SOME GIBONACCI CONVOLUTIONS WITH DIVIDENDS

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Abstract. We develop convolution formulas linking the Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials, and then deduce the corresponding ones for Pell-Jacobsthal polynomials, and their numeric counterparts. Using the numeric Fibonacci-Jacobsthal hybridity, we show how the corresponding Fibonacci-Jacobsthal-Lucas, Lucas-Jacobsthal, and Lucas-Jacobsthal-Lucas convolution formulas can be derived. We also construct combinatorial models for the Fibonacci-Jacobsthal, Fibonacci-Jacobsthal-Lucas, and Lucas-Jacobsthal-Lucas convolutions.

1. Introduction

Generalized *gibonacci polynomials* $g_n(x)$ are defined by the second-order recurrence $g_{n+2}(x) = a(x)g_{n+1}(x)+b(x)g_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $g_0(x)$, and $g_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [4, 10].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [9, 10].

On the other hand, let $a(x) = 1$ and $b(x) = 2x$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $g_0(x) = 2$ and $g_1(x) = 1$, $g_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [7, 8]. Correspondingly, $J_n = J_n(1)$ and $j_n = j_n(1)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1/2) = F_n$; and $j_n(1/2) = L_n$.

Extending these definitions to negative integers n, it follows that $F_{-1} = 1 = -F_{-2} = -L_{-1}$ and $J_{-1} = 1/2$; we need these values later.

In the interest of brevity and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$.

1.1. Binet-like Formulas.

Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials can also be defined by *Binet-like formulas*:

$$\begin{aligned} f_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad & l_n &= \alpha^n + \beta^n; \\ p_n &= \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad & q_n &= \gamma^n + \delta^n; \\ J_n(x) &= \frac{u^n - v^n}{u - v} \quad \text{and} \quad & j_n(x) &= u^n + v^n, \end{aligned}$$

where $2\alpha = x+\sqrt{x^2+4}$, $2\beta = x-\sqrt{x^2+4}$, $\gamma = x+\sqrt{x^2+1}$, $\delta = x-\sqrt{x^2+1}$, $2u = 1+\sqrt{8x+1}$, and $2v = 1-\sqrt{8x+1}$. In the interest of conciseness, we let $\Delta = \alpha - \beta = \sqrt{x^2+4}$, and $\omega = u - v = \sqrt{8x+1}$.

Using the Binet-like formulas and the respective recurrences, we can extract a multitude of identities. For example, $f_{n+1} + f_{n-1} = l_n$, $l_{n+1} + l_{n-1} = \Delta^2 f_n$, $l_n + x f_n = 2f_{n+1}$, $J_{n+1}(x) + 2xJ_{n-1}(x) = j_n(x)$, and $j_{n+1}(x) + 2xj_{n-1}(x) = \omega^2 J_n(x)$.

2. Generating Functions

Generating functions also play an important role in the development of identities:

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} f_n t^n &&= \frac{t}{1 - xt - t^2} &&= \frac{1}{\Delta}\left(\frac{1}{1-\alpha t} - \frac{1}{1-\beta t}\right);\\
l(t) &= \sum_{n=0}^{\infty} l_n t^n &&= \frac{2 - xt}{1 - xt - t^2} &&= \frac{1}{1-\alpha t} + \frac{1}{1-\beta t};\\
J(t) &= \sum_{n=0}^{\infty} J_n(x) t^n &&= \frac{t}{1 - t - 2xt^2} &&= \frac{1}{\omega}\left(\frac{1}{1-ut} - \frac{1}{1-vt}\right);\\
j(t) &= \sum_{n=0}^{\infty} j_n(x) t^n &&= \frac{2 - t}{1 - t - 2xt^2} &&= \frac{1}{1-ut} + \frac{1}{1-vt}.
\end{aligned}$$

2.1. **Applications.** One application of generating functions is in finding convolution formulas. The following theorem gives one such formula. The proof involves plenty of algebraic manipulation; so in the interest of brevity, we give only the key steps.

