CONNECTION COEFFICIENTS FOR HIGHER-ORDER BERNOULLI AND EULER POLYNOMIALS: A RANDOM WALK APPROACH

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ABSTRACT. We use random walks as an approach to obtain connection coefficients for higher-order Bernoulli and Euler polynomials. In particular, we study the cases of a 1-dimensional linear reflected Brownian motion and of a 3-dimensional Bessel process. By considering the successive hitting times of two, three, and four fixed levels of these random walks, we obtain non-trivial identities that involve higher-order Bernoulli and Euler polynomials.

1. INTRODUCTION

Different types of random walks have been studied in the literature, together with their connections to different fields of mathematics and physics; for a modern introduction, see, e.g., [6]. In this paper, we focus on two specific random walks: the 1-dimensional linear reflected Brownian motion and the 3-dimensional Bessel process. It would seem that there is no relation between these processes and special functions such as Bernoulli and Euler polynomials that appear mostly in number theory and combinatorics. However, we will show how the study of the hitting times of these two processes allows us to derive non-elementary identities for higher-order Bernoulli and Euler polynomials, denoted by $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$, respectively. They are defined through their generating functions as

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n\geq 0} B_n^{(p)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n\geq 0} E_n^{(p)}(x)\frac{t^n}{n!}. \tag{1.1}$$

The special case $p = 1$ yields the usual Bernoulli and Euler polynomials: $B_n^{(1)}(x) = B_n(x)$ and $E_n^{(1)}(x) = E_n(x)$; in addition, Bernoulli and Euler numbers are the special evaluations $B_n = B_n(0)$ and $E_n = 2^n E_n(1/2)$. See, e.g., [8, Chpt. 24] for their properties.

One of the key tools of this work is the interpretation of the polynomials $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$ as probabilistic moments of certain random variables that are related to the hitting times of some random walks, which will be introduced in Section 2.

This study arises from early work on the higher-order Euler polynomials. In a previous article [5, eq. 3.8, p. 781], we obtained the following expansion for the usual Euler polynomials as a linear combination of higher-order Euler polynomials: for any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{l\geq N} p_l^{(N)} E_n^{(l)}\left(\frac{l-N}{2} + Nx\right), \tag{1.2}$$

where the positive coefficients $p_l^{(N)}$ are defined by the generating function

$$\frac{1}{T_N(1/t)} = \sum_{l\geq 0} p_l^{(N)} t^l,$$

via the N-th Chebychev polynomial $T_N(t)$ [8, Table 18.3.1]. These coefficients $p_l^{(N)}$ also appear as transition probabilities in the context of a random walk over a finite number of sites [5,

Note 4.8, p. 787]. This characterization provides an alternate interpretation of (1.2) in terms of a Markov process.

For the two random walks considered in this paper, we shall focus on their consecutive hitting times: namely, for a process starting from the origin at time $t = 0$, we consider the first epochs — called hitting times — at which the process reaches successive levels $0 = a_0 < a_1 < a_2 < \cdots < a_N$. In particular, we study some special cases for $N = 2$, 3, and 4 levels. From the moment generating functions of these hitting times, we derive identities involving $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$, which will be presented in Section 3 and Section 4.

2. Probabilistic Preliminaries and Principle

2.1. **The symbols $\mathcal{B}$, $\mathcal{E}$ and $\mathcal{U}$.** We shall make full use of the classical umbral symbols $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{U}$, defined as follows — for an introduction to the classical umbral calculus, see, for example, [4, 9]. The Bernoulli symbol $\mathcal{B}$ satisfies the evaluation rule

$$(x + \mathcal{B})^n = B_n(x). \tag{2.1}$$

Equivalently, this symbol $\mathcal{B}$ can be interpreted as a random variable [3, Thm. 2.3, p. 384]: if L_B is a random variable distributed according to the squared hyperbolic secant density $p_B(t) = \pi \operatorname{sech}^2(\pi t)/2$, then, for any suitable function f, with $i^2 = -1$,

$$f(x + \mathcal{B}) = \mathbb{E}\left[f\left(x + iL_B - \frac{1}{2}\right)\right] = \int_{\mathbb{R}} f\left(x + it - \frac{1}{2}\right) p_B(t)\mathrm{d}t.$$

