DIFFERENCES OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDERS 2, 3, AND 4

THOMAS KOSHY

Abstract. We present the extended gibonacci polynomial family; and then investigate the differences of some special gibonacci products of orders 2, 3, and 4, and their polynomial and numeric implications to the Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev subfamilies.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$ [5, 6].

Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials, denoted by $f_n(x), l_n(x), p_n(x), q_n(x), J_n(x)$, and $j_n(x)$, belong to the gibonacci family $\{z_n(x)\}$; their numeric counterparts are denoted by F_n, L_n, P_n, Q_n, J_n, and j_n, respectively. Vieta and Vieta-Lucas polynomials V_n and v_n, and Chebyshev polynomials $T_n(x)$ and $U_n(x)$ also belong to the same family [5, 6].

1.1. Relationships Among the Subfamilies.

By virtue of the relationships in Table 1, every ginonacci result has a Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev counterpart, where $i = \sqrt{-1}$ [5, 6].

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(x) = U_{n-1}(x/2)$	$v_n(x) = 2T_n(x/2)$.

Table 1: Links Among the Gibonacci Subfamilies

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$. Again, for brevity, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $c_n = J_n(x)$ or $j_n(x)$; $d_n = V_n$ or v_n; and $e_n = T_n$ or U_n; and correspondingly, we let $G_n = F_n$ or L_n; $B_n = P_n$ or Q_n; and $C_n = J_n$ or j_n. We also *omit* a lot of basic algebra.

Again for brevity and convenience, we let

$$\gamma = \begin{cases} 1, & \text{if } G_n = F_n, \\ 2, & \text{if } G_n = L_n; \end{cases} \quad \kappa = \begin{cases} 1, & \text{if } B_n = P_n, \\ 3, & \text{if } B_n = Q_n; \end{cases} \quad \nu = \begin{cases} 1, & \text{if } C_n = J_n, \\ 5, & \text{if } C_n = j_n; \end{cases} \text{ and } \Delta = \sqrt{x^2+4}.$$

We can develop an explicit Binet-like formula for g_n. To this end, we need the following result; its proof is straightforward, so we omit it.

Lemma 1.1. *Let g_n denote the nth gibonacci polynomial. Then $g_n = af_{n-2} + bf_{n-1}$, where $n \geq 0$.* □

The next theorem gives the promised explicit formula. Its proof follows by the lemma, so we omit that also.

Theorem 1.2 (Binet-like formula)**.** *Let $c = c(x) = a+(ax-b)\beta$ and $d = d(x) = a+(ax-b)\alpha$, where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1 = 0$. Then,*

$$g_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}. \qquad \square$$

2. Differences of Gibonacci Products of Order 2

A *gibonacci product of order m* is a product of gibonacci polynomials g_{n+i} of the form $\prod_{i\geq 0} g_{n+i}^{s_j}$, where $\sum_{s_j\geq 1} s_j = m$. We now briefly study differences of gibonacci products of order 2.

Using Theorem 1.2, we can establish the following differences of gibonacci products of order 2:

$$\begin{aligned} g_{n+h}g_{n+k} - g_n g_{n+h+k} &= \mu(-1)^n f_h f_k; \\ g_{m+k}g_{n-k} - g_m g_n &= (-1)^{n-k+1}\mu f_k f_{m-n+k}; \\ g_{n+k}g_{n-k} - g_n^2 &= (-1)^{n-k+1}\mu f_k^2, \end{aligned} \tag{2.1}$$

where $\mu = \mu(x) = a^2 + abx - b^2$; μ equals 1 when $g_n = f_n$; and $-(x^2+4)$ when $g_n = l_n$.

In particular, we have

$$F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k; \tag{2.2}$$

$$F_{n+k}F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2; \tag{2.3}$$

$$F_m F_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n}. \tag{2.4}$$

A. Tagiuri discovered the beautiful formula (2.2) in 1901 [1]. About 60 years later, D. Everman *et al.* re-discovered it [2, 8]. E. C. Catalan developed identity (2.3) in 1879 [4]. G. D. Cassini found identity (2.3) in 1680 with $k = 1$; R. Simson discovered it independently in 1753 [4]. P. M. d'Ocagne found identity (2.4) [4].

