A CONNECTION BETWEEN HYPER-FIBONACCI NUMBERS AND FISSIONS OF POLYNOMIAL SEQUENCES

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ABSTRACT. We prove a new formula for hyper-Fibonacci numbers, $F_n^{[k]}$, using fissions of certain polynomials. The result is a concise description of the entries of the matrix of hyper-Fibonacci numbers.

1. Introduction

Let $\{F_n\}$ denote the sequence of Fibonacci numbers, defined as usual by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The hyper-Fibonacci numbers $F_n^{[k]}$ were introduced by Dil and Mező [2] as follows. For $k \geq 0$ and $n \geq 0$, let $F_n^{[0]} = F_n$ and $F_0^{[k]} = 0$, and

$$F_n^{[k]} = F_{n-1}^{[k]} + F_n^{[k-1]}, \qquad k, n > 0.$$

It is natural to arrange these numbers in an infinite matrix having $F_n^{[k]}$ in row k and column n, so that $F_n^{[k]}$ is the sum of the first $n+1$ elements (from the 0th to the nth) of row $k-1$, i.e., $F_n^{[k]} = \sum_{i=0}^{n} F_i^{[k-1]} (n \geq 0,\ k \geq 1)$. A consequence of Proposition 2 in [2] is

$$F_n^{[k]} = \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} F_j. \tag{1.1}$$

A different expression for $F_n^{[k]}$ is known [4]:

$$F_n^{[k]} = F_{n+2k} - p_k(n), \tag{1.2}$$

where $p_k(n)$ is a polynomial with rational coefficients, given [5] explicitly by

$$p_k(x) = \sum_{t=1}^{k-1} \left(\sum_{j=1}^{t} \frac{(-1)^{t-j}}{(k-j)!} \begin{bmatrix} k-j \\ k-t \end{bmatrix} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \right) x^{k-t} + F_{2k}, \tag{1.3}$$

where $\left[{k-j \atop k-t}\right]$ denotes an unsigned Stirling number of first kind. The first few polynomials are

$$\begin{aligned}
p_0(x) &= 0,\\
p_1(x) &= 1,\\
p_2(x) &= x+3,\\
p_3(x) &= \frac{x^2+7x+16}{2},\\
p_4(x) &= \frac{x^3+12x^2+59x+126}{6},\\
p_5(x) &= \frac{x^4+18x^3+143x^2+630x+1320}{24},\\
p_6(x) &= \frac{x^5+25x^4+285x^3+1955x^2+8294x+17280}{120}.
\end{aligned}$$

Using (1.2), we have

$$F_n^{[1]} = F_{n+2} - 1, \qquad F_n^{[2]} = F_{n+4} - (n+3), \qquad F_n^{[3]} = F_{n+6} - \frac{n^2+7n+16}{2},$$

and so on.

For nonnegative k and n, Belbachir and Belkhir proved that

$$F_n^{[k]} = F_{n+2k} - \sum_{t=0}^{k-1} \binom{n-1+2k-t}{t}, \tag{1.4}$$

which resembles (1.2); see Theorem 10 in [1]. However, in contrast to (1.1) and (1.4), the representation (1.3) gives an explicit representation of the coefficients of the polynomial $p_k(x)$. Possibly, this sort of representation will also prove useful in studying hyper-Lucas sequences and others.

2. Main Result: Connection to t-sion of Polynomial Sequences

A surprising connection to *fission* of two polynomial sequences enables a reformulation of (1.2) and (1.3). Kimberling [3] defined the notion of fission (and *fusion*), and later, Kimberling and Szalay [4] generalized it by introducing the *t-sion* of two polynomial sequences. First we state the new result, and then give details.

Theorem 2.1. *Let $k \geq 1$. Then, $p_k(x)$ is the $(k-1)$th element of the sequence of polynomials*

$$\mathrm{FIS}\left(x \cdot \mathrm{FIS}\left((x+1)^r, \sum_{i=0}^{r} F_{i+1}x^{r-i}\right), \binom{x}{r}\right). \tag{2.1}$$

2.1. *t*-sion of polynomial sequences.