In particular, choosing $f(x) = x^n$ produces the Bernoulli polynomial (2.1), so that $\mathcal{B}$ coincides with the random variable $iL_B - 1/2$. More generally, the p-th order Bernoulli polynomial can be expressed as

$$B_n^{(p)}(x) = (x + \mathcal{B}^{(p)})^n,$$

where $\mathcal{B}^{(p)} := \mathcal{B}_1 + \cdots + \mathcal{B}_p$, for a set of p independent umbral symbols (or random variables) $(\mathcal{B}_i)_{1 \leq i \leq p}$ satisfying the two following evaluation rules:

1: if $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent symbols then

$$\mathcal{B}_1^n \mathcal{B}_2^m = B_n B_m;$$

2: if $\mathcal{B}_1 = \mathcal{B}_2$, then

$$\mathcal{B}_1^n \mathcal{B}_2^m = \mathcal{B}_1^{n+m} = B_{n+m}.$$

From the generating function (1.1), we deduce that

$$e^{\mathcal{B}t} = \frac{t}{e^t - 1}, \quad e^{t(2\mathcal{B}+1)} = \frac{t}{\sinh t}, \quad \text{and} \quad e^{t(2\mathcal{B}^{(p)}+p)} = \left(\frac{t}{\sinh t}\right)^p.$$

Similarly, we let L_E be a random variable distributed according to the hyperbolic secant density $p_E(t) = \operatorname{sech}(\pi t)$, and define the Euler symbol $\mathcal{E}$ by

$$f(x + \mathcal{E}) = \mathbb{E}\left[f\left(x + iL_E - \frac{1}{2}\right)\right] = \int_{\mathbb{R}} f\left(x + it - \frac{1}{2}\right) p_E(t)\mathrm{d}t.$$

Then, denoting $\mathcal{E}^{(p)} = \mathcal{E}_1 + \cdots + \mathcal{E}_p$, the sum of p independent Euler symbols, the Euler polynomial of order p is expressed as

$$E_n^{(p)}(x) = (x + \mathcal{E}^{(p)})^n. \tag{2.2}$$

Also, we have

$$e^{t\mathcal{E}} = \frac{2}{e^t + 1}, \quad e^{t(2\mathcal{E}+1)} = \operatorname{sech} t, \quad \text{and} \quad e^{t(2\mathcal{E}^{(p)}+p)} = \operatorname{sech}^p t.$$

From the generating function

$$e^{2\mathcal{B}t} = \frac{2t}{e^{2t}-1} = \frac{t}{e^t-1} \cdot \frac{2}{e^t+1} = e^{t(\mathcal{B}+\mathcal{E})},$$

we deduce that the symbols $\mathcal{B}$ and $\mathcal{E}$ satisfy the rule

$$2\mathcal{B} = \mathcal{B} + \mathcal{E}, \tag{2.3}$$

in the sense that for any suitable function f,

$$f(x + 2\mathcal{B}) = f(x + \mathcal{B} + \mathcal{E}).$$

The third and final special symbol that is useful to us is the uniform symbol $\mathcal{U}$ with the evaluation rule

$$\mathcal{U}^n = \frac{1}{n+1}.$$

It can be interpreted as a random variable U uniformly distributed over the interval $[0,1]$, namely, for any suitable function f,

$$f(x + \mathcal{U}) = \mathbb{E}[f(x + U)] = \int_0^1 f(x+t)\mathrm{d}t.$$

Thus, we have in terms of generating functions

$$e^{t\mathcal{U}} = \sum_{n\geq 0} \frac{t^n}{(n+1)!} = \frac{e^t-1}{t}, \qquad e^{t(2\mathcal{U}-1)} = \frac{\sinh t}{t}, \qquad \text{and} \qquad e^{t(2\mathcal{U}^{(p)}-p)} = \left(\frac{\sinh t}{t}\right)^p,$$

where $\mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p$ denotes the sum of p independent uniform symbols.

An important link between the Bernoulli symbol $\mathcal{B}$ and the uniform symbol $\mathcal{U}$ is deduced from the identity

$$e^{t(\mathcal{U}+\mathcal{B})} = e^{t\mathcal{U}}e^{t\mathcal{B}} = \frac{t}{e^t-1} \cdot \frac{e^t-1}{t} = 1;$$

this shows that, for any suitable function f,

$$f(x + \mathcal{B} + \mathcal{U}) = f(x), \tag{2.4}$$

so that the actions of these two symbols cancel each other.

In what follows, we will use independent copies of Bernoulli, Euler and uniform symbols. In order to distinguish them, we shall denote independent uniform symbols by $\mathcal{U}, \mathcal{U}', \dots$ and $\mathcal{U}^{(p)}, \mathcal{U}'^{(p)}, \dots$ and similarly for the other two symbols.

