SEQUENCES OF CONSECUTIVE HAPPY NUMBERS IN NEGATIVE BASES

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Abstract. For $b \leq -2$ and $e \geq 2$, let $S_{e,b} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be the function taking an integer to the sum of the e-powers of the digits of its base b expansion. An integer a is a *b-happy number* if there exists $k \in \mathbb{Z}^+$ such that $S_{2,b}^k(a) = 1$. We prove that an integer is -2-happy if and only if it is congruent to 1 modulo 3 and that it is -3-happy if and only if it is odd. Defining a d-sequence to be an arithmetic sequence with constant difference d and setting $d = \gcd(2, b-1)$, we prove that for odd $b \leq -3$ and for $b \in \{-4, -6, -8, -10\}$, there exist arbitrarily long finite sequences of d-consecutive b-happy numbers.

1. Introduction

As is standard, a positive integer a can be uniquely expanded in the base $b \geq 2$ as $a = \sum_{i=0}^{n} a_i b^i$, where $0 \leq a_i \leq b-1$ and $a_n \neq 0$. This definition can be extended to negative bases $b \leq -2$ in an analogous manner, with $0 \leq a_i \leq |b| - 1$. The study of negative bases was introduced in the 1885 work of Vittorio Grünwald [4]. It is known that for any base $b > 0$, each nonnegative integer has a unique base b representation. Similarly, for any base $b' < 0$, every nonzero integer has a unique base b' representation (with no need for a leading negative sign). Note that any integer written in a negative base with an odd number of digits is necessarily positive, whereas any written with an even number of digits is necessarily negative. For example, converting between base 10 and base -10, we have $2018 = (18198)_{(-10)}$ and $-2018 = (2022)_{(-10)}$.

We begin by adapting the definition of generalized happy numbers and the corresponding function given in [3] to include the case of negative bases. It is natural, in this case, to extend the domain of the function to include all integers.

Definition 1. Let $b \leq -2$ and $e \geq 2$ be integers, and let $a \in \mathbb{Z} - \{0\}$ be given by $a = \sum_{i=0}^{n} a_i b^i$ where $0 \leq a_i \leq |b| - 1$, for each $0 \leq i \leq n$. Define the function $S_{e,b} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by $S_{e,b}(0) = 0$ and, for $a \neq 0$,

$$S_{e,b}(a) = \sum_{i=0}^{n} a_i^e.$$

Further, let $S_{e,b}^0(a) = a$ and for each $k \in \mathbb{Z}^+$, $S_{e,b}^k(a) = S_{e,b}(S_{e,b}^{k-1}(a))$.

Definition 2. An integer a is an *e-power b-happy number* if, for some $k \in \mathbb{Z}^+$, $S_{e,b}^k(a) = 1$. A *b-happy number* is a 2-power b-happy number.

For example, if $b = -8$, then $554 = 17132_{(-8)}$ is a -8-happy number since $S_{2,-8}^2(17132_{(-8)}) = S_{2,-8}(100_{(-8)}) = 1$; $46 = 136_{(-8)}$ is a fixed point since $S_{2,-8}(46) = S_{2,-8}(136_{(-8)}) = 46$; and the integers 11 and 59 form a cycle since $S_{2,-8}(11) = S_{2,-8}(173_{(-8)}) = 59$ and $S_{2,-8}(59) = S_{2,-8}(113_{(-8)}) = 11$.

The following definition is from [2].

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Definition 3. A *d-consecutive* sequence is defined to be an arithmetic sequence with constant difference d.

In Section 2, we determine the fixed points and cycles of the functions $S_{2,b}$ for $-10 \leq b \leq -2$. In Section 3, we generalize the work of El-Sedy and Siksek [1] and the work of Grundman and Teeple [2] on sequences of consecutive b-happy numbers. In particular, Grundman and Teeple showed that there exist arbitrarily long finite d-consecutive sequences of b-happy numbers, where $b \geq 2$ and $d = \gcd(2, b-1)$ [2, Corollary 2]. We prove that this result does not hold for $b = -2$, but does hold for all odd negative bases and for even negative bases $-10 \leq b \leq -4$.

