CLOSED FORMULAS FOR FINITE SUMS OF FRACTIONAL EXPRESSIONS THAT INVOLVE THE SINE AND COSINE FUNCTIONS

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Abstract. In this paper, we give closed formulas for a number of fractional expressions that involve the sine and cosine functions. In each case, the argument of sine/cosine is in arithmetic progression or geometric progression.

1. Introduction

In this paper, we present closed formulas for families of finite sums of fractional expressions that involve the sine and cosine functions. In Theorems 2.1 to 2.4, the arguments of sine and cosine are in arithmetic progression, and in Theorem 2.5, the argument of sine is in geometric progression.

In Section 2, we state our main results, and in Section 3, we give a sample proof.

2. The Main Results

In this section, we present our main results in five theorems. In each case, the constraints that we place on a, b, and k eliminate the possibility of a zero denominator.

Theorem 2.1. *Let $a \neq 0$ and b be real numbers such that $\cos(2ai+b) \pm \cos a \neq 0$ for $i \geq 0$. Then,*

$$\sum_{i=0}^{n} \frac{1}{\cos(2ai+b)+\cos a} = \frac{1}{2\sin a \cos((a-b)/2)} \times \frac{\sin(a(n+1))}{\cos(an+(a+b)/2)},$$

$$\sum_{i=0}^{n} \frac{1}{\cos(2ai+b)-\cos a} = \frac{1}{2\sin a \sin((a-b)/2)} \times \frac{\sin(a(n+1))}{\sin(an+(a+b)/2)}.$$

Theorem 2.2. *Let $a \neq 0$ and b be real numbers such that $\cos(2ai+b) \pm \cos a \neq 0$ for $i \geq 1$. Then,*

$$\sum_{i=1}^{n} \frac{(-1)^i \sin(2ai+b)}{\cos(2ai+b)+\cos a} = \begin{cases} \frac{1}{2\cos((a+b)/2)} \times \frac{\sin(an)}{\cos(an+(a+b)/2)}, & \text{if } n \text{ is even;} \\ \frac{-1}{2\cos((a+b)/2)} \times \frac{\sin(a(n+1)+b)}{\cos(an+(a+b)/2)}, & \text{if } n \text{ is odd.} \end{cases}$$

$$\sum_{i=1}^{n} \frac{(-1)^i \sin(2ai+b)}{\cos(2ai+b)-\cos a} = \begin{cases} \frac{1}{2\sin((a+b)/2)} \times \frac{\sin(an)}{\sin(an+(a+b)/2)}, & \text{if } n \text{ is even;} \\ \frac{1}{2\sin((a+b)/2)} \times \frac{\sin(a(n+1)+b)}{\sin(an+(a+b)/2)}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.3. *Let $a \neq 0$ and b be real numbers such that $\cos(2ai+b) \pm \cos a \neq 0$ for $i \geq 0$. Then,*

$$\sum_{i=0}^{n} \frac{\sin(2ai+b)}{(\cos(2ai+b)+\cos a)^2} = \frac{1}{4\sin a}\left(\frac{1}{\cos^2(an+(a+b)/2)} - \frac{1}{\cos^2((a-b)/2)}\right),$$

$$\sum_{i=0}^{n} \frac{\sin(2ai+b)}{(\cos(2ai+b)-\cos a)^2} = \frac{1}{4\sin a}\left(\frac{1}{\sin^2((a-b)/2)} - \frac{1}{\sin^2(an+(a+b)/2)}\right).$$

Theorem 2.4. *Let $a \neq 0$ and b be real numbers such that $\cos(2ai+b) \pm \cos a \neq 0$, and $\sin(2ai+b) \pm \sin a \neq 0$ for $i \geq 0$. Then,*

$$\sum_{i=0}^{n} \frac{\cos(2ai+b)}{\cos^2(2ai+b) - \cos^2 a} = \frac{1}{2\sin a}\left(\frac{1}{\sin(a(2n+1)+b)} + \frac{1}{\sin(a-b)}\right), \tag{2.1}$$

$$\sum_{i=0}^{n} \frac{(-1)^i \sin(2ai+b)}{\sin^2(2ai+b) - \sin^2 a} = \frac{1}{2\cos a}\left(\frac{(-1)^n}{\sin(a(2n+1)+b)} - \frac{1}{\sin(a-b)}\right), \tag{2.2}$$

$$\sum_{i=0}^{n} \frac{1}{\sin^2(2ai+b) - \sin^2 a} = \frac{1}{\sin(2a)\sin(b-a)} \times \frac{\sin(2a(n+1))}{\sin(a(2n+1)+b)}. \tag{2.3}$$

Theorem 2.5. *Let $k \neq 0$ be a real number that is not a rational multiple of π. Then,*

$$\sum_{i=1}^{n} \frac{1}{\sin(k2^i)} = \frac{1}{\sin 1}\left(\frac{\sin(k2^n - 1)}{\sin(k2^n)} - \frac{\sin(k-1)}{\sin k}\right). \tag{2.4}$$

For $k=1$, (2.4) becomes

$$\sum_{i=1}^{n} \frac{1}{\sin(2^i)} = \frac{\sin(2^n - 1)}{\sin 1 \sin(2^n)}.$$

3. A Sample Proof

In this section, we give a proof of (2.1). This proof is typical of the method by which all results in Section 2 can be proved.

Proof. We require three identities from high school trigonometry. These are

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right), \tag{3.1}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right), \tag{3.2}$$

$$\cos(2\alpha) - \cos(2\beta) = 2\left(\cos^2\alpha - \cos^2\beta\right). \tag{3.3}$$

Let $r(n,a,b)$ denote the right side of (2.1), and let $l(n,a,b)$ denote the left side of (2.1). Then,

$$\begin{aligned}
&r(n+1,a,b) - r(n,a,b)\\
&= \frac{1}{2\sin a}\left(\frac{1}{\sin(a(2n+3)+b)} - \frac{1}{\sin(a(2n+1)+b)}\right)\\
&= \frac{1}{2\sin a} \times \frac{\sin(a(2n+1)+b) - \sin(a(2n+3)+b)}{\sin(a(2n+3)+b)\sin(a(2n+1)+b)}\\
&= -\frac{\cos(2a(n+1)+b)}{\sin(a(2n+3)+b)\sin(a(2n+1)+b)} && \text{by (3.1)}\\
&= \frac{2\cos(2a(n+1)+b)}{\cos(4a(n+1)+2b) - \cos(2a)} && \text{by (3.2)}\\
&= \frac{\cos(2a(n+1)+b)}{\cos^2(2a(n+1)+b) - \cos^2 a} && \text{by (3.3)}\\
&= l(n+1,a,b) - l(n,a,b).
\end{aligned}$$

Likewise, we see that

$$r(0,a,b) = l(0,a,b) = \frac{\cos b}{\cos^2 b - \cos^2 a},$$

and this completes the proof of (2.1). □

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