2.2. Level sites with 1 loop and 2 loops.

The general setting of this paper is a random walk starting from the origin (in $\mathbb{R}$ or $\mathbb{R}^3$) and hitting some defined levels — either some points on the line or some sphere in the 3-dimensional space, called sites. Looping back and forth, each random walk can reach each site multiple times, and we are interested in the first time the process reaches each of these sites. Before we derive identities, we shall first consider the contribution of loops to the hitting times of these sites.

Consider one possible loop between sites a and b, with $a < b$. Let $\phi_{a\to b}$ be the moment generating function of the hitting time of site b starting from site a; also let $\phi_{b\to a}$ be the counterpart from b to a of $\phi_{a\to b}$. Let us further denote

$$I_{a,b} = \phi_{a\to b}\phi_{b\to a};$$

the random walk can loop an arbitrary number of times $k \geq 0$ between sites a and b so that the overall contribution of these visits to the generating function is

$$\sum_{k \geq 0} I_{a,b}^k = \frac{1}{1 - I_{a,b}}. \tag{2.5}$$

Next, consider one possible loop between sites a and b, and another loop between sites c and d, with $a < b \leq c < d$. The possible contributions are as follows:

- k loops between sites a and b, followed by l loops between sites c and d, with $k, l = 0, 1, \ldots$, contributing

$$\sum_{k,\, l \geq 0} I_{a,b}^k I_{c,d}^l = \frac{1}{1 - I_{a,b}} \cdot \frac{1}{1 - I_{c,d}};$$

- k_1 loops between sites a and b, followed by l_1 loops between sites c and d, then followed by k_2 loops between sites a, and b and finally l_2 loops between sites c and d, with $k_1,\, l_2$ nonnegative and $k_2,\, l_1$ positive, contributing (by letting $s = k_1 = l_2$ and $t = k_2 = l_1$)

$$\sum_{s \geq 0,\, t \geq 1} I_{a,b}^s I_{c,d}^t I_{a,b}^t I_{c,d}^s = \frac{I_{a,b} I_{c,d}}{(1 - I_{a,b})^2 (1 - I_{c,d})^2};$$

- the general term will consist of k_1 loops between sites a and b, followed by l_1 loops between sites c and d and so on, followed by k_n loops between sites a and b and l_n loops between sites c and d, with $k_1,\, l_n$ nonnegative and the other indices being positive, contributing

$$\frac{I_{a,b}^{n-1} I_{c,d}^{n-1}}{(1 - I_{a,b})^n (1 - I_{c,d})^n}$$

to the generating function.

Therefore, the overall contribution of all possible loops over sites a and b, followed by loops over sites c and d is

$$\sum_{n \geq 1} \frac{I_{a,b}^{n-1} I_{c,d}^{n-1}}{(1 - I_{a,b})^n (1 - I_{c,d})^n} = \frac{1}{1 - (I_{a,b} + I_{c,d})} = \sum_{k \geq 0} (I_{a,b} + I_{c,d})^k. \tag{2.6}$$

3. One-Dimensional Reflected Brownian Motion

3.1. **Introduction.** Consider the 1-dimensional reflected Brownian motion on $\mathbb{R}_+$. For simplicity, we let

- $\phi_{r \to s}(z)$ be the generating function of the hitting time of site s starting from site r;
- $\phi_{r \to s | t}(z)$ be the generating function of the hitting time of site s starting from site r without reaching site t.

In the case of three consecutive sites a, b and c with $0 < a < b < c$, the generating functions of the corresponding hitting times can be found in [1, p. 198 and p. 355]: with $w = \sqrt{2z}$,

$$\phi_{a \to b}(z) = \frac{\cosh(aw)}{\cosh(bw)}, \tag{3.1}$$

$$\phi_{b \to a | c}(z) = \frac{\sinh((c - b)w)}{\sinh((c - a)w)}, \tag{3.2}$$

$$\phi_{b \to c | a}(z) = \frac{\sinh((b - a)w)}{\sinh((c - a)w)}. \tag{3.3}$$

Let us now label $(N + 1)$ sites on the real positive line as $0 = a_0 < a_1 < a_2 < \cdots < a_N$.

3.2. **The case of** 3 **sites.** We consider here 3 sites $0 = a_0 < a_1 < a_2$; the study of hitting times yields the following identity.