2. Cycles and Fixed Points

In this section, we first determine a bound, dependent on the given base $b \leq -2$, such that each fixed point and at least one point in every cycle is less than this bound. We then use this result to compute all cycles and fixed points of $S_{2,b}$ for $-10 \leq b \leq -2$. Note that if $b \leq -2$ and $k > 0$, for each $a \neq 0$, $S_{2,b}^k(a) > 0$. Hence, there are no negative fixed points.

For larger values of a, we have the following result.

Theorem 1. *Let $b \leq -2$. If $a > (|b|-1)(|b|^2 - |b| + 1)$, then $0 < S_{2,b}(a) < a$.*

Proof. Let a and b be as in the hypothesis. Then $a = \sum_{i=0}^{n} a_i b^i$ with n even, $0 \leq a_i \leq |b| - 1$, $a_n \neq 0$. Observe that

$$\begin{aligned} a - S_{2,b}(a) &= \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^2 = \sum_{i=0}^{n} a_i (b^i - a_i) \\ &= \sum_{j=1}^{\frac{n}{2}} a_{2j}(|b|^{2j} - a_{2j}) - \sum_{j=1}^{\frac{n}{2}} a_{2j-1}\left(|b|^{2j-1} + a_{2j-1}\right) + a_0(|b|^0 - a_0). \end{aligned} \tag{2.1}$$

<u>Case $n \geq 4$</u>. Since, $a_n \geq 1$ and, for each i, $0 \leq a_i \leq |b| - 1$, minimizing each term in (2.1) yields

$$a - S_{2,b}(a) \geq 1(b^n - 1) - \sum_{j=1}^{\frac{n}{2}} (|b|-1)\left(|b|^{2j-1} + (|b|-1)\right) + (|b|-1)(1 - (|b|-1)).$$

Noting that

$$\sum_{j=1}^{\frac{n}{2}} |b|^{2j-1} = \frac{|b|}{|b|^2 - 1}\left(|b|^n - 1\right),$$

we have

$$\begin{aligned} a - S_{2,b}(a) &\geq (b^n - 1) - (|b|-1)\left(\frac{|b|}{|b|^2-1}\left(|b|^n - 1\right) + \frac{n}{2}(|b|-1)\right) - (|b|-1)(|b|-2) \\ &= \frac{b^n - 1}{|b|+1} - \frac{n}{2}(|b|-1)^2 - (|b|-1)(|b|-2) \\ &= \frac{1}{|b|+1}\left(|b|^n - \frac{n}{2}(|b|^2-1)(|b|-1) - (|b|^2-1)(|b|-2) - 1\right) && (2.2) \\ &> \frac{1}{|b|+1}\left(|b|^n - \frac{n}{2}|b|^3 - |b|^3\right) && (2.3) \\ &> \frac{1}{|b|+1}\left(|b|^{n-3} - \frac{n}{2} - 1\right). && (2.4) \end{aligned}$$

Note that the function $f(x) = 2^{x-3} - x/2 - 1$ is an increasing function for $x \geq 5$ and that $f(5) > 0$. Thus, for $n \geq 5$, since $b \leq -2$,

$$|b|^{n-3} - n/2 - 1 \geq 2^{n-3} - n/2 - 1 > 0,$$

and so, by inequality (2.4), $a - S_{2,b}(a) > 0$. Now, for $n = 4$ and $b < -2$, using inequality (2.3), $a - S_{2,b}(a) > \frac{1}{|b|+1}\left(|b|^4 - 3|b|^3\right) \geq 0$, and for $n = 4$ and $b = -2$, using inequality (2.2), $a - S_{2,b}(a) > 0$.