Theorem 2.1.

$$\sum_{k=0}^{n} f_k J_{n-k}(x) = \frac{[(1-2x)J_{n+1}(x) - 2x(x-1)J_n(x)] + (2xf_{n+1} - f_n - f_{n-1})}{2x^3 - 6x^2 + 3x}. \tag{2.1}$$

Proof. Let S_n denote the sum. Then,

$$\begin{aligned}
\Delta\omega f(t)J(t) &= \left(\frac{1}{1-\alpha t} - \frac{1}{1-\beta t}\right)\left(\frac{1}{1-ut} - \frac{1}{1-vt}\right)\\
&= \frac{1}{(1-\alpha t)(1-ut)} - \frac{1}{(1-\alpha t)(1-vt)} - \frac{1}{(1-\beta t)(1-ut)} + \frac{1}{(1-\beta t)(1-vt)}\\
&= \left[\frac{\alpha}{(\alpha-u)(1-\alpha t)} - \frac{u}{(\alpha-u)(1-ut)}\right] - \left[\frac{\alpha}{(\alpha-v)(1-\alpha t)} - \frac{v}{(\alpha-v)(1-vt)}\right]\\
&\quad - \left[\frac{\beta}{(\beta-u)(1-\beta t)} - \frac{u}{(\beta-u)(1-ut)}\right] + \left[\frac{\beta}{(\beta-v)(1-\beta t)} - \frac{v}{(\beta-v)(1-vt)}\right]\\
&= \frac{\alpha\omega}{(1-\alpha t)(\alpha^2-\alpha-2x)} - \frac{\beta\omega}{(1-\beta t)(\beta^2-\beta-2x)}\\
&\quad - \frac{u\Delta}{(1-ut)(1+xu-u^2)} + \frac{v\Delta}{(1-vt)(1+xv-v^2)}.
\end{aligned}$$

Equating the coefficients of t^n from both sides, we get

$$\begin{aligned}
\Delta\omega S_n &= \omega\left(\frac{\alpha^{n+1}}{\alpha^2-\alpha-2x} - \frac{\beta^{n+1}}{\beta^2-\beta-2x}\right) - \Delta\left(\frac{u^{n+1}}{1+xu-u^2} - \frac{v^{n+1}}{1+xv-v^2}\right)\\
S_n &= \frac{2xf_{n+1} - f_n - f_{n-1}}{2x^3-6x^2+3x} + \frac{J_{n+1}(x) - 2x^2J_n(x) - 4x^2J_{n-1}(x)}{2x^3-6x^2+3x}.
\end{aligned}$$

This yields the desired result. □

In particular, formula (2.1) yields

$$\sum_{k=0}^{n} F_k J_{n-k} = J_{n+1} - F_{n+1}. \tag{2.2}$$

The next theorem gives three additional convolution formulas. Their proofs follow similar steps; again, in the interest of brevity, we omit them.

Theorem 2.2. *Let* $A = (1-2x)j_{n+1}(x) - 2x(x-1)j_n(x)$, $B = (x^2-5x+2)J_{n+2}(x) - x(3x-4)J_{n+1}(x)$, *and* $C = (x^2-5x+2)j_{n+2}(x) - x(3x-4)j_{n+1}(x)$. *Then,*

$$\sum_{k=0}^{n} f_k j_{n-k}(x) = \frac{A + (4x-1)f_{n+2} - (x+1)f_{n+1} - xf_n}{2x^3 - 6x^2 + 3x}; \tag{2.3}$$

$$\sum_{k=0}^{n} l_k J_{n-k}(x) = \frac{B + 2xl_{n+1} - l_n - l_{n-1}}{2x^3 - 6x^2 + 3x}; \tag{2.4}$$

$$\sum_{k=0}^{n} l_k j_{n-k}(x) = \frac{C + (4x-1)l_{n+2} - (x+1)l_{n+1} - xl_n}{2x^3 - 6x^2 + 3x}. \tag{2.5}$$

It follows from Theorem 2.2 that

$$\sum_{k=0}^{n} F_k j_{n-k} = j_{n+1} - L_{n+1}; \tag{2.6}$$

$$\sum_{k=0}^{n} L_k J_{n-k} = j_{n+1} - L_{n+1}; \tag{2.7}$$

$$\sum_{k=0}^{n} L_k j_{n-k} = 9J_{n+1} - 5F_{n+1}. \tag{2.8}$$

Griffiths and Bramham discovered formula (2.7) [6] and gave a nice combinatorial interpretation later [5].