This section sets the stage for proving Theorem 2.1 in the next section. Let $u \geq 1$ be an integer, and let

$$\omega(x) = \omega_u x^u + \omega_{u-1}x^{u-1} + \cdots + \omega_1 x + \omega_0 \in \mathbb{C}[x]$$

be a polynomial; further let $B = (b_r(x))_{r=0}^{\infty}$ be an arbitrary sequence of polynomials all in $\mathbb{C}[x]$, and let t be an arbitrary integer. For $u+t \geq 0$, the (B,t)*-step of* $\omega(x)$ is the polynomial in $\mathbb{C}[x]$ defined by

$$h_t(\omega(x)) = \omega_u b_{u+t}(x) + \omega_{u-1} b_{u-1+t}(x) + \cdots + \omega_\tau b_{\tau+t}(x),$$

where

$$\tau = \begin{cases} 0, & \text{if } t \geq 0; \\ |t|, & \text{if } t < 0. \end{cases} \tag{2.2}$$

If $u+t<0$, then $h_t(\omega(x))$ is defined to be the zero polynomial. Now taking another arbitrary sequence $A=(a_n(x))_{n=0}^{\infty}$ in $\mathbb{C}[x]$, we define the *t-sion of A by B*, denoted by $A \circ_t B$, as the sequence $C=(c_n(x))_{n=0}^{\infty}$ of polynomials given by

$$c_r(x) = h_t(a_r(x)).$$

For $t=1$ and $t=-1$, the sequences $P \circ_t Q$ are the fusion and fission of P and Q, respectively, as defined in [3]. In the present paper, we apply only fission. Note that in the statement of Theorem 2.1, we have written the fission as $\mathrm{FIS}(A,B)$ instead of $A \circ_{(-1)} B$.

As an example, choose $A=((x+1)^r)_{r=0}^{\infty}$ and $B=(\sum_{i=0}^{r} F_{i+1}x^{r-i})_{r=0}^{\infty}$, so that their fission is $C=\mathrm{FIS}\left((x+1)^r, \sum_{i=0}^{r} F_{i+1}x^{r-i}\right)$. Then, the fission of $x \cdot C$ and $\left(\binom{x}{r}\right)_{r=0}^{\infty}$ yields (2.1). This example will be developed in the next section.

Continuing now with sequences A and B of arbitrary polynomials in $\mathbb{C}[x]$, for $k \geq 0$, we shall determine a representation for the finite sequence $(c_r(x))_{r=0}^{k}$ of initial terms of C. Put

$$D_k = \max_{r=0\ldots k}\{\deg(a_r(x))\} - \tau,$$

with τ as in (2.2). Clearly, it suffices to assume that $D_k \geq 0$. The nth row $(1 \leq n \leq k+1)$ of the matrix $\mathcal{A}_{k,t} \in \mathbb{C}^{(k+1)\times(D_k+1)}$ consists of the coefficients of

$$a_{n-1}(x) = a_{n-1,u}x^u + a_{n-1,u-1}x^{u-1} + \cdots + a_{n-1,1}x + a_{n-1,0},$$

and their positions from right to left, starting with the coefficient of the term of least degree, are given by

$$[0 \quad \cdots \quad 0 \quad a_{n-1,u} \quad a_{n-1,u-1} \quad \cdots \quad a_{n-1,\tau+1} \quad a_{n-1,\tau}].$$

(Each entry is zero if $u+t<0$.) Let

$$D'_k = \max_{j=0\ldots D_k}\{\deg(b_{\tau+t+j}(x))\}.$$

We define the matrix $\mathcal{B}_{k,t} \in \mathbb{C}^{(D_k+1)\times(D'_k+1)}$ from the coefficients of the polynomials in B, as follows: for $k \geq 0$, column n, for $n \geq 0$, is the (D_k+1)-dimensional vector

$$[b_n, b_{n-1}, \ldots, b_0, 0, \ldots, 0].$$

Clearly,

$$\mathcal{A}_{k,t}\mathcal{B}_{k,t} = \mathcal{C}_{k,t} \in \mathbb{C}^{(k+1)\times(D'_k+1)}.$$