It follows from the Catalan-like identity (2.1) that $(g_{n+k}g_{n-k} - g_n^2)^2 = \mu^2 f_k^4$; consequently,

$$4g_{n+k}g_n^2 g_{n-k} + \mu^2 f_k^4 = (g_{n+k}g_{n-k} + g_n^2)^2. \tag{2.5}$$

Thus, $4g_{n+k}g_n^2 g_{n-k} + \mu^2 f_k^4$ is a square.

It follows from identity (2.5) that

$$\begin{aligned} 4G_{n+k}G_n^2 G_{n-k} + \nu^2\, G_k^4 &= (G_{n+k}G_{n-k} + G_n^2)^2; \\ 4B_{n+k}B_n^2 B_{n-k} + \gamma^2\, B_k^4 &= (B_{n+k}B_{n-k} + B_n^2)^2. \end{aligned}$$

3. Differences of Gibonacci Products of Order 3

With these tools, we now investigate differences of gibonacci products of order 3. The next theorem gives one such formula.

Theorem 3.1. *Let $n \geq 0$. Then,*

$$g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 = \mu(-1)^n(x^3 g_{n+2} - g_{n+1}). \tag{3.1}$$

Proof. By the gibonacci recurrence, we have

$$\begin{aligned} g_{n+6} &= (x^4+3x^2+1)g_{n+2} + (x^3+2x)g_{n+1}; \\ g_{n+1}g_{n+2}g_{n+6} &= (x^4+3x^2+1)g_{n+2}^2 g_{n+1} + (x^3+2x)g_{n+2}g_{n+1}^2; \\ g_{n+3}^3 &= x^3 g_{n+2}^3 + 3x^2 g_{n+2}^2 g_{n+1} + 3x g_{n+2} g_{n+1}^2 + g_{n+1}^3. \end{aligned}$$

Then, by identity (2.1) and some basic algebra, we have

$$\begin{aligned} \text{LHS} &= g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 \\ &= (x^4+1)g_{n+2}^2 g_{n+1} + (x^3-x)g_{n+2}g_{n+1}^2 - x^3 g_{n+2}^3 - g_{n+1}^3 \\ &= x^3 g_{n+2}^2(xg_{n+1} - g_{n+2}) + g_{n+2}g_{n+1}(g_{n+2} - xg_{n+1}) + x^3 g_{n+2} g_{n+1}^2 - g_{n+1}^3 \\ &= -x^3 g_{n+2}^2 g_n + g_{n+2}g_{n+1}g_n + x^3 g_{n+1}^2(xg_{n+1} + g_n) - g_{n+1}^3 \\ &= -x^3 g_{n+2}\left[g_{n+1}^2 + \mu(-1)^{n+1}\right] + g_{n+1}\left[g_{n+1}^2 + \mu(-1)^{n+1}\right] + x^3 g_{n+2}g_{n+1}^2 - g_{n+1}^3 \\ &= \mu(-1)^n(x^3 g_{n+2} - g_{n+1}), \end{aligned}$$

as desired. □

It follows by Theorem 3.1 that

$$g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 = \begin{cases} (-1)^n(x^3 g_{n+2} - g_{n+1}), & \text{if } g_n = f_n, \\ (-1)^{n+1}\Delta^2(x^3 g_{n+2} - g_{n+1}), & \text{if } g_n = l_n; \end{cases} \tag{3.2}$$

$$b_{n+1}b_{n+2}b_{n+6} - b_{n+3}^3 = \begin{cases} (-1)^n(8x^3 b_{n+2} - b_{n+1}), & \text{if } b_n = p_n, \\ (-1)^{n+1}\, 4(x^2+1)(8x^3 b_{n+2} - b_{n+1}), & \text{if } b_n = q_n. \end{cases}$$

Consequently,

$$G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = \begin{cases} (-1)^n G_n, & \text{if } G_n = F_n, \\ (-1)^{n+1} 5G_n, & \text{if } G_n = L_n; \end{cases} \tag{3.3}$$

$$B_{n+1}B_{n+2}B_{n+6} - B_{n+3}^3 = \begin{cases} (-1)^n(8B_{n+2} - B_{n+1}), & \text{if } B_n = P_n, \\ (-1)^{n+1}2(8B_{n+2} - B_{n+1}), & \text{if } B_n = Q_n. \end{cases}$$

Melham discovered the formula (3.3) with $G_n = F_n$ [7].