Next, we present combinatorial interpretations of formulas (2.2), (2.6), and (2.8).

3. Combinatorial Models

The number of n-tilings of a $1\times n$ board with 1×1 white tiles and 1×2 white tiles (dominoes) is F_{n+1} [2, 5, 11]. Likewise, a circular board with n cells can be tiled with (flexible) square tiles and (flexible) dominoes in L_n different ways [2, 5, 11]; such a tiling is called an *n-bracelet*. An n-bracelet is *out-of-phase* if a domino occupies cells n and 1; otherwise, it is *in-phase*.

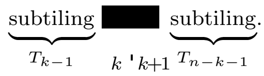
3.1. A Model for Formula (2.2).

A $1 \times n$ board can be tiled with white squares, and black and white dominoes in J_{n+1} different ways [2, 5, 11]. Consequently, the number of n-tilings with white squares, and black and white dominoes such that each tiling contains at least one black domino equals $J_{n+1} - F_{n+1}$, where $n \geq 0$. With this tool at hand, we can combinatorially establish formula (2.2).

The proof hinges on the well known *Fubini's principle* [1], named after the Italian mathematician Guido Fubini (1879–1943): *Counting the elements of a set in two different ways yields the same result.*

Proof. The number of n-tilings T_n with white squares, and black and white dominoes such that each tiling contains at least one black domino equals $J_{n+1} - F_{n+1}$. We will now count such tilings in a different way.

Suppose the first such black domino B occurs in cells k and $k+1$, where $1 \le k \le n-1$. Then, B partitions T_n into subtilings T_{k-1}, B, and T_{n-k-1}, where T_{k-1} contains only white squares and white dominoes, and T_{n-k-1} may contain white squares, and black and white dominoes:



There are F_k tilings T_{k-1} and J_{n-k} tilings T_{n-k-1}, so there are $F_k J_{n-k}$ such tilings T_n for every k. Consequently, there are

$$\sum_{k=1}^{n-1} F_k J_{n-k} = \sum_{k=0}^{n} F_k J_{n-k}$$

n-tilings T_n such that every tiling contains at least one black domino.

This result, coupled with the earlier count, gives the desired result. □

3.2. **A Model for Formula (2.6).** A circular board with n-cells can be tiled with 1×1 white tiles, and 1×2 white dominoes, and 1×2 black dominoes in j_n ways [2, 5]. It can be tiled with 1×1 white tiles, and 1×2 white dominoes in L_n ways. So there are $S = j_n - L_n$ such n-bracelets, each containing at least one black domino.

To compute this sum S in a different way, consider an arbitrary n-bracelet B.

Case 1. Assume B is in-phase. Suppose the first black domino D occurs in cells k and $k+1$, where $1 \le k \le n-1$; see Figure 1. There are $F_k J_{n-k}$ such n-bracelets. Consequently, the total number of in-phase bracelets B equals $\sum_{k=1}^{n-1} F_k J_{n-k} = \sum_{k=1}^{n} F_k J_{n-k}$.

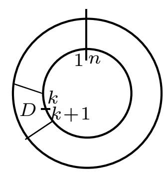


Figure 1

Case 2. Suppose B is out-of-phase. Assume a white domino W occupies cells n and 1; see Figure 2. It follows by Case 1 that the number of such n-bracelets, where each contains at least one black domino, is $\sum_{k=1}^{n-2} F_k J_{n-k-2}$.

