ARITHMETIC FUNCTIONS OF BALANCING NUMBERS

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Abstract. Two inequalities involving the Euler totient function and the sum of the kth powers of the divisors of balancing numbers are explored.

1. Introduction

For any positive integer n, the Euler totient function $\phi(n)$ is defined as the number of positive integers less than n and relatively prime to n, and $\sigma_k(n)$ denotes the sum of the kth powers of divisors of n. If $k = 0$, $\sigma_k(n)$ reduces to the function $\tau(n)$, which counts the number of positive divisors of n. For many centuries, mathematicians were more concerned about the arithmetic functions of natural numbers and solved many Diophantine equations concerning these functions. Subsequently, some researchers have focused their attention on the study of arithmetic functions related to binary recurrence sequences such as the Fibonacci sequence, the Lucas sequence, the Pell sequence, and the associated Pell sequence.

In 1997, Luca [5] showed that the Euler totient function for the homogeneous binary recurrence sequences $\{u_n\}_{n\geq 0}$ satisfies the inequality $\phi(|u_n|) \geq |u_{\phi(n)}|$ for those binary recurrences with characteristic equations having real roots and the inequality is not valid for those recurrences with characteristic equations having complex roots. In [6], he proved that the nth Fibonacci number satisfies $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ and $\tau(F_n) \geq F_{\tau(n)}$ for all $k, n \geq 1$. Motivated by these works, we study two similar inequalities involving arithmetic functions of balancing numbers.

Recall that a natural number B is a balancing number with balancer R if the pair (B, R) satisfies the Diophantine equation $1 + 2 + \cdots + (B - 1) = (B + 1) + \cdots + (B + R)$. If B is a balancing number, then $8B^2 + 1$ is a perfect square and its positive square root is called a Lucas-balancing number. The nth balancing number is denoted by B_n, whereas the nth Lucas-balancing number is denoted by C_n. The balancing numbers satisfy the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$, $B_0 = 0$, $B_1 = 1$, which holds for $n \geq 1$, whereas the Lucas-balancing numbers satisfy a binary recurrence identical with that of the balancing numbers, but with initial values $C_0 = 1$, $C_1 = 3$. The characteristic equation of these recurrences is given by $x^2 - 6x + 1 = 0$, and the roots of this equation are $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. The Binet forms of balancing and Lucas-balancing numbers are given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad C_n = \frac{\alpha^n + \beta^n}{2}$$

(see [1, 10]).

Given a natural number $A > 2$, the sequence arising from the class of binary recurrence $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0$, $x_1 = 1$ is known as a balancing-like sequence because the case $A = 6$ corresponds to the balancing sequence [9]. It is interesting to note that when $A = 2$, the above recurrence relation generates the sequence of natural numbers. Further, when $A = 3$, the corresponding balancing-like sequence coincides with the sequence of even indexed Fibonacci numbers. The balancing-like sequences (and hence the balancing sequence) satisfy certain identities in which they behave like natural numbers [8, 9] and hence, these sequences are considered as generalizations of the sequence of natural numbers.

2. Auxiliary Results

To establish the inequalities concerning arithmetic functions of balancing numbers, we need the following results. Some results of this section are new and hence we provide proofs of such results.

The following lemma presents some basic properties of balancing numbers.

Lemma 2.1. ([8], Theorem 2.5, [10], Theorem 5.2.6) *If m and n are natural numbers, then*

(1) $B_{m+n} = B_m C_n + C_m B_n$.

(2) $5^{n-1} < B_n < 6^{n-1}$ *for* $n \geq 3$.

The following two lemmas address the divisibility property of balancing numbers.

Lemma 2.2. ([8], Theorem 2.8) *If m and n are natural numbers, then B_m divides B_n if and only if m divides n.*

Lemma 2.3. ([8], Theorem 2.13) *If m and n are natural numbers, then $(B_m, B_n) = B_{(m,n)}$, where (x, y) denotes the greatest common divisor of x and y.*

Given any two nonzero integers A and B, we consider the second order linear recurrence sequence $\{w_n\}_{n\geq 0}$ defined by $w_{n+1} = Aw_n + Bw_{n-1}$ with initial terms $w_0 = 0$ and $w_1 = 1$. If $A^2 + 4B > 0$, then the characteristic equation $x^2 - Ax - B = 0$ has distinct real roots $\alpha = \frac{A+\sqrt{A^2+4B}}{2}$, $\beta = \frac{A-\sqrt{A^2+4B}}{2}$ and the Binet form is given by $w_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$. A prime p is called a primitive divisor of w_n if p divides w_n but does not divide w_m for $0 < m < n$.