Case $n < 4$. In this case, $(|b|-1)(|b|^2-|b|+1) < a \leq (|b|-1)(|b|^2+1)$. So, $a = a_2b^2+a_1b+a_0$ with $a_2 = |b| - 1$, $0 \leq a_1 \leq |b| - 2$, and $0 \leq a_0 \leq |b| - 1$. Thus,

$$\begin{aligned} a - S_{2,b}(a) &= a_2(|b|^2 - a_2) - a_1\left(|b| + a_1\right) + a_0(1 - a_0) \\ &\geq (|b| - 1)(|b|^2 - (|b| - 1)) - (|b| - 2)(|b| + (|b| - 2)) + (|b| - 1)(1 - (|b| - 1)) \\ &= |b|^3 - 5|b|^2 + 11|b| - 7 > 0, \end{aligned}$$

since $b \leq -2$. □

Note that when $b = -2$, the bound in Theorem 1 is 4. Since 3 is a fixed point of $S_{2,-2}$, the given bound is best possible.

The following corollary is immediate.

Corollary 2. *Let $b \leq -2$. Every fixed point of $S_{2,b}$ is less than or equal to $(|b|-1)(|b|^2-|b|+1)$ and every cycle of $S_{2,b}$ contains a number that is less than or equal to $(|b| - 1)(|b|^2 - |b| + 1)$.*

Using Corollary 2 and a direct computer search, we determine all fixed points and cycles in the bases $-10 \leq b \leq -2$. The results are given in Table 1.

Base	Fixed points	Cycles	Smallest happy number > 1	Largest happy number < -1
-2	1,2,3	None	4	-2
-3	1	(2,4,6)	3	-1
-4	1	(6,14)	16	-4
-5	1,10,11	(2,4,16,6,18,14,26), (9,33,29,17)	25	-5
-6	1	(2,4,16,33,11,51,29,30)	36	-6
-7	1,41	(2,4,16,30,14,26,42), (5,25,33,35), (6,36)	49	-7
-8	1,46	(11,59), (30,62,38,53)	64	-8
-9	1	(6,36,26,114,76,18,50,42,62,74), (9,65), (27,37)	5	-5
-10	1	(19,163,29,146,76,46,73), (35,75)	100	-10

TABLE 1. Base 10 representation of fixed points and cycles of $S_{2,b}$ for $-10 \leq b \leq -2$.

Definition 4. For $e \geq 2$ and $b \leq -2$, let

$$U_{e,b} = \{a \in \mathbb{Z}^+ |\text{ for some } m \in \mathbb{Z}^+,\ S^m_{e,b}(a) = a\}.$$

The following straightforward lemmas are used throughout this work.

Lemma 3. *Fix $b \leq -2$. For each $a \neq 0$, there exists some $k \in \mathbb{Z}^+$ such that $S^k_{2,b}(a) \in U_{2,b}$.*

Lemma 4. *Fix $b \leq -2$, $a \in \mathbb{Z}$, and $k \in \mathbb{Z}^+$. If b is odd, then*

$$S^k_{2,b}(a) \equiv a \pmod 2.$$

Proof. Fix a, b, and k as in the lemma. Noting that the result is trivial if $a = 0$. Let $a = \sum_{i=0}^{n} a_i b^i$. Then,

$$a = \sum_{i=0}^{n} a_i b^i \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i^2 \equiv S_{2,b}(a) \pmod 2.$$

A simple induction argument completes the proof. □

3. Consecutive b-Happy Numbers

In this section, we consider sequences of consecutive b-happy numbers for negative b. Grundman and Teeple [2] proved, for each base $b \geq 2$, that, letting $d = \gcd(2, b-1)$, there exist arbitrarily long finite sequences of d-consecutive b-happy numbers. We prove the following theorem using ideas from [1, 2]. Note that part (1) of the theorem demonstrates that the results in [2] do not generalize directly to negative bases.