In case $\deg(a_n(x)) = \deg(b_n(x)) = n$, we find for $k \geq \tau$, excluding trivial cases, that

$$D_k = \begin{cases} k, & \text{if } t \geq 0; \\ k-\tau = k+t, & \text{if } t < 0, \end{cases} \qquad \text{and} \qquad D'_k = k+t.$$

In particular, $t=-1$ gives $D_k = D'_k = k-1$. Moreover, the constant term (and only the constant term) of the polynomials $a_r(x)$ does not appear in $\mathcal{A}_{k,t}$.

Example 2.2. *Assume that $t=-1$, so that $\tau=1$. Let $k=4$, $a_r(x)=(x+1)^r$ for $r=0,\ldots,4$, and $b_r(x)=\sum_{i=0}^{r} F_{i+1}x^{r-i}$. Then, $D_k=D'_k=3$. The matrix product $\mathcal{A}_{k,t}\mathcal{B}_{k,t}=\mathcal{C}_{k,t}$ is then*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 1 & 4 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 1 & 5 & 12 & 21 \end{bmatrix},$$

so that $c_0(x) = 0$, $c_1(x) = 1$, $c_2(x) = x+3$, $c_3(x) = x^2+4x+8$, *and* $c_4(x) = x^3+5x^2+12x+21$.

2.2. **Proof of Theorem 2.1.** As in Section 2.1, let

$$C = \mathrm{FIS}\left((x+1)^r, \sum_{i=0}^{r} F_{i+1}x^{r-i}\right).$$

The initial terms $c_r(x))_{r=0}^{k}$ can be represented as follows. Consider the matrix product (2.3), in which the first matrix has dimensions $(k+1)\times k$ and the second has $k\times k$:

$$\begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \binom{1}{0} \\ 0 & \dots & \binom{2}{0} & \binom{2}{1} \\ \vdots & \iddots & \vdots & \vdots \\ \binom{k}{0} & \dots & \binom{k}{k-2} & \binom{k}{k-1} \end{bmatrix} \cdot \begin{bmatrix} F_1 & F_2 & \dots & F_{k-1} & F_k \\ 0 & F_1 & \dots & F_{k-2} & F_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & F_1 & F_2 \\ 0 & 0 & \dots & 0 & F_1 \end{bmatrix}. \tag{2.3}$$

The two matrices consist of the coefficients (apart from the constant term) of the polynomials $(x+1)^r$ $(r = 0,\dots,k)$, and the coefficients $\sum_{i=0}^{r} F_{i+1}x^{r-i}$ $(r = 0,\dots,k)$, respectively, and the product determines the coefficients of the first few polynomials of the fission C. If we multiply these polynomials by x, the constant terms of the resulting polynomials are zero. Accordingly, in the matrix representation of the fission, the constant terms of the first polynomial sequences must be suppressed, so that the required coefficients in C are obtained by multiplying the product (2.3) by the $k\times k$ matrix

$$\begin{bmatrix} {k-1 \brack k-1}^* & {k-1 \brack k-2}^* & \dots & {k-1 \brack 1}^* & {k-1 \brack 0}^* \\ 0 & {k-2 \brack k-2}^* & \dots & {k-2 \brack 1}^* & {k-2 \brack 0}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & {1 \brack 1}^* & {1 \brack 0}^* \\ 0 & 0 & \dots & 0 & {0 \brack 0}^* \end{bmatrix},$$

where ${n \brack k}^* = \frac{(-1)^{n-k}}{n!}{n \brack k}$. The rows of the $(k+1)\times k$ matrix product give the coefficients of the first few polynomials of the sequence

$$\mathrm{FIS}\left(x\cdot \mathrm{FIS}\left((x+1)^r, \sum_{i=0}^{r} F_{i+1}x^{r-i}\right), \binom{x}{r}\right).$$

More precisely, row j contains the coefficients of $c_{j-1}(x) = p_{j-1}(x)$ $(1 \le j \le k+1)$. Carrying the matrix products out leads immediately to (1.3).

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