Theorem 3.1. *Let $0 < a_1 < a_2$, then for any positive integer n, the expansion of Euler polynomials as a convex combination of higher-order Bernoulli polynomials is given by*

$$E_n \left(\frac{x}{2a_2} + \frac{3}{2} - 2\frac{a_1}{a_2} \right) - E_n \left(\frac{x}{2a_2} + \frac{1}{2} \right) = K_n \sum_{k \geq 0} p_k B_n^{(k+1)} \left(\frac{x}{4a_1} + \frac{a_2}{4a_1} + \frac{k}{2} \right) \tag{3.4}$$

where $K_n = (n+1)(1 - 2a_1/a_2)2^n a_1^n / a_2^n$ and the coefficients

$$p_k = \frac{a_1}{a_2} \left(1 - \frac{a_1}{a_2} \right)^k, \quad k \geq 0$$

are the probability weights of a geometric distribution with parameter a_1/a_2.

Proof. From (3.1)–(3.3), we have

$$\phi_{0 \to a_1}(z) = \operatorname{sech}(a_1 w), \qquad \phi_{0 \to a_2}(z) = \operatorname{sech}(a_2 w),$$

$$\phi_{a_1 \to a_2 | 0}(z) = \frac{\sinh(a_1 w)}{\sinh(a_2 w)}, \qquad \phi_{a_1 \to 0 | a_2}(z) = \frac{\sinh((a_2 - a_1)w)}{\sinh(a_2 w)}.$$

The process includes one possible loop between sites 0 and a_1, so that by (2.5), we have

$$\phi_{0 \to a_2}(z) = \phi_{0 \to a_1}(z) \phi_{a_1 \to a_2 | 0}(z) \sum_{k \geq 0} \left(\phi_{0 \to a_1}(z) \phi_{a_1 \to 0 | a_2}(z) \right)^k$$

$$= \operatorname{sech}(a_1 w) \cdot \frac{\sinh(a_1 w)}{\sinh(a_2 w)} \sum_{k \geq 0} \left(\operatorname{sech}(a_1 w) \frac{\sinh((a_2 - a_1)w)}{\sinh(a_2 w t)} \right)^k \tag{3.5}$$

$$= \frac{\sinh(a_1 w)}{\cosh(a_1 w) \sinh(a_2 w)} \cdot \frac{1}{1 - \frac{\sinh((a_2 - a_1)w)}{\cosh(a_1 w) \sinh(a_2 w)}}$$

$$= \frac{\sinh(a_1 w)}{\cosh(a_1 w) \sinh(a_2 w) - \sinh((a_2 - a_1)w)},$$

which coincides with $\phi_{0 \to a_2}(z) = \operatorname{sech}(a_2 w)$, since

$$\sinh((a_2 - a_1)w) = \sinh(a_2 w) \cosh(a_1 w) - \cosh(a_2 w) \sinh(a_1 w).$$

Meanwhile, we transform (3.5) into a form involving $\mathcal{B}$, $\mathcal{U}$ and $\mathcal{E}$ as follows:

$$e^{a_2 w(2\mathcal{E}+1)} = \frac{a_1}{a_2} e^{a_1 w(2\mathcal{U}-1)} e^{a_2 w(2\mathcal{B}+1)}$$

$$\times \sum_{k \geq 0} e^{a_1 w \left(2\mathcal{E}^{(k+1)} + k + 1 \right)} \left(\frac{a_2 - a_1}{a_2} \right)^k e^{(a_2 - a_1)w(2\mathcal{U}^{(k)} - k)} e^{a_2 w(2\mathcal{B}^{(k)} + k)}$$

$$= e^{a_1 w(2\mathcal{U}-1) + a_2 w(2\mathcal{B}+1)}$$

$$\times \sum_{k \geq 0} p_k e^{a_1 w(2\mathcal{E}^{(k+1)} + k + 1) + (a_2 - a_1)w(2\mathcal{U}^{(k)} - k) + a_2 w(2\mathcal{B}^{(k)} + k)}.$$