Theorem 5. *Let $b \leq -2$.*

1. *There exist infinitely long sequences of 3-consecutive -2-happy numbers. In particular, $a \in \mathbb{Z}$ is -2-happy if and only if $a \equiv 1 \pmod 3$.*
2. *There exist infinitely long sequences of 2-consecutive -3-happy numbers. In particular, $a \in \mathbb{Z}$ is -3-happy if and only if $a \equiv 1 \pmod 2$.*
3. *For $b \in \{-4, -6, -8, -10\}$, there exist arbitrarily long finite sequences of consecutive b-happy numbers.*
4. *For b odd, there exist arbitrarily long finite sequences of 2-consecutive b-happy numbers.*

The smaller even negative bases not covered by Theorem 5 are addressed in the following conjecture, a proof for which would extend Pan's theorem [5] to all integral bases for $e = 2$.

Conjecture 6. *For $b \leq -12$ and even, there exist arbitrarily long finite sequences of consecutive b-happy numbers.*

We begin by proving the first two cases of Theorem 5. The other two cases follow immediately from Corollary 13 and are stated and proved at the end of this section.

Lemma 7. *An integer a is -2-happy if and only if $a \equiv 1 \pmod 3$ and is -3-happy if and only if a is odd.*

Proof. If $a = \sum_{i=0}^{n} a_i(-2)^i$ with $a_i \in \{0, 1\}$ for all $0 \leq i \leq n$, then

$$S_{2,-2}(a) = \sum_{i=0}^{n} a_i^2 = \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i(-2)^i \equiv a \pmod 3.$$

Thus, if a is -2-happy, $a \equiv 1 \pmod 3$. Now, suppose that $a \equiv 1 \pmod 3$. By Lemma 3, there exists a $k \in \mathbb{Z}^+$ such that $S_{2,-2}^k(a) \in U_{2,-2} = \{1, 2, 3\}$. Since $S_{2,-2}^k(a) \equiv a \equiv 1 \pmod 3$, $S_{2,-2}^k(a) = 1$ and so a is a -2-happy number.

By Lemma 4, if a is a -3-happy number, then a is odd. Since $U_{2,-3} = \{1, 2, 4, 6\}$, Lemmas 3 and 4 together imply that if $a \equiv 1 \pmod 2$, then a is a -3-happy number. □

The following definitions are from [2].

Definition 5. Let $e \geq 2$ and $b \leq -2$. A finite set T is (e, b)-*good* if, for each $u \in U_{e,b}$, there exist n, $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S_{e,b}^k(t + n) = u$.

Definition 6. Let $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $I(t) = t + 1$.

We will prove that for each odd $b \le -5$ and for $b \in \{-4,-6,-8,-10\}$, a finite set T of positive integers is $(2,b)$-good if and only if all of the elements of T are congruent modulo $d = \gcd(2, b-1)$. Lemma 8 and its proof are analogous to [2, Lemma 4 and proof].

Lemma 8. *Fix $e \ge 2$ and $b \le -2$. Let $T \subseteq \mathbb{Z}^+$ be finite. Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the composition of a finite sequence of the functions $S_{e,b}$ and I. If $F(T)$ is (e,b)-good, then T is (e,b)-good.*

Proof. Fix $e \ge 2$, $b \le -2$, and a finite set of positive integers, T. Clearly, if $I(T)$ is (e,b)-good, then T is (e,b)-good. Using a simple induction argument, it suffices to show that if $S_{e,b}(T)$ is (e,b)-good, then T is (e,b)-good.

Let $S_{e,b}(T)$ be (e,b)-good and $u \in U_{e,b}$. Then, by the definition of (e,b)-good, there exist n' and k' such that for each $s \in S_{e,b}(T)$, $S^{k'}_{e,b}(s+n') = u$. Let ℓ be the number of base b digits of the largest element of T and let $\ell' = \ell$ or $\ell+1$ such that $n' + \ell'$ is odd. Let

$$n = \sum_{i=\ell'}^{n'+\ell'-1} b^i = \underbrace{11\cdots11}_{n'}\underbrace{00\cdots00}_{\ell'} \in \mathbb{Z}^+.$$

