. By Lemma 2.3, $x$ and $y$ are separated by a single zero of $U_k$ for some $k \le n-1$. Then $\operatorname{sgn}(U_k(x,1)) = -\operatorname{sgn}(U_k(y,1))$. This means that depending on the parity of $k$, there will be an $i$ with either $S_{i,i+1}$ or $S_{i,i+2}$ differing from $\pi(x)$ to $\pi(y)$. In the case where $n$ is odd, we must also show that for positive $x$ and $y$, $\pi(x) \neq \pi(-y)$. Since $S_{1,2} = \operatorname{sgn}(x)$, when $x$ is positive, 1 will be to the left of 2 but for negative $x$, 1 is to the right of 2, so these permutations are all distinct. □

If there are at least three necklaces, then the necklaces arising from the cases where $Q < 0$ and $Q > 0$, $T < 0$ are distinct from each other. When $n \ge 4$ is even, these permutations are also distinct from those with $Q > 0$, $T > 0$, since the first three permutations will not start with 1 but those with $Q > 0$ and $T > 0$ will. Thus, for even $n \ge 4$ we have $m+4$ permutations.

If $n = 3$, a simple check shows that both orders $\langle 1,2\rangle$ and $\langle 2,1\rangle$ occur. When $n \ge 5$, the four permutations associated with $Q < 0$ and $Q > 0$, $T < 0$ also occur among the permutations of Theorem 4.3. If $x_1$ is the smallest element of $M$, and $0 < x < x_1$ then $\pi(x)$ and $\pi(-x)$ are the permutations that arise in Lemma 4.1. This is because an easy calculation shows $S_{i,j} = (-1)^j$, which does not depend on $i$, and the discussion following Lemma 4.2 applies to this case. Similarly, the two permutations from Lemma 4.2 are $\pi(x)$ and $\pi(-x)$ where $x > x_m$, the largest of the zeros in $M$. In this case, $U_k(x,1) > 0$ for all $k$, so $\pi(x)$ is the identity permutation and $\pi(-x)$ is its reverse, as in Lemma 4.2. Consequently, when $n$ is odd, the only permutations we have are those arising from Theorem 4.3. Consequently, we have a count on the number of permutations. It is

$$\begin{cases} 4+m, & \text{when } n \text{ is even;} \\ 2+2m, & \text{when } n \text{ is odd.} \end{cases}$$

The proof of Theorem 1.1 follows from the observation that for $M$ as in Theorem 4.3, $|M| = \frac{1}{2}\sum_{k=3}^{n-1}\phi(k) = -1 + \frac{1}{2}\sum_{k=1}^{n-1}\phi(k)$. This, in turn, follows from

$$M = \left\{2\cos\frac{k\pi}{m} \mid 1 \le k \le \frac{1}{2}(m-1),\ 3 \le m \le n-1\right\}.$$

That is, given that $M$ contains the zeros for $U_k(x,1)$, with $3 \le k < m$, the contribution of the zeros of $U_m$ to $M$ will consist of those numbers $2\cos\frac{j\pi}{m}$, with $j$ prime to $m$. There are $\frac{1}{2}\phi(m)$ of these, by periodicity and that we seek only positive zeros. This concludes the proof of Theorem 1.1.

## 5. Comments

Products of $A$'s and $B$'s with at least three $A$'s and three $B$'s are more complicated for two reasons. First, for products of $2 \times 2$ matrices $A$ and $B$, there is another trace symmetry in addition to cyclic permutations: If a product is written in reverse order, it has the same trace, as proved in [6] or [3]. That is,

$$\mathrm{Tr}(AABBAB) = \mathrm{Tr}(BABBAA)$$

for all $2 \times 2$ matrices $A$ and $B$. If a product consists of just two $A$'s, then the reverse of a product is in the same necklace, but for larger numbers of $A$'s, as in the example, this need not be the case. This makes ordering the necklaces more challenging.