The following two lemmas address the existence of primitive divisors of the sequence $\{w_n\}_{n\geq 0}$ described in the last paragraph and the balancing sequence $\{B_n\}_{n\geq 0}$.

Lemma 2.4. ([13], Theorem 1). *If the roots α and β are real and $n \neq 1, 2, 6, 12$, then w_n contains at least one primitive divisor.*

Lemma 2.5. *A primitive prime factor of B_n exists if $n > 1$.*

Proof. In Section 1, we have seen that the characteristic roots $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$ corresponding to the binary recurrence of the balancing sequence are real. Hence, by virtue of Lemma 2.4, B_n has a primitive divisor for all $n \in \mathbb{Z}$ except possibly $n \in \{1, 2, 6, 12\}$. But one can easily check that $B_2 = 6$, $B_6 = 6930$, and $B_{12} = 271669860$ have primitive divisors 3, 11, and 1153, respectively. □

Lemma 2.6. ([11], Theorem 3.2) *If p is a prime of the form $8x \pm 1$, then p divides B_{p-1}. Furthermore, if the prime p is of the form $8x \pm 3$, then p divides B_{p+1}.*

The following lemma provides bounds for the ratio of two consecutive balancing numbers.

Lemma 2.7. *For any natural number n, $\frac{B_{n+1}}{B_n} > \alpha$.*

Proof. Using $\alpha\beta = 1$, we get

$$B_{n+1} - \alpha B_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - \alpha\frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\beta^{n-1} - \beta^{n+1}}{\alpha - \beta}$$
$$= \frac{\beta^{n-1}(1 - \beta^2)}{\frac{1}{\beta} - \beta} = \beta^n > 0.$$

□

The following corollary is a direct consequence of Lemma 2.7.

Corollary 2.8. *For all natural numbers $n \geq 2$, $B_n > \alpha^{n-1}$.*

The following lemma provides an upper bound for the nth balancing number.

Lemma 2.9. *For all natural numbers $n \geq 1$, $B_n < \alpha^n$.*

Proof. It follows from the Binet formula for balancing numbers that

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} < \frac{\alpha^n}{4\sqrt{2}} < \alpha^n.$$

□

The following lemma gives a comparison of the $(m+n)$th and $(m-n)$th balancing numbers with the product and ratio of the mth and nth balancing numbers, respectively.

Lemma 2.10. *If m and n are two natural numbers, then*

$$B_{m+n} > B_m B_n \quad \text{and} \quad B_{m-n} < \frac{B_m}{B_n}.$$

Proof. Let m and n be natural numbers. Since by Lemma 2.1, $B_{m+n} = B_m C_n + C_m B_n$ and from the definition of Lucas-balancing numbers $B_n < C_n$, it follows that $B_{m+n} > B_m B_n$. Since $B_m = B_{(m-n)+n} > B_{m-n} B_n$, the inequality $B_{m-n} < \frac{B_m}{B_n}$ follows. □

The next lemma gives a comparison of n^kth balancing number with the kth power of nth balancing number.

Lemma 2.11. $B_{n^k} > {B_n}^k$ *for* $n \geq 2$ *and* $k \geq 1$.

Proof. Let m, n, and k be natural numbers. Since $B_m \geq B_n$ whenever $m \geq n$, and $n^k \geq nk$ for all $n \geq 2$, it follows that $B_{n^k} \geq B_{nk}$. Now, using Lemma 2.10 and simple mathematical induction, it is easy to see that $B_{nk} > {B_n}^k$. □

The following lemma gives certain bounds involving the arithmetic functions. For the proof of this lemma, the readers are advised to go through [5] and [12].

Lemma 2.12. ([5], Lemma 3) *Let m and n be natural numbers.*

(1) *If $n \geq 2 \cdot 10^9$, then $\phi(n) > \frac{n}{\log n}$.*

(2) *If $1 \leq n < 2 \cdot 10^9$, then $\phi(n) > \frac{n}{6}$.*

(3) *If $m \geq 2$ and $k \geq 1$, then $\frac{m}{\phi(m)} > \frac{\sigma_k(m)}{m^k}$.*

(4) *If n is not prime, then $n - \phi(n) \geq \sqrt{n}$.*

(5) *If n is not prime, then $\sigma_k(n) - n^k \geq \sqrt{n^k}$.*

The following lemma addresses an inequality involving the Euler totient function of balancing numbers.