Then, $S_{e,b}(n) = n'$ and for each $t \in T$, $S_{e,b}(t+n) = S_{e,b}(t) + n'$. Let $k = k'+1$. Then, for each $t \in T$,

$$S^k_{e,b}(t+n) = S^{k'}_{e,b}(S_{e,b}(t+n)) = S^{k'}_{e,b}(S_{e,b}(t)+n') = u.$$

So, T is (e,b)-good. □

Lemma 9. *Let $b \in \{-4,-6,-8,-10\}$, and let $0 < t_2 < t_1$ be integers. Then, there exists a function F of the type described in Lemma 8 with $e = 2$ such that $F(t_1) = F(t_2)$.*

Proof. Let $k \in \mathbb{Z}^+$ such that $S^k_{2,b}(t_1), S^k_{2,b}(t_2) \in U_{2,b}$, and let $F_1 = S^k_{2,b}$. If $F_1(t_1) = F_1(t_2)$, we are done, and so we assume otherwise. From Table 1, we have

$$\begin{aligned} U_{2,-4} &= \{1,6,14\},\\ U_{2,-6} &= \{1,2,4,11,16,29,30,33,51\},\\ U_{2,-8} &= \{1,11,30,38,46,53,59,62\},\\ U_{2,-10} &= \{1,19,29,35,46,73,75,76,146,163\}. \end{aligned}$$

Case $b = -4$. Let $F_2 = S^2_{2,-4}I$ and $F_3 = S^5_{2,-4}I^3$. Note that

$$\begin{aligned} F_2(6) &= S^2_{2,-4}(7) = S^2_{2,-4}(133_{(-4)}) = S_{2,-4}(19) = S_{2,-4}(103_{(-4)}) = 10 \text{ and}\\ F_2(1) &= S^2_{2,-4}(2) = S_{2,-4}(4) = S_{2,-4}(130_{(-4)}) = 10. \end{aligned}$$

Thus, if $\{F_1(t_1), F_1(t_2)\} = \{1,6\}$, then let $F = F_2F_1$, so that $F(t_1) = F(t_2)$. And if $\{F_1(t_1), F_1(t_2)\} = \{1,14\}$, then, noting that $S_{2,-4}(14) = 6$, let $F = F_2F_1S_{2,-4}$.

Finally, observe that

$$F_3(6) = S^5_{2,-4}(9) = S^5_{2,-4}(121_{(-4)}) = S^4_{2,-4}(6) = S^4_{2,-4}(132_{(-4)}) = S^3_{2,-4}(14) = S^2_{2,-4}(6) = 6 \text{ and}$$
$$F_3(14) = S^5_{2,-4}(17) = S^5_{2,-4}(101_{(-4)}) = S^4_{2,-4}(2) = S^3_{2,-4}(4) = S^2_{2,-4}(10) = S_{2,-4}(9) = 6.$$

Hence, if $\{F_1(t_1), F_1(t_2)\} = \{6,14\}$, let $F = F_3F_1$.

Case $b = -6$. Let $F_2 = S^{\ell}_{2,-6}I^{6^4 - F_1(t_1)}$, where $\ell \in \mathbb{Z}^+$ such that $F_2F_1(t_2) \in U_{2,-6}$. Note that $F_2F_1(t_1) = S^{\ell}_{2,-6}(6^4) = 1$, regardless of the choice of ℓ. If $F_2F_1(t_2) = 1$, we are done. If not, since $(2,4,16,33,11,51,29,30)$ is a cycle, we can modify our choice of ℓ (making it larger, if necessary) to guarantee that $F_2F_1(t_2) = 2$.

Now, let $F_3 = S^6_{2,-6}I^7$. Noting that $F_3(1) = F_3(2)$, we set $F = F_3F_2F_1$.