Theorem 3.1 has a byproduct that follows from identity (3.3) that $G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = (-1)^n\mu(1)G_n$, so $(G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3)^2 = \nu^2 G_n^2$. This implies

$$4G_{n+1}G_{n+2}G_{n+3}^3 G_{n+6} + \nu^2 G_n^2 = (G_{n+1}G_{n+2}G_{n+6} + G_{n+3}^3)^2.$$

Similarly, we have

$$\begin{aligned} 4B_{n+1}B_{n+2}B_{n+3}^3 B_{n+6} + 4(8B_{n+2} - B_{n+1})^2 &= (B_{n+1}B_{n+2}B_{n+6} + B_{n+3}^3)^2; \\ 4C_{n+1}C_{n+2}C_{n+3}^3 C_{n+6} + \kappa^4 4^{n+1}(C_{n+2} - 4C_{n+1})^2 &= (C_{n+1}C_{n+2}C_{n+6} + C_{n+3}^3)^2. \end{aligned}$$

Next, we pursue the implications of Theorem 3.1 to the Jacobsthal family.

3.1. **Jacobsthal Implications.** By virtue of the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$, Theorem 3.1 has Jacobsthal consequences. To see them, first replace x with $u = 1/\sqrt{x}$ in identity (3.1). We then get

$$g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 = (-1)^n\mu\left(\frac{1}{x\sqrt{x}} g_{n+2} - g_{n+1}\right), \tag{3.4}$$

where $g_n = g_n(u)$ and $\mu = \mu(u)$.

Suppose $g_n = f_n$. Then, (3.4) yields

$$f_{n+1}f_{n+2}f_{n+6} - f_{n+3}^3 = (-1)^n \mu \left(f_{n+2} - f_{n+1} \right),$$

where $f_n = f_n(u)$. Multiplying this equation with $x^{(3n+6)/2}$ results in the Jacobsthal identity

$$J_{n+1}(x)J_{n+2}(x)J_{n+6}(x) - J_{n+3}^3(x) = (-1)^n x^{n+1} \left[J_{n+2}(x) - x^2 J_{n+1}(x) \right].$$

Similarly, when $g_n = l_n$, we get

$$j_{n+1}(x)j_{n+2}(x)j_{n+6}(x) - j_{n+3}^3(x) = (-1)^{n+1}(4x+1)x^{n+1} \left[j_{n+2}(x) - x^2 j_{n+1}(x) \right].$$

Combining the two cases, we have

$$c_{n+1}c_{n+2}c_{n+6} - c_{n+3}^3 = \begin{cases} -(-x)^{n+1} \left(c_{n+2} - x^2 c_{n+1} \right), & \text{if } c_n = J_n(x), \\ (4x+1)(-x)^{n+1} \left(c_{n+2} - x^2 c_{n+1} \right), & \text{if } c_n = j_n(x). \end{cases}$$

Consequently,

$$C_{n+1}C_{n+2}C_{n+6} - C_{n+3}^3 = \begin{cases} -(-2)^{n+1} \left(C_{n+2} - 4C_{n+1} \right), & \text{if } C_n = J_n, \\ 9(-2)^{n+1} \left(C_{n+2} - 4C_{n+1} \right), & \text{if } C_n = j_n. \end{cases}$$

The next theorem gives a companion formula for a difference of gibonacci products of order 3.