Upon multiplying both sides by e^{xw} and comparing coefficients of w^n, we get

$$(x + 2a_2\mathcal{E} + a_2)^n = \sum_{k \geq 0} p_k \left[x + a_1(2\mathcal{U} - 1) + a_2(2\mathcal{B} + 1) + a_1 \left(2\mathcal{E}^{(k+1)} + k + 1 \right) \right.$$

$$+(a_2 - a_1)\left(2\mathcal{U}^{(k)} - k\right) + a_2\left(2\mathcal{B}^{(k)} + k\right)\Big]^n$$

$$= \sum_{k \geq 0} p_k \left(x + 2(a_2 - a_1)\mathcal{B} + 2a_1\mathcal{E}^{(k+1)} + 2a_1\mathcal{B}^{(k)} + a_2 + 2a_1 k\right)^n,$$

where, in the last step, we have used the fact that $\mathcal{B}$ and $\mathcal{U}$ cancel each other according to (2.4). Now, the substitution $x \mapsto x + 2(a_2 - 2a_1)\mathcal{U}$ yields

$$\left(x + 2(a_2 - 2a_1)\mathcal{U} + 2a_2\mathcal{E} + a_2\right)^n = \sum_{k \geq 0} p_k \left(x + 2a_1\mathcal{E}^{(k+1)} + 2a_1\mathcal{B}^{(k+1)} + a_2 + 2a_1 k\right)^n.$$

Here, the left-hand side can be computed as

$$2^n a_2^n \left(\mathcal{E} + \frac{x}{2a_2} + \frac{1}{2} + \left(1 - 2\frac{a_1}{a_2}\right)\mathcal{U}\right)^n = 2^n a_2^n E_n\left(\frac{x}{2a_2} + \frac{1}{2} + \left(1 - 2\frac{a_1}{a_2}\right)\mathcal{U}\right)$$

$$= 2^n a_2^n \int_0^1 E_n\left(\frac{x}{2a_2} + \frac{1}{2} + \left(1 - 2\frac{a_1}{a_2}\right)t\right) \mathrm{d}t$$

$$= \frac{2^n a_2^n \left[E_n\left(\frac{x}{2a_2} + \frac{3}{2} - 2\frac{a_1}{a_2}\right) - E_n\left(\frac{x}{2a_2} + \frac{1}{2}\right)\right]}{\left(1 - 2\frac{a_1}{a_2}\right)(n+1)},$$

while, by (2.3) the right-hand side is

$$\sum_{k \geq 0} p_k \left(x + 2a_1\mathcal{E}^{(k+1)} + 2a_1\mathcal{B}^{(k+1)} + a_2 + 2a_1 k\right)^n = \sum_{k \geq 0} p_k \left(x + 4a_1\mathcal{B}^{(k+1)} + a_2 + 2a_1 k\right)^n$$

$$= \sum_{k \geq 0} p_k 4^n a_1^n B_n^{(k+1)}\left(\frac{x}{4a_1} + \frac{a_2}{4a_1} + \frac{k}{2}\right).$$

Further simplification completes the proof. $\qquad\square$

Remark 3.2. *For uniformly spaced levels $a_1 = a_2/2 = a/2$, identity (3.4) collapses to the trivial identity $0 = 0$.*

3.3. The case of 4 sites. We consider now the case of four sites $0 = a_0 < a_1 < a_2 < a_3$ with two possible loops, one between a_0 and a_1 and one between a_1 and a_2. For the sake of simplicity, we consider only the case where all sites are uniformly distributed, with $a_i = i$ for $i = 0, 1, 2, 3$, and obtain the following identity.

Theorem 3.3. *For any positive integer n, the Euler polynomial of degree n can be expressed as a linear combination of higher-order Euler polynomials of the same degree as*

$$E_n(x) = \sum_{k \geq 0} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}(3x + k). \tag{3.6}$$

Proof. By (2.6) and the basic identity $\sinh(2w) = 2\sinh(w)\cosh(w)$, we have

$$\phi_{0 \to 3}(z) = \phi_{0 \to 1}(z)\phi_{1 \to 2|0}(z)\phi_{2 \to 3|1}(z) \sum_{k \geq 0} \left(\phi_{0 \to 1}(z)\phi_{1 \to 0|2}(z) + \phi_{1 \to 2|0}(z)\phi_{2 \to 1|3}(z)\right)^k$$

$$= \operatorname{sech}(w)\left(\frac{\sinh(w)}{\sinh(2w)}\right)^2 \sum_{k \geq 0} \left(\operatorname{sech}(w)\frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)}\right)^k$$

$$= \sum_{k \geq 0} \frac{1}{4}\left(\frac{3}{4}\right)^k \operatorname{sech}^{2k+3}(w).$$

Therefore, we deduce

$$e^{3w(2\mathcal{E}+1)} = \sum_{k\geq 0} \frac{3^k}{4^{k+1}} e^{w(2\mathcal{E}^{(2k+3)}+2k+3)},$$

namely,

$$e^{6w\mathcal{E}} = \sum_{k\geq 0} \frac{3^k}{4^{k+1}} e^{2w\mathcal{E}^{(2k+3)}+2wk}.$$