A second issue is that with at least three $A$'s and three $B$'s, the matrices interact more than just through the trace of $ABAB - A^2B^2$. For example,

$$\mathrm{Tr}(ABABAB - A^2BAB^2) = \mathrm{Tr}(AB)\mathrm{Tr}(ABAB - A^2B^2).$$

We do not have an analog for Theorem 3.1 when there are more than two $A$'s. However, we at least have the following weak version.

**Theorem 5.1.** *Suppose that $M_1$ and $M_2$ are each products of $m$ $A$'s and $n$ $B$'s where $A$ and $B$ are $2 \times 2$ matrices. Then*

$$\mathrm{Tr}(M_1 - M_2) = c\mathrm{Tr}(ABAB - A^2B^2),$$

*where $c$ is a polynomial in the entries of $A$ and $B$.*

*Proof.* We may induct on $m+n$, the number of matrices in the two products. If $m+n \le 4$, the trace is zero when $m$ is 0, 1, 3, or 4, and for $m = 2$, the result is true by Theorem 3.1. For larger $m+n$, we first note that by cyclic permutation we may write

$$M_1 = A^{a_1}B^{b_1}A^{a_2}B^{b_2}\cdots A^{a_j}B^{b_j},$$

$$M_2 = A^{c_1}B^{d_1}A^{c_2}B^{d_2}\cdots A^{c_k}B^{d_k},$$

where each of the exponents is a positive integer, $a_1 + \cdots + a_j = m = c_1 + \cdots + c_k$, and $b_1 + \cdots + b_j = n = d_1 + \cdots + d_k$. Moreover, we may take $a_1$ to be the largest of the $a$'s and $c_1$ to be the largest of the $c$'s. If $a_1 \ge 2$ and $c_1 \ge 2$, we may use $A^2 = \mathrm{Tr}(A)A - \det(A)I$ and induct. Similarly, if one of the $b$'s and one of the $d$'s is at least 2, we may induct. If no $A$ has an exponent larger than 1, then $j = k$, and the only way to prevent some exponent of $B$ to be at least 2 is to have $M_1 = M_2$. Thus, we may assume that some $c$, say $c_1$, is at least 2, and $a_1 = a_2 = \cdots = a_j = 1$. In this case, $j = m \le n$ and $k < m$. Consequently, the largest $b$ and largest $d$ will both be at least 2, unless $m = n$ and $M_1 = (AB)^m$. Since $m+n > 4$, $m \ge 3$. Let $M_3 = A(AB)^{m-1}B = A^2(BA)^{m-2}B^2$ and consider

$$\mathrm{Tr}(M_1 - M_2) = \mathrm{Tr}(M_1 - M_3) + \mathrm{Tr}(M_3 - M_2).$$

By the previous discussion, $\mathrm{Tr}(M_3 - M_2) = c_1\mathrm{Tr}(ABAB - A^2B^2)$, since each matrix contains $A^2$. Since $m - 1 \geq 2$ we may use $(AB)^2 = \mathrm{Tr}(AB)AB - \det(AB)I$ to write

$$\begin{aligned}\mathrm{Tr}(M_1 - M_3) = \mathrm{Tr}(AB)((AB)^{m-1} - A(AB)^{m-2}B)\\ - \det(AB)((AB)^{m-2} - A(AB)^{m-3}B),\end{aligned}$$

and the inductive hypothesis gives $\mathrm{Tr}(M_1 - M_3) = c_2\mathrm{Tr}(ABAB - A^2B^2)$, from which the proof follows. □

We note that the polynomial $c$ in this proof can be thought of as a polynomial in the five variables $\mathrm{Tr}(A)$, $\mathrm{Tr}(B)$, $\mathrm{Tr}(AB)$, $\det(A)$, $\det(B)$, rather than the eight entries of $A$ and $B$.