Case $b = -8$. Let $F_2 = S_{2,-8}^{\ell} I^{64-F_1(t_1)}$, where $\ell \in \mathbb{Z}^+$ such that $F_2F_1(t_2) \in U_{2,-8}$. Since $F_2F_1(t_1) = S_{2,-8}^{\ell}(64) = 1$, if $F_2F_1(t_2) = 1$, we are done. Otherwise, using Table 1, we can choose a possibly larger value of ℓ so that $F_2F_1(t_2) \in \{30, 59, 46\}$. If $F_2F_1(t_2) \in \{30, 59\}$, let $F_3 = S_{2,-8}^{8} I^2$. Noting that $F_3(1) = F_3(30) = F_3(59)$, we set $F = F_3F_2F_1$. If instead $F_2F_1(t_2) = 46$, then let $F_4 = S_{2,-8}^{9} I^7$. Since $F_4(1) = F_4(46)$, setting $F = F_4F_2F_1$ completes this case.

Case $b = -10$. Let $F_2 = S_{2,-10}^{\ell} I^{10000-F_1(t_1)}$, where $\ell \in \mathbb{Z}^+$ such that $F_2F_1(t_2) \in U_{2,-10}$. Since $F_2F_1(t_1) = 1$, if $F_2F_1(t_2) = 1$, we are done. If not, we can choose ℓ so that $F_2F_1(t_2) \in \{19, 35\}$. If $F_2F_1(t_2) = 19$, let $F_3 = S_{2,-10}^{3} I^{22}$ and set $F = F_3F_2F_1$. If instead $F_2F_1(t_2) = 35$, let $F_4 = S_{2,-10}^{16} I$ and set $F = F_4F_2F_1$, completing the proof. □

We now apply the methods in [2] to odd negative bases, noting that the original proof does not carry over, since, for b negative, $b - 1 \neq |b| - 1$.

Lemma 10. *Fix $b \leq -5$ odd, $v' \in 2\mathbb{Z}^+$, and $r' \in \mathbb{Z}^+$ such that $b^{2r'} > v'$. There exists $0 \leq c < |b| - 1$ such that*

$$2c \equiv 4r' - S_{2,b}\left((|b|-1)\sum_{i=0}^{r'-1} b^{2i+1} + v' - 1\right) - 1 \pmod{b-1}. \tag{3.1}$$

Proof. Since b is odd and v' is even, the input to $S_{2,b}$ in (3.1) is odd. Thus, by Lemma 4, the output is also odd. Hence, we can choose

$$c \equiv 2r' - \frac{1}{2}\left(S_{2,b}\left((|b|-1)\sum_{i=0}^{r'-1} b^{2i+1} + v' - 1\right) + 1\right) \left(\text{mod } \frac{b-1}{2}\right),$$

with $0 \leq c < |\frac{b-1}{2}| < |b| - 1$, since $b \leq -5$. □

Lemma 11. *Fix $b \leq -5$ odd and let $t_1, t_2 \in \mathbb{Z}^+$ be congruent modulo 2 with $t_2 < t_1$. Then, there exists a function F of the type described in Lemma 8 with $e = 2$ such that $F(t_1) = F(t_2)$.*

Proof. First note that if t_1 and t_2 have the same non-zero digits, then $S_{2,b}(t_1) = S_{2,b}(t_2)$, and so $F = S_{2,b}$ suffices.

Next, if $t_1 \equiv t_2 \pmod{b-1}$, let $v \in \mathbb{Z}^+$ such that $t_2 - t_1 = (b-1)v$. Choose $r \in \mathbb{Z}^+$ so that $b^{2r} > b^2 v + t_1$, and let $m = b^{2r} + v - t_1 > 0$. Then,

$$I^m(t_1) = t_1 + b^{2r} + v - t_1 = b^{2r} + v$$

and

$$I^m(t_2) = t_2 + b^{2r} + v - t_1 = b^{2r} + v + (b-1)v = b^{2r} + bv.$$

Since $b^{2r} > b^2 v$, it follows that $I^m(t_1)$ and $I^m(t_2)$ have the same non-zero digits. Thus, it suffices to let $F = S_{2,b} I^m$.