Lemma 2.13. *For any natural numbers n, $\phi(B_n) \geq B_{\phi(n)}$ and equality holds only if $n = 1$.*

Proof. Consider the binary recurrence sequence $\{w_n\}_{n \geq 0}$ defined just after Lemma 2.3. Luca [5] proved that if the characteristic roots α and β are real, then $\phi(|w_n|) \geq |w_{\phi(n)}|$. Since the characteristic roots $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$ corresponding to the recurrence relation of the balancing sequence are real, the inequality $\phi(B_n) \geq B_{\phi(n)}$ holds for all $n \geq 1$. □

The following lemma will play a crucial role in proving an important result of this paper.

Lemma 2.14. *If n is an odd prime and $B_n = p_1^{\gamma_1} \cdots p_t^{\gamma_t}$ is the canonical decomposition of B_n, then $p_i \geq 2n-1$ for $i = 1, \ldots, t$. Furthermore, if the inequality $\sigma_k(B_n) > B_{\sigma_k(n)}$ is satisfied for all natural numbers k and $n \geq 2$, then $t > 2(n-1)\log 5$.*

Proof. Let p be any odd prime. By virtue of Lemma 2.6, $p|B_{p+1}$ or $p|B_{p-1}$. Since $p+1$ and $p-1$ both divide p^2-1, it follows from Lemma 2.2 that both B_{p-1} and B_{p+1} divide B_{p^2-1} and hence $p|B_{p^2-1}$. If p is one of the primes $p_1, p_2, \ldots, p_t$, then $p|B_n$ and hence $p|(B_{p^2-1}, B_n)$. Since $(B_{p^2-1}, B_n) = B_{(p^2-1,n)}$, by virtue of Lemma 2.3, it follows that $p|B_{(p^2-1,n)}$. If $n \nmid p^2-1$, then $(p^2-1, n) = 1$ and then $p|B_1 = 1$, which is not possible. Thus, $n|p^2-1$ and since n is a prime, $n|p+1$ or $n|p-1$, and hence $p \equiv \pm 1 \pmod n$. Clearly $p \neq n \pm 1$ since p and n are both primes and $n > 2$. Hence, $p \geq 2n-1$. This proves the first part.

We next prove the second part assuming that the inequality $\sigma_k(B_n) > B_{\sigma_k(n)}$ holds for all natural numbers k and $n \geq 2$. Since

$$\frac{B_n}{\phi(B_n)} = \frac{B_n}{B_n \prod_{i=1}^t \left(1 - \frac{1}{p_i}\right)} = \prod_{i=1}^t \left(1 + \frac{1}{p_i - 1}\right),$$

using Lemma 2.7, Lemma 2.11, and Lemma 2.12, we get

$$\prod_{i=1}^t \left(1 + \frac{1}{p_i - 1}\right) = \frac{B_n}{\phi(B_n)} > \frac{\sigma_k(B_n)}{{B_n}^k} > \frac{B_{\sigma_k(n)}}{{B_n}^k} > \frac{B_{1+n^k}}{{B_n}^k} \geq \frac{B_{1+n^k}}{B_{n^k}} > \alpha > 5 \tag{2.1}$$

Taking logarithms on both sides, we get

$$\sum_{i=1}^t \log\left(1 + \frac{1}{p_i - 1}\right) > \log 5.$$

Since $\log(1+x) < x$ for all $x > 0$, we conclude that

$$\sum_{i=1}^t \frac{1}{p_i - 1} > \log 5. \tag{2.2}$$

In view of the first part of the lemma, it follows that

$$\frac{t}{2(n-1)} > \log 5, \tag{2.3}$$

which is equivalent to $t > 2(n-1)\log 5$. □

3. Main Results

In this section, we provide two important theorems addressing arithmetic functions of the balancing sequence. In the first theorem, we establish an inequality concerning the sum of kth powers of divisors of balancing numbers.

Theorem 3.1. *The balancing numbers satisfy $\sigma_k(B_n) \leq B_{\sigma_k(n)}$ for all $n \geq 1$. Equality holds only if $n = 1$.*

Proof. Since for $k \geq 1$, $\sigma_k(B_1) = \sigma_k(1) = B_{\sigma_k(1)}$, the assertion of the theorem holds for $n = 1$ and all $k \geq 1$. For $n \geq 2$, assume to the contrary that

$$\sigma_k(B_n) > B_{\sigma_k(n)} \tag{3.1}$$

for some $k \geq 1$ and $n \geq 2$. First, we show that Inequality (3.1) holds only if n is prime. Assume that Inequality (3.1) holds for some composite number $n \geq 2$.