We only briefly investigated cases with a higher number of $A$'s. When there are three $A$'s or four $A$'s we obtained the following table.

Table 4. The number of necklace orderings in numerical simulations.

| # of $A$'s | # of $B$'s | Necklaces | Possible orders | Orders occurring |
|---|---|---|---|---|
| 3 | 3 | 3 | 6 | 6 |
| 3 | 4 | 4 | 24 | 24 |
| 3 | 5 | 5 | 120 | 52 |
| 3 | 6 | 7 | 5040 | 175 |
| 3 | 7 | 8 | 40,320 | 246 |
| 4 | 4 | 8 | 40,320 | 616 |

We have not verified that the entries in the fifth column are the true numbers of possible orders. However, it is not too difficult to show that certain orders do not occur. For example, when there are three $A$'s and five $B$'s, if one labels the necklaces via $1 \leftrightarrow ABAB^2AB^2$, $2 \leftrightarrow ABABAB^3$, $3 \leftrightarrow A^2B^2AB^3$, $4 \leftrightarrow A^2BAB^4$, $5 \leftrightarrow A^3B^5$, then we may construct a table of polynomials as in the previous section and use this to show, for example, that the trace order $\langle 1, 2, 3, 4, 5\rangle$ does not occur. Let $a = \mathrm{Tr}(A)$, $b = \mathrm{Tr}(AB)$, $x = \mathrm{Tr}(B)$, $y = \det(B)$, $z = \mathrm{Tr}(ABAB - A^2B^2)$. Then

$$1 \text{ is left of } 2 \ \rightarrow byz > 0, \qquad \text{and} \qquad 1 \text{ is left of } 4 \ \rightarrow bx^2z > 0.$$

These require that $bz > 0$ and $y > 0$. Next,

$$2 \text{ is left of } 4 \ \rightarrow b(x^2 - y)z > 0, \qquad 3 \text{ is left of } 5 \ \rightarrow ax(x^2 - y)z > 0, \qquad \text{and}$$
$$4 \text{ is left of } 3 \ \rightarrow axyz < 0,$$

and multiplying these three inequalities together, we have $a^2x^2(x^2 - y)^2z^2(byz) < 0$, contradicting $byz > 0$.

Finally, it might be expected that restrictions on trace orders will occur with matrices of any size. We briefly investigated this for $3 \times 3$ and $4 \times 4$ matrices but the extra degrees of freedom in larger matrices mean that a large number of necklaces must be involved before such restrictions occur, making such restrictions hard to find numerically.

## Acknowledgement

I thank the reviewer for a careful reading of this paper and for many fine comments that improved format and quality of exposition in the final product.

## References


[1] G. Andrews, R. Askey, and R. Roy, *Special Functions, Encyclopedia of Mathematics and its Applications 71*, Cambridge University Press, McGraw Hill, Cambridge, 1999.

[2] J. Greene, *Traces of matrix products*, Electronic Journal of Linear Algebra, **27** (2014), 716–734.

[3] J. Greene and K. Niedzielski, *Periodic solutions to some difference equations over the integers*, Difference Equations and Their Applications, **16** (2010), 321–346.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth ed., Oxford University Press, Oxford, 1979.

[5] K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, Englewood Cliffs, 1971.

[6] Z. Huang, *Traces of Matrix Products*, Master's Thesis, University of Minnesota Duluth, 2009.

[7] C. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.

[8] P. Ribenboim, *The Book of Prime Number Records*, Springer-Verlag, New York, 1988.

[9] H. Riesel, *Prime Numbers and Computer Methods for Factorization*, Progress in Mathematics 126 (2nd ed.), Birkhäuser, Boston, 1994.

[10] A. Schneider, *Investigating traces of matrix products*, Master's Project, University of Minnesota Duluth, 2015.


MSC2010: 11B39, 15A15

Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, Minnesota, 55812

*E-mail address*: `jgreene@d.umn.edu`