Finally, if $t_1 \not\equiv t_2 \pmod{b-1}$, let $v' = t_1 - t_2 \in 2\mathbb{Z}^+$. Choose $r' \in \mathbb{Z}^+$ such that $b^{2r'} > b^2 t_1$. By Lemma 10, since $b^2 t_1 > v'$, there exists $0 \leq c < |b| - 1$ such that congruence (3.1) holds.

Let

$$m' = cb^{2r'} + \sum_{i=0}^{r'-1} (|b|-1)b^{2i} - t_2 \geq 0.$$

Then,

$$S_{2,b}(t_2 + m') = S_{2,b}\left(cb^{2r'} + \sum_{i=0}^{r'-1} (|b|-1)b^{2i}\right) = c^2 + r'(|b|-1)^2.$$

And

$$\begin{aligned} S_{2,b}\left(t_1+m'\right) &= S_{2,b}\left(cb^{2r'}+\sum_{i=0}^{r'-1}(|b|-1)b^{2i}+v'\right) \\ &= S_{2,b}\left((c+1)b^{2r'}+\sum_{i=0}^{r'-1}(|b|-1)b^{2i+1}+v'-1\right) \\ &= (c+1)^2+S_{2,b}\left(\sum_{i=0}^{r'-1}(|b|-1)b^{2i+1}+v'-1\right). \end{aligned}$$

It follows that

$$S_{2,b}(t_1+m')-S_{2,b}(t_2+m') = 2c+1+S_{2,b}\left(\sum_{i=0}^{r'-1}(|b|-1)b^{2i+1}+v'-1\right)-r'(b+1)^2.$$

Using congruence (3.1), this yields

$$S_{2,b}(t_1+m')-S_{2,b}(t_2+m') \equiv 4r'-r'(b+1)^2 \equiv 0 \pmod{b-1}.$$

Therefore, $S_{2,b}(I^{m'}(t_1)) \equiv S_{2,b}(I^{m'}(t_2)) \pmod{b-1}$. Applying the earlier argument to these two numbers, we obtain an appropriate value of $m \in \mathbb{Z}^+$ and let $F = S_{2,b}I^m S_{2,b}I^{m'}$. □

Theorem 12. *Fix $b \le -5$ odd or $b \in \{-4,-6,-8,-10\}$. Let $d = \gcd(2, b-1)$. A finite set T of positive integers is $(2,b)$-good if and only if all of the elements of T are congruent modulo d.*

Proof. Fix a finite set of positive integers, T. First, assume that T is $(2,b)$-good. If b is even, then $d = 1$, and the congruence result is trivial. If b is odd, fix $u \in U_{2,b}$. Then there exists n, $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S_{2,b}^k(t+n) = u$. It follows from Lemma 4 that, for each $t \in T$, $t+n \equiv u \pmod 2$. Hence, the elements of T are congruent modulo $d = 2$.

For the converse, assume that the elements of T are congruent modulo d. If T is empty, then vacuously it is $(2,b)$-good. If $T = \{t\}$, then given $u \in U_{2,b}$, by definition, there exist $x \in \mathbb{Z}^+$ such that $S_{2,b}(x) = u$. Fix some $r \in 2\mathbb{Z}^+$ such that $t \le b^r x$. Then, letting $n = b^r x - t$ and $k = 1$, since $S_{2,b}^k(t+n) = S_{2,b}(t+(b^r x - t)) = S_{2,b}(x) = u$, T is $(2,b)$-good.