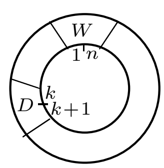


Figure 2

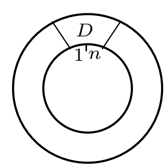


Figure 3

On the other hand, suppose a black domino D occupies cells n and 1; see Figure 3. Clearly, there are J_{n-1} such bracelets, with each containing at least one black domino.

Since $J_{n+2} + 2J_n = j_{n+1}$, it follows that the total number of n-bracelets S with the desired property, is given by

$$\begin{aligned}
S &= \sum_{k=1}^{n-1} F_k J_{n-k} + \sum_{k=1}^{n-2} F_k J_{n-k-2} + J_{n-1} \\
&= \sum_{k=1}^{n-1} F_k (J_{n-k} + 2J_{n-k-2}) - 2F_{n-1}J_{-1} - \sum_{k=1}^{n-2} F_k J_{n-k-2} + J_{n-1} \\
&= \sum_{k=1}^{n-1} F_k j_{n-k-1} - F_{n-1} - (J_{n-1} - F_{n-1}) + J_{n-1} \\
&= \sum_{k=1}^{n-1} F_k j_{n-k-1}.
\end{aligned}$$

The desired formula now follows by combining the two counts.

3.3. **A Model for Formula (2.8).** A combinatorial proof of convolution identity (2.8) is more complicated than it is for (2.2) or (2.6). So, first, we prepare the needed ground work by gathering several facts.

Proof. A combinatorial argument in [3] involving boards, bracelets, and uncolored squares and dominoes shows that $L_{n+2} + L_n = 5F_{n+1}$. We can rewrite this identity as

$$5F_{n+1} = L_{n+2} + L_n = L_{n+2} + (L_{n+2} - L_{n+1}) = 2L_{n+2} - L_{n+1}. \tag{3.1}$$

Using elements from the set $\mathcal{D} = \{d_1, \ldots, d_r, s\}$ comprising r 1×2 dominoes of different colors (denoted d_1 to d_r) and uncolored 1×1 squares (denoted s), it is reasonably straightforward to extend the aforementioned argument to obtain the following general identity:

$$rm_{n,r} + m_{n+2,r} = (4r+1)M_{n,r},$$

where $M_{n,r}$ and $m_{n,r}$ enumerate the tilings of an $(n-1)$-board and an n-bracelet, respectively, using elements from $\mathcal{D}$.

When $r = 2$, we have $2j_n + j_{n+2} = 9J_{n+1}$. By the Jacobsthal recurrence, this yields

$$9J_{n+1} = j_{n+2} + 2j_n = j_{n+2} + (j_{n+2} - j_{n+1}) = 2j_{n+2} - j_{n+1}.$$

In [5], it is shown via combinatorial means that

$$\sum_{k=0}^{n} L_k J_{n-k} = j_{n+1} - L_{n+1}. \tag{3.2}$$

Next, we establish combinatorially that

$$j_n + J_n = 2J_{n+1}. \tag{3.3}$$

To this end, notice that the expression $j_n - J_{n+1}$ gives the number of out-of-phase tilings of an n-bracelet using white squares, and black and white dominoes, which is equal to $2J_{n-1}$. From the Jacobsthal recurrence, we have $J_{n+1} - J_n = 2J_{n-1}$. Consequently, $j_n - J_{n+1} = J_{n+1} - J_n$. This yields the desired identity.

Using identities (5.3) through (3.3), we now have

$$
\begin{aligned}
9J_{n+1} - 5F_{n+1} &= (2j_{n+2} - j_{n+1}) - (2L_{n+2} - L_{n+1}) = 2\left(j_{n+2} - L_{n+2}\right) - \left(j_{n+1} - L_{n+1}\right) \\
&= 2\sum_{k=0}^{n+1} L_k J_{n+1-k} - \sum_{k=0}^{n} L_k J_{n-k} = 2\sum_{k=0}^{n} L_k J_{n+1-k} + L_{n+1}J_0 - \sum_{k=0}^{n} L_k J_{n-k} \\
&= \sum_{k=0}^{n} \left(2L_k J_{n+1-k} - L_k J_{n-k}\right) = \sum_{k=0}^{n} L_k\left(2J_{n+1-k} - J_{n-k}\right) \\
&= \sum_{k=0}^{n} L_k j_{n-k},
\end{aligned}
$$