Theorem 3.2. *Let $n \geq 0$. Then,*

$$g_n g_{n+4} g_{n+5} - g_{n+3}^3 = \mu(-1)^{n+1}(x^3 g_{n+4} + g_{n+5}).$$

Proof. By the gibonacci recurrence, we have $g_n = (x^2+1)g_{n+4} - (x^3+2x)g_{n+3}$. Then,

$$g_n g_{n+4} g_{n+5} = (x^2+1)g_{n+4}^2 g_{n+5} - (x^3+2x)g_{n+3}g_{n+4}g_{n+5}.$$

We also have

$$\begin{aligned} g_{n+3}^3 &= (g_{n+5} - x g_{n+4})^3 \\ &= g_{n+5}^3 - 3x g_{n+4} g_{n+5}^2 + 3x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3 \\ &= (g_{n+5} - x g_{n+4})(g_{n+5} - 2x g_{n+4}) g_{n+5} + x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3 \\ &= g_{n+3}(g_{n+5} - 2x g_{n+4}) g_{n+5} + x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3. \end{aligned}$$

Consequently,

$$\begin{aligned} g_n g_{n+4} g_{n+5} - g_{n+3}^3 &= g_{n+4}^2 g_{n+5} - x^3 g_{n+3} g_{n+4} g_{n+5} - g_{n+3} g_{n+5}^2 + x^3 g_{n+4}^3 \\ &= (g_{n+4}^2 - g_{n+3} g_{n+5})(x^3 g_{n+4} + g_{n+5}) \\ &= (-1)^{n+1} \mu (x^3 g_{n+4} + g_{n+5}), \end{aligned}$$

as claimed. □

It follows by Theorem 3.2 that

$$g_n g_{n+4} g_{n+5} - g_{n+3}^3 = \begin{cases} (-1)^{n+1}(x^3 g_{n+4} + g_{n+5}), & \text{if } g_n = f_n, \\ (-1)^{n+1} \mu (x^3 g_{n+4} + g_{n+5}), & \text{if } g_n = l_n; \end{cases}$$

$$b_n b_{n+4} b_{n+5} - b_{n+3}^3 = \begin{cases} (-1)^{n+1}(8x^3 b_{n+4} + b_{n+5}), & \text{if } b_n = p_n, \\ (-1)^n 4(x^2+1)(8x^3 b_{n+4} + b_{n+5}), & \text{if } b_n = q_n; \end{cases}$$

Consequently, we have

$$G_nG_{n+4}G_{n+5} - G_{n+3}^3 = \begin{cases} (-1)^{n+1}G_{n+6}, & \text{if } G_n = F_n, \\ (-1)^n 5G_{n+6}, & \text{if } G_n = L_n; \end{cases} \tag{3.5}$$

$$B_nB_{n+4}B_{n+5} - B_{n+3}^3 = \begin{cases} (-1)^{n+1}(8B_{n+4} + B_{n+5}), & \text{if } B_n = P_n, \\ (-1)^n 2(8B_{n+4} + B_{n+5}), & \text{if } B_n = Q_n. \end{cases}$$

S. Fairgrieve and H. W. Gould discovered the delightful identity (3.5) when $G_n = F_n$ [3]. Next, we study the consequences of Theorem 3.2 to the Jacobsthal subfamily.

3.2. **Jacobsthal Consequences.** Replacing x with $u = 1/\sqrt{x}$ in (3.2), we get

$$g_ng_{n+4}g_{n+5} - g_{n+3}^3 = \mu(-1)^{n+1}\left(\frac{1}{x\sqrt{x}}g_{n+4} + g_{n+5}\right).$$

Suppose $g_n = f_n$. Multiplying the resulting equation with $x^{(3n+6)/2}$ gives

$$J_n(x)J_{n+4}(x)J_{n+5}(x) - J_{n+3}^3(x) = -(-x)^n[J_{n+4}(x) + xJ_{n+5}(x)].$$

Similarly, when $g_n = l_n$, we get

$$j_n(x)j_{n+4}(x)j_{n+5}(x) - j_{n+3}^3(x) = (-x)^n(4x+1)[j_{n+4}(x) + xj_{n+5}(x)].$$

Thus, we have

$$c_nc_{n+4}c_{n+5} - c_{n+3}^3 = \begin{cases} -(-x)^n(c_{n+4} + xc_{n+5}), & \text{if } c_n = J_n(x), \\ (4x+1)(-x)^n(c_{n+4} + xc_{n+5}), & \text{if } c_n = j_n(x); \end{cases}$$

$$C_nC_{n+4}C_{n+5} - C_{n+3}^3 = \begin{cases} -(-2)^n(C_{n+4} + 2C_{n+5}), & \text{if } C_n = J_n, \\ 9(-2)^n(C_{n+4} + 2C_{n+5}), & \text{if } C_n = j_n. \end{cases}$$