Now assume that $|T| = N > 1$ and assume, by induction, that any set of fewer than N elements all of which are congruent modulo d is $(2,b)$-good. Let $t_1 > t_2 \in T$. By Lemmas 9 and 11, there exists a function F as in Lemma 8 with $e = 2$ such that $F(t_1) = F(t_2)$. This implies that $F(T)$ has fewer than N elements. Further, since the elements of T are congruent modulo d, the same holds for $I(T)$ and, by Lemma 4, for $S_{2,b}(T)$, implying that the same holds for $F(T)$. Thus, by the induction hypothesis, $F(T)$ is $(2,b)$-good and so, by Lemma 8, T is $(2,b)$-good. □

Corollary 13. *For $b \le -3$ odd or $b \in \{-4,-6,-8,-10\}$ and $d = \gcd(2, b-1)$, there exist arbitrarily long finite sequences of d-consecutive b-happy numbers.*

Proof. By Lemma 7, every odd integer is -3-happy. So the corollary holds for $b = -3$. For $b < -3$, given $N \in \mathbb{Z}^+$, let $T = \{1 + dt \mid 0 \le t \le N-1\}$. By Theorem 12, T is $(2,b)$-good. By Definition 5, there exist n, $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S_{2,b}^k(t+n) = 1$. Thus, $\{1+n+dt \mid 0 \le t \le N-1\}$ is a sequence of N d-consecutive b-happy numbers, as desired. □

Finally, we prove that for $b \le -3$ we can choose the arbitrarily long finite sequence of d-consecutive b-happy numbers to consist entirely of negative numbers.

Corollary 14. *For $b \leq -3$ odd or $b \in \{-4, -6, -8, -10\}$ and $d = \gcd(2, b-1)$, there exist arbitrarily long finite sequences of d-consecutive b-happy numbers all less than zero.*

Proof. By Lemma 7, every odd integer is -3-happy. So the corollary holds for $b = -3$. For $b < -3$, given $N \in \mathbb{Z}^+$, set $r \geq 2$ such that $b^{2r} > 4N$. Let $T = \{b^{2r} + 1 + ds \mid 0 \leq s \leq 2N - 2\}$. By Theorem 12, T is $(2, b)$-good. By Definition 5, there exist n, $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S^k_{2,b}(t + n) = 1$. Thus,

$$\mathcal{S} = \{b^{2r} + 1 + n + ds \mid 0 \leq s \leq 2N - 2\}$$

is a sequence of $2N - 1$ d-consecutive b-happy numbers.

Now, $(b^{2r} + 1 + n + (2N - 2)d) - (b^{2r} + 1 + n) = (2N - 2)d < 4N < b^{2r}$. Thus, there is at most one number between $b^{2r} + 1 + n$ and $b^{2r} + 1 + n + (2N - 2)d$, inclusive, that is congruent to $(|b| - 1) \sum_{i=0}^{r-1} b^{2i} \pmod{b^{2r}}$. If $b^{2r} + 1 + n + i \equiv (|b| - 1) \sum_{0}^{r-1} b^{2i} \pmod{b^{2r}}$ for some $0 \leq i < dN$, then let $C = b^{2r} + 1 + n + dN$. Otherwise, let $C = b^{2r} + 1 + n$. Then, in either case, no integer in the closed interval $[C, C + d(N - 1)]$ is congruent to $(|b| - 1) \sum_{i=0}^{r-1} b^{2i} \pmod{b^{2r}}$. Since $C > b^{2r}$, it follows that all of integers in the subsequence

$$\mathcal{S}' = \{C + ds \mid 0 \leq s \leq N - 1\}$$

have the same leading digit.

Let a be the leading digit of C and fix m such that $C = ab^{2m} + R$, with $0 < a \leq |b| - 1$ and $-b^{2m} < R < b^{2m}$. Since all of the numbers of $\mathcal{S}'$ have the same leading digit, for each $0 \leq s \leq N - 1$, $-b^{2m} < R + ds < b^{2m}$.

Define $C^- = ab^{2m+1} + R$, which is negative, since C is positive. Then, $S_{2,b}(C^- + ds) = S_{2,b}(ab^{2m+1} + R + ds) = a^2 + S_{2,b}(R + ds) = S_{2,b}(ab^{2m} + R + ds) = S_{2,b}(C + ds)$. Hence, the sequence

$$\{C^- + ds \mid 0 \leq s \leq N - 1\}$$

is a sequence of N d-consecutive b-happy numbers, each of which is negative. □

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