as required. □

4. Convolutions Revisited

Using convolution formula (2.2), we can establish algebraically formulas (2.6), (2.7), and (2.8). To this end, we need the following identities: $F_{n+1} + F_{n-1} = L_n, J_{n+1} + 2J_{n-1} = j_n$, and $L_{n+1} + L_{n-1} = 5F_n$. We give only the key steps involved in each case.

Proof of Formula (2.6).

$$
\begin{aligned}
\sum_{k=0}^{n} F_k j_{n-k} &= \sum_{k=0}^{n} F_k J_{n-k+1} + 2\sum_{k=0}^{n} F_k J_{n-k-1} = \sum_{k=0}^{n+1} F_k J_{n-k+1} + 2\sum_{k=0}^{n-1} F_k J_{n-k-1} + F_n \\
&= (J_{n+2} - F_{n+2}) + 2(J_n - F_n) + F_n = j_{n+1} - L_{n+1},
\end{aligned}
$$

as desired. □

Formula (2.7) follows similarly.

Proof of Formula (2.8).

$$
\begin{aligned}
\sum_{k=0}^{n} L_k j_{n-k} &= \sum_{k=0}^{n} L_k J_{n-k+1} + 2\sum_{k=0}^{n} L_k J_{n-k-1} \\
&= \sum_{k=-1}^{n-1} L_{k+1} J_{n-k} + 2\sum_{k=1}^{n+1} L_{k-1} J_{n-k} \\
&= \sum_{k=0}^{n} L_{k+1} J_{n-k} + 2\sum_{k=0}^{n} L_{k-1} J_{n-k} + L_n + 2J_{n+1} + 2J_n \\
&= \sum_{k=0}^{n} (L_{k+1} + L_{k-1}) J_{n-k} + \sum_{k=0}^{n} (F_k + F_{k-2}) J_{n-k} + L_n + 2J_{n+1} + 2J_n \\
&= 6\sum_{k=0}^{n} F_k J_{n-k} + \sum_{k=0}^{n} F_{k-2} J_{n-k} + L_n + 2J_{n+1} + 2J_n \\
&= 6(J_{n+1} - F_{n+1}) + \left(\sum_{k=0}^{n-2} F_k J_{n-k-2} - J_n + J_{n-1}\right) + L_n + 2J_{n+1} + 2J_n \\
&= 8J_{n+1} - 6F_{n+1} + (J_{n-1} - F_{n-1}) + J_n + J_{n-1} + (F_{n+1} + F_{n-1}) \\
&= 8J_{n+1} - 5F_{n+1} + (J_n + 2J_{n-1}).
\end{aligned}
$$

This gives the desired result. □

We add that by invoking the identity $j_{n+2}+2j_n = 9J_{n+1}$ and the summation formula (2.7), we can confirm formula (2.8) in fewer steps.

4.1. **Pell Dividends.** Since $p_n = f_n(2x)$ and $q_n = l_n(2x)$, it follows that Theorems 2.1 and 2.2 yield interesting Pell byproducts. For brevity, we let $D = (1-4x)J_{n+1}(2x)-4x(2x-1)J_n(2x)$, $E = (1-4x)j_{n+1}(2x)-4x(2x-1)j_n(2x)$, $F = (4x^2-10x+2)J_{n+2}(2x)-4x(3x-2)J_{n+1}(2x)$, and $G = (4x^2-10x+2)j_{n+2}(2x)-4x(3x-2)j_{n+1}(2x)$. Then,