3.3. **Additional Consequences.** Theorem 3.2 has additional consequences. It follows from identity (3.2) that
$G_nG_{n+4}G_{n+5} - G_{n+3}^3 = (-1)^{n+1}\mu(1)G_{n+6}$; so $(G_nG_{n+4}G_{n+5} - G_{n+3}^3)^2 = \nu^2G_{n+6}^2$. Consequently,

$$4G_nG_{n+3}^3G_{n+4}G_{n+5} + \nu^2G_{n+6}^2 = (G_nG_{n+4}G_{n+5} + G_{n+3}^3)^2.$$

Likewise,

$$4B_nB_{n+3}^3B_{n+4}B_{n+5} + \gamma^2(8B_{n+4} + B_{n+5})^2 = (B_nB_{n+4}B_{n+5} + B_{n+3}^3)^2;$$
$$4C_nC_{n+3}^3C_{n+4}C_{n+5} + 4^n\nu^4(C_{n+4} + 2C_{n+5})^2 = (C_nC_{n+4}C_{n+5} + C_{n+3}^3)^2.$$

The next theorem presents another difference of gibonacci products of order 3.

Theorem 3.3. *Let $n \geq 0$. Then,*

$$g_ng_{n+3}^2 - g_{n+2}^3 = \mu(-1)^{n+1}(x^2g_{n+2} - g_n). \tag{3.6}$$

Proof. By the gibonacci recurrence, we have

$$\begin{aligned} g_ng_{n+3}^2 &= g_n(xg_{n+2} + g_{n+1})^2 \\ &= x^2g_ng_{n+2}^2 + 2xg_ng_{n+1}g_{n+2} + g_ng_{n+1}^2. \end{aligned}$$

But,

$$\begin{aligned} 2xg_ng_{n+1}g_{n+2} &= (g_{n+2} - xg_{n+1})(g_{n+2} - g_n)g_{n+2} + g_n(g_{n+2} - g_n)g_{n+2} \\ &= g_{n+2}^3 - xg_{n+1}g_{n+2}(g_{n+2} - g_n) - g_n^2 g_{n+2} \\ &= g_{n+2}^3 - x^2 g_{n+1}^2 g_{n+2} - g_n^2 g_{n+2}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_ng_{n+3}^2 - g_{n+2}^3 &= x^2 g_n g_{n+2}^2 - x^2 g_{n+1}^2 g_{n+2} - g_n^2 g_{n+2} + g_n g_{n+1}^2 \\ &= (g_n g_{n+2} - g_{n+1}^2)(x^2 g_{n+2} - g_n) \\ &= (-1)^{n+1}\mu(x^2 g_{n+2} - g_n), \end{aligned}$$

as desired. □

As can be predicted, this theorem also has Pell and Jacobsthal ramifications:

$$g_ng_{n+3}^2 - g_{n+2}^3 = \begin{cases} (-1)^{n+1}(x^2 g_{n+2} - g_n), & \text{if } g_n = f_n, \\ (-1)^n\, \Delta^2(x^2 g_{n+2} - g_n), & \text{if } g_n = l_n; \end{cases} \tag{3.7}$$

$$b_nb_{n+3}^2 - b_{n+2}^3 = \begin{cases} (-1)^{n+1}(4x^2 b_{n+2} - b_n), & \text{if } b_n = p_n, \\ (-1)^n\, 4(x^2+1)(4x^2 b_{n+2} - b_n), & \text{if } b_n = q_n; \end{cases}$$

$$c_nc_{n+3}^2 - c_{n+2}^3 = \begin{cases} -(-x)^n(c_{n+2} - x^2 c_n), & \text{if } c_n = J_n(x), \\ (-x)^n(4x+1)(c_{n+2} - x^2 c_n), & \text{if } c_n = j_n(x); \end{cases}$$

the Jacobsthal identities can be established as before.