Case 1. Suppose that $B_n < 2 \cdot 10^9$. It is only possible when $n < 13$. From Lemmas 2.10, 2.11, 2.12, and Inequality (3.1), it follows that

$$B_2 = 6 > \frac{B_n}{\phi(B_n)} > \frac{\sigma_k(B_n)}{{B_n}^k} > \frac{B_{\sigma_k(n)}}{B_{n^k}} > B_{\sigma_k(n)-n^k}, \tag{3.2}$$

which implies that $2 > \sigma_k(n) - n^k$. Since n is not prime, it follows from Lemma 2.12 that

$$2 > \sqrt{n^k}. \tag{3.3}$$

One can easily check that Inequality (3.3) does not hold for any composite number n.

Case 2. Suppose that $B_n \geq 2 \cdot 10^9$. Then certainly $n \geq 14$. From Lemmas 2.10, 2.11, 2.12, and Inequality (3.1), it follows that

$$\log B_n > \frac{B_n}{\phi(B_n)} > \frac{\sigma_k(B_n)}{{B_n}^k} \geq \frac{B_{\sigma_k(n)}}{B_{n^k}} > B_{\sigma_k(n)-n^k}. \tag{3.4}$$

Since $\alpha^n > B_n > \alpha^{n-1}$ by Lemmas 2.8 and 2.9 and $\sigma_k(n) - n^k \geq \sqrt{n^k}$, it follows that

$$n \log \alpha > \log B_n > B_{\sigma_k(n)-n^k} > B_{\sqrt{n^k}} \geq \alpha^{\sqrt{n^k}-1}. \tag{3.5}$$

Furthermore, since $\sqrt{n^k} \geq n$ for $k \geq 2$, Inequality (3.5) gives

$$n \log \alpha > \alpha^{n-1}, \tag{3.6}$$

which holds only when $n = 1$, contradicting $n \geq 14$. Thus, the only possibility left is $k = 1$. But $k = 1$ implies

$$n \log \alpha > \alpha^{\sqrt{n}-1},$$

which is true for $n < 5$ and again contradicts $n \geq 14$. Hence, Inequality (3.1) doesn't hold for any composite number. Hence, n is prime.

Let n be any odd prime. From Lemma 2.14, it follows that

$$\begin{aligned} n \log \alpha > \log B_n &\geq \sum_{i=1}^{t} \log p_i \\ &\geq t \log(2n-1) > 2(n-1)\log(2n-1)\log 5. \end{aligned}$$

Hence,

$$\frac{n \log \alpha}{2(n-1)\log(2n-1)} - \log 5 > 0,$$

which does not hold for any odd prime n. Hence, $\sigma_k(B_n) \leq B_{\sigma_k(n)}$ for all natural numbers k and odd primes n. For $n = 2$, we need to show that $\sigma_k(B_2) = \sigma_k(6) = 1+2^k+3^k+6^k \leq B_{1+2^k}$. It is sufficient to prove that $4 \cdot 6^k < B_{1+2^k}$. Since $2k+2 \leq 2^k$ for all natural number $k \geq 3$, it follows that

$$4 \cdot 6^k = 2^{k+2}3^k < \alpha^{2k+2} \leq \alpha^{2^k} < B_{1+2^k}$$

and for $k = 1, 2$, one can easily check that $\sigma_k(B_2) < B_{1+2^k}$. This completes the proof. □

In the following theorem, we present an inequality involving another arithmetic function, namely the tau function of balancing numbers. We denote the number of distinct prime divisors of B_n by $\omega(B_n)$.

Theorem 3.2. *For any natural number n, $\tau(B_n) > B_{\lfloor \frac{\tau(n)}{3} \rfloor}$, where $\lfloor \cdot \rfloor$ denote the floor function.*

Proof. Let n be a natural number. By virtue of Lemma 2.2, corresponding to each divisor m of n, there exist a primitive divisor of B_m that divides B_n and hence, the number of distinct prime divisors of B_n is at least the total number of divisors of n, i.e., $\omega(B_n) \geq \tau(n)$ for $n > 1$. For each natural number n, it is easy to see that $\tau(n) \geq 2^{\omega(n)}$. Thus,

$$\tau(B_n) \geq 2^{\omega(B_n)} \geq 2^{\tau(n)}. \tag{3.7}$$

Since for each natural number n, $B_n \leq 6^{n-1} < 8^{n-1} = 2^{3n-3}$, it follows that

$$B_{\lfloor \frac{n}{3} \rfloor} < 2^{n-3}. \tag{3.8}$$

Now, from Inequality (3.7), we have

$$\tau(B_n) \geq 2^{\tau(n)} > 2^{\tau(n)-3} > B_{\lfloor \frac{\tau(n)}{3} \rfloor}. \tag{3.9}$$

This completes the proof. □

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