$$\sum_{k=0}^{n} p_k J_{n-k}(2x) = \frac{D+4xp_{n+1}-p_n-p_{n-1}}{16x^3-24x^2+6x};$$
$$\sum_{k=0}^{n} p_k j_{n-k}(2x) = \frac{E+(8x-1)p_{n+2}-(2x+1)p_{n+1}-2xp_n}{16x^3-24x^2+6x};$$
$$\sum_{k=0}^{n} q_k J_{n-k}(2x) = \frac{F+4xq_{n+1}-q_n-q_{n-1}}{16x^3-24x^2+6x};$$
$$\sum_{k=0}^{n} q_k j_{n-k}(2x) = \frac{G+(8x-1)q_{n+2}-(2x+1)q_{n+1}-2xq_n}{16x^3-24x^2+6x}.$$

It follows from these hybrid formulas that

$$\sum_{k=0}^{n} P_k J_{n-k}(2) = 2J_n(2) - P_{n+1} + \frac{3J_{n+1}(2)-Q_{n+1}}{2};$$
$$\sum_{k=0}^{n} P_k j_{n-k}(2) = 2j_n(2) - 3P_{n+2} + \frac{3j_{n+1}(2)+Q_{n+1}}{2};$$
$$\sum_{k=0}^{n} Q_k J_{n-k}(2) = J_{n+2}(2) + J_{n+1}(2) - 2Q_{n+1} + P_n;$$
$$\sum_{k=0}^{n} Q_k j_{n-k}(2) = j_{n+2}(2) + j_{n+1}(2) - 3Q_{n+2} + P_{n+1}.$$

Note that $3J_{n+1}(2) - Q_{n+1}$ and $3j_{n+1}(2) + Q_{n+1}$ are even integers.

5. Additional Convolutions

The next theorem gives five additional convolution formulas. They can be confirmed using the property $2J_{n+1}(x) = J_n(x) + j_n(x)$, and generating functions (or Binet-like formulas); again, for the sake of brevity, we omit them.

Theorem 5.1. *Let n be a nonnegative integer. Then,*

$$(x^2+4)\sum_{k=0}^{n} f_k f_{n-k} = nl_n - xf_n; \tag{5.1}$$

$$\sum_{k=0}^{n} l_k l_{n-k} = (n+2)l_n + xf_n; \tag{5.2}$$

$$(8x+1)\sum_{k=0}^{n} J_k(x)J_{n-k}(x) = nj_n(x) - J_n(x);$$

$$\sum_{k=0}^{n} J_k(x)j_{n-k}(x) = (n+1)J_n(x);$$

$$\sum_{k=0}^{n} j_k(x)j_{n-k}(x) = (n+2)j_n(x) + J_n(x).$$

It then follows that

$$5\sum_{k=0}^{n} F_k F_{n-k} = nL_n - F_n; \tag{5.3}$$

$$\sum_{k=0}^{n} L_k L_{n-k} = (n+2)L_n + F_n; \tag{5.4}$$

$$9\sum_{k=0}^{n} J_k J_{n-k} = nj_n - J_n; \tag{5.5}$$

$$\sum_{k=0}^{n} J_k j_{n-k} = (n+1)J_n;$$

$$\sum_{k=0}^{n} j_k j_{n-k} = (n+2)j_n + J_n,$$

respectively.

It follows from formulas (5.3) and (5.5) that $nL_n \equiv F_n \pmod{5}$ and $nj_n \equiv J_n \pmod{9}$, respectively.

5.1. **Pell Consequences.** Clearly, formulas (5.1) and (5.2) have Pell consequences:

$$4(x^2+1)\sum_{k=0}^{n} p_k p_{n-k} = nq_n - 2xp_n;$$

$$\sum_{k=0}^{n} q_k q_{n-k} = (n+2)q_n + 2xp_n,$$

respectively.

Consequently, we have

$$4\sum_{k=0}^{n} P_k P_{n-k} = nQ_n - P_n;$$

$$2\sum_{k=0}^{n} Q_k Q_{n-k} = (n+2)Q_n + P_n,$$

respectively. It then follows that $nQ_n \equiv P_n \pmod 4$.

References

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MSC2010: 11B37, 11B39

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