Their numeric counterparts are:

$$G_nG_{n+3}^2 - G_{n+2}^3 = \begin{cases} (-1)^{n+1}G_{n+1}, & \text{if } G_n = F_n, \\ (-1)^n\, 5G_{n+1}, & \text{if } G_n = L_n; \end{cases} \tag{3.8}$$

$$B_nB_{n+3}^2 - B_{n+2}^3 = \begin{cases} (-1)^{n+1}(4B_{n+2} - B_n), & \text{if } B_n = P_n, \\ (-1)^n\, 2(4B_{n+2} - B_n), & \text{if } B_n = Q_n; \end{cases}$$

$$C_nC_{n+3}^2 - C_{n+2}^3 = \begin{cases} -2^n, & \text{if } C_n = J_n, \\ -27 \cdot 2^n, & \text{if } C_n = j_n, \end{cases}$$

where we have used $J_{n+2} - 4J_n = (-1)^n$ and $j_{n+2} - 4j_n = 3(-1)^{n+1}$.

Fairgrieve and Gould also found the identity (3.8) when $G_n = F_n$ [3].

It also follows from identity (3.8) that $G_nG_{n+3}^2 - G_{n+2}^3 = (-1)^{n+1}\mu(1)G_{n+1}$. This implies

$$4G_nG_{n+2}^3G_{n+3}^2 + \nu^2 G_{n+1}^2 = (G_nG_{n+3}^2 + G_{n+2}^3)^2.$$

Similarly,

$$\begin{aligned} 4B_nB_{n+2}^3B_{n+3}^2 + \gamma^2(4B_{n+2} - B_n)^2 &= (B_nB_{n+3}^2 + B_{n+2}^3)^2; \\ 4C_nC_{n+2}^3C_{n+3}^2 + \kappa^6\, 4^n &= (C_nC_{n+3}^2 + C_{n+2}^3)^2. \end{aligned}$$

Fairgrieve and Gould also discovered that $F_n^2F_{n+3} - F_{n+1}^3 = (-1)^{n+1}F_{n+2}$ [3]. The next theorem extends this identity to the gibonacci family. Its proof is also short and neat.

Theorem 3.4. *Let $n \geq 0$. Then,*

$$g_n^2 g_{n+3} - g_{n+1}^3 = \mu(-1)^{n+1}(g_{n+3} - x^2 g_{n+1}). \tag{3.9}$$

Proof. By the gibonacci recurrence, we have

$$\begin{aligned} g_n^2 g_{n+3} - g_{n+1}^3 &= (g_{n+2} - x g_{n+1})^2 g_{n+3} - g_{n+1}(g_{n+3} - x g_{n+2})^2 \\ &= g_{n+2}^2 g_{n+3} + x^2 g_{n+1}^2 g_{n+3} - g_{n+1} g_{n+3}^2 - x^2 g_{n+1} g_{n+2}^2 \\ &= (g_{n+1} g_{n+3} - g_{n+2}^2)(x^2 g_{n+1} - g_{n+3}) \\ &= (-1)^{n+1} \mu (g_{n+3} - x^2 g_{n+1}). \end{aligned}$$

□

It follows from identity (3.9) that

$$g_n^2 g_{n+3} - g_{n+1}^3 = \begin{cases} (-1)^{n+1}(g_{n+3} - x^2 g_{n+1}), & \text{if } g_n = f_n, \\ (-1)^n \Delta^2 (g_{n+3} - x^2 g_{n+1}), & \text{if } g_n = l_n; \end{cases}$$

$$b_n^2 b_{n+3} - b_{n+1}^3 = \begin{cases} (-1)^{n+1}(b_{n+3} - 4x^2 b_{n+1}), & \text{if } b_n = p_n, \\ (-1)^n \, 4(x^2+1)(b_{n+3} - 4x^2 b_{n+1}), & \text{if } b_n = q_n; \end{cases}$$

$$c_n^2 c_{n+3} - c_{n+1}^3 = \begin{cases} (-x)^{n-1}(c_{n+3} - c_{n+1}), & \text{if } c_n = J_n(x), \\ -(4x+1)(-x)^{n-1}(c_{n+3} - c_{n+1}), & \text{if } c_n = j_n(x). \end{cases}$$

In particular, we have

$$G_n^2 G_{n+3} - G_{n+1}^3 = \begin{cases} (-1)^{n+1} G_{n+2}, & \text{if } G_n = F_n, \\ (-1)^n \, 5 G_{n+2}, & \text{if } G_n = L_n; \end{cases}$$

$$B_n^2 B_{n+3} - B_{n+1}^3 = \begin{cases} (-1)^{n+1}(B_{n+3} - 4B_{n+1}), & \text{if } B_n = P_n, \\ (-1)^n \, 2(B_{n+3} - 4B_{n+1}), & \text{if } B_n = Q_n; \end{cases}$$

$$C_n^2 C_{n+3} - C_{n+1}^3 = \begin{cases} -(-4)^n, & \text{if } C_n = J_n, \\ 27(-4)^n, & \text{if } C_n = j_n, \end{cases}$$

where we have used the Jacobsthal properties that $J_{n+3} - J_{n+1} = 2^{n+1}$ and $j_{n+3} - j_{n+1} = 3 \cdot 2^{n+1}$.

3.4. **Additional Consequences.** It follows from the above numeric identities that

$$\begin{aligned} 4G_n^2 G_{n+1}^3 G_{n+3} + \nu^2 G_{n+2}^2 &= (G_n^2 G_{n+3} + G_{n+1}^3)^2; \\ 4B_n^2 B_{n+1}^3 B_{n+3} + \gamma^2 (B_{n+3} - 4B_{n+1})^2 &= (B_n^2 B_{n+3} + B_{n+1}^3)^2; \\ 4C_n^2 C_{n+1}^3 C_{n+3} + \kappa^6 \, 16^n &= (C_n^2 C_{n+3} + C_{n+1}^3)^2. \end{aligned}$$

Next, we investigate differences of gibonacci products of order 4.

4. Differences of Gibonacci Products of Order 4

The next theorem highlights an interesting difference of two gibonacci products of order 4. It is a straightforward application of the Catalan-like identity (2.2).

Theorem 4.1. *Let $n \geq 0$. Then,*

$$g_{n+2} g_{n+1} g_{n-1} g_{n-2} - g_n^4 = \mu[(1 - x^2)(-1)^n g_n^2 - \mu x^2]. \tag{4.1}$$

Proof. We have

$$\begin{aligned} \text{LHS} &= (g_{n+2} g_{n-2})(g_{n+1} g_{n-1}) - g_n^4 \\ &= [g_n^2 - \mu(-1)^n x^2][g_n^2 + \mu(-1)^n] - g_n^4 \\ &= [\mu(-1)^n - \mu(-1)^n x^2] g_n^2 - \mu^2 x^2 \\ &= \mu(1 - x^2)(-1)^n g_n^2 - \mu^2 x^2. \end{aligned}$$

□

It follows Theorem 4.1 that

$$g_{n+2}g_{n+1}g_{n-1}g_{n-2} - g_n^4 = \begin{cases} (-1)^n(1-x^2)g_n^2 - x^2, & \text{if } g_n = f_n, \\ \Delta^2\left[(-1)^n(x^2-1)g_n^2 - \Delta^2 x^2\right], & \text{if } g_n = l_n; \end{cases} \tag{4.2}$$

$$b_{n+2}b_{n+1}b_{n-1}b_{n-2} - b_n^4 = \begin{cases} (-1)^n(1-4x^2)b_n^2 - 4x^2, & \text{if } b_n = p_n, \\ 4(x^2+1)[(-1)^n(4x^2-1)b_n^2 - 16x^2(x^2+1)], & \text{if } b_n = q_n. \end{cases} \tag{4.3}$$

Next, we pursue the Jacobsthal implications of Theorem 4.1.

4.1. **Jacobsthal Implications.** Letting $u = 1/\sqrt{x}$, equation (4.1) becomes

$$g_{n+2}g_{n+1}g_{n-1}g_{n-2} - g_n^4 = \frac{\mu}{x}\left[(x-1)(-1)^n g_n^2 - \mu\right],$$

where $g_n = g_n(u)$ and $\mu = \mu(u)$.

Suppose $g_n = f_n$, where $f_n = f_n(u)$. Multiplying the resulting equation with x^{2n-2}, we get

$$J_{n+2}(x)J_{n+1}(x)J_{n-1}(x)J_{n-2}(x) - J_n^4(x) = x^{n-2}\left[(-1)^n(x-1)J_n^2(x) - x^{n-1}\right].$$

On the other hand, suppose $g_n = l_n$. This time, multiply the corresponding equation with x^{2n}; this yields

$$j_{n+2}(x)j_{n+1}(x)j_{n-1}(x)j_{n-2}(x) - j_n^4(x) = x^{n-2}(4x+1)\left[(-1)^n(1-x)j_n^2(x) - (4x+1)x^{n-1}\right].$$

Combining the two cases, we have

$$c_{n+2}c_{n+1}c_{n-1}c_{n-2} - c_n^4 = \begin{cases} x^{n-2}\left[(-1)^n(x-1)c_n^2 - x^{n-1}\right], & \text{if } c_n = J_n(x), \\ x^{n-2}(4x+1)\left[(-1)^n(1-x)c_n^2 - (4x+1)x^{n-1}\right], & \text{if } c_n = j_n(x). \end{cases} \tag{4.4}$$

4.2. **Additional Byproducts.** It follows from the polynomial identities (4.2), (4.3), and (4.4) that

$$G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = -\nu^2; \tag{4.5}$$

$$B_{n+2}B_{n+1}B_{n-1}B_{n-2} - B_n^4 = \begin{cases} 3(-1)^{n+1}B_n^2 - 4, & \text{if } B_n = P_n, \\ 2[3(-1)^n B_n^2 - 8], & \text{if } B_n = Q_n; \end{cases}$$

$$C_{n+2}C_{n+1}C_{n-1}C_{n-2} - C_n^4 = \begin{cases} 2^{n-2}\left[(-1)^n C_n^2 - 2^{n-1}\right], & \text{if } C_n = J_n, \\ 9\cdot 2^{n-2}\left[(-1)^{n+1}C_n^2 - 9\cdot 2^{n-1}\right], & \text{if } C_n = j_n, \end{cases}$$

respectively.

Identity (4.5) with $G_n = F_n$ is the *Gelin-Cesàro identity*, stated by E. Gelin, but proved by E. Cesàro (1859–1906) [1, 3].

It follows from identity (4.5) that $(G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4)^2 = \nu^4$. Consequently,

$$4G_{n+2}G_{n+1}G_n^4G_{n-1}G_{n-2} + \nu^4 = (G_{n+2}G_{n+1}G_{n-1}G_{n-2} + G_n^4)^2.$$

Similarly, we have

$$(B_{n+2}B_{n+1}B_{n-1}B_{n-2} + B_n^4)^2 = \begin{cases} 4B_{n+2}B_{n+1}B_n^4B_{n-1}B_{n-2} + [4 + 3(-1)^n B_n^2]^2, & \text{if } B_n = P_n, \\ 4B_{n+2}B_{n+1}B_n^4B_{n-1}B_{n-2} + 4[8 - 3(-1)^n B_n^2]^2, & \text{if } B_n = Q_n; \end{cases}$$

$$(C_{n+2}C_{n+1}C_{n-1}C_{n-2} + C_n^4)^2 = \begin{cases} 4C_{n+2}C_{n+1}C_n^4C_{n-1}C_{n-2} + A, & \text{if } C_n = J_n, \\ 4C_{n+2}C_{n+1}C_n^4C_{n-1}C_{n-2} + B, & \text{if } C_n = j_n, \end{cases}$$

where $A = 4^{n-2}[(-1)^n C_n^2 - 2^{n-1}]^2$ and $B = 81\cdot 4^{n-2}[(-1)^n C_n^2 + 9\cdot 2^{n-1}]^2$.

5. Vieta and Chebyshev Implications

Finally, it follows by the relationships in Table 1 that Theorems 3.1 through 4.1 have implications to the Vieta and Chebyshev subfamilies also. In the interest of brevity, we leave the work for interested gibonacci enthusiasts.

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Department of Mathematics, Framingham State University, Framingham, MA 01701, USA
E-mail address: tkoshy@emeriti.framingham.edu