Multiplying both sides by e^{xw} and comparing coefficients of w^n, we obtain

$$(x + 6\mathcal{E})^n = \sum_{k\geq 0} \frac{3^k}{4^{k+1}} \left(x + 2k + 2\mathcal{E}^{(2k+3)} \right)^n,$$

where the left-hand side is

$$(x + 6\mathcal{E})^n = 6^n \left(\mathcal{E} + \frac{x}{6} \right)^n = 6^n E_n\left(\frac{x}{6} \right),$$

and the right-hand side is

$$\sum_{k\geq 0} \frac{3^k \left(x + 2k + 2\mathcal{E}^{(2k+3)} \right)^n}{4^{k+1}} = \sum_{k\geq 0} \frac{3^k 2^n \left(\frac{x}{2} + k + \mathcal{E}^{(2k+3)} \right)^n}{4^{k+1}} = \sum_{k\geq 0} \frac{3^k}{4^{k+1}} 2^n E_n^{(2k+3)} \left(\frac{x}{2} + k \right).$$

Simplification completes the proof. □

In the general case of 4 sites that are not uniformly spaced, we obtain a much more complicated result as follows:

Theorem 3.4. *For any $0 < a_1 < a_2 < a_3$ and arbitrary positive integer n, we have*

$$E_n\left(\frac{x}{2a_3} + \frac{1}{2} \right) = \sum_{k\geq l\geq 0} q_{k,l} \left(\frac{a_1}{2a_3} \right)^n E_n^{(l)} \left(\frac{x}{a_1} + 2\frac{a_2 - a_1}{a_1}\mathcal{B} + 2\frac{a_3 - a_2}{a_1}\mathcal{B}' + 2\frac{a_2 - a_1}{a_1}\mathcal{U}^{(l)} \right.$$

$$\left. + 2\mathcal{U}'^{(k-l)} + 2\frac{a_2 - a_1}{a_1}\mathcal{B}'^{(k-l)} + \frac{r_{k,l}}{a_1} \right),$$

with the coefficients

$$q_{k,l} = \binom{k}{l} \frac{(a_2 - a_1)^{l+1} a_1^{k-l+1} (a_3 - a_2)^{k-l}}{a_2^{k+1} (a_3 - a_1)^{k-l+1}},$$

and

$$r_{k,l} = a_3 + (2k - 2l)a_2 + (3l - k + 1)a_1.$$

Proof. Apply (2.6) to obtain

$$\phi_{0\to a_3}(z) = \phi_{0\to a_1}(z)\phi_{a_1\to a_2|0}(z)\phi_{a_2\to a_3|a_1}(z)$$

$$\times \sum_{k\geq 0} \left(\phi_{0\to a_1}(z)\phi_{a_1\to 0|a_2}(z) + \phi_{a_1\to a_2|0}(z)\phi_{a_2\to a_1|a_3}(z) \right)^k$$

$$= \operatorname{sech}(a_1 w) \cdot \frac{\sinh(a_1 w)}{\sinh(a_2 w)} \cdot \frac{\sinh((a_2 - a_1)w)}{\sinh((a_3 - a_1)w)}$$

$$\times \sum_{k\geq 0} \left(\operatorname{sech}(a_1 w) \cdot \frac{\sinh((a_2 - a_1)w)}{\sinh(a_2 w)} + \frac{\sinh(a_1 w)}{\sinh(a_2 w)} \cdot \frac{\sinh((a_3 - a_2)w)}{\sinh((a_3 - a_1)w)} \right)^k$$

$$= \sum_{k \geq 0} \frac{1}{\sinh^{k+1}(a_2 w)} \sum_{l=0}^{k} \binom{k}{l} \operatorname{sech}^{l+1}(a_1 w) \sinh^{l+1}((a_2 - a_1)w)$$

$$\times \frac{\sinh^{k-l+1}(a_1 w) \sinh^{k-l}((a_3 - a_2)w)}{\sinh^{k-l+1}((a_3 - a_1)w)}.$$

Now, we multiply both sides by e^{xw} and compare coefficients of w^n, with the notation

$$q_{k,l} = \binom{k}{l} \frac{(a_2 - a_1)^{l+1} a_1^{k-l+1}(a_3 - a_2)^{k-l}}{a_2^{k+1}(a_3 - a_1)^{k-l+1}}$$

and

$$r_{k,l} = a_3 + (2k - 2l)a_2 + (3l - k + 1)a_1,$$

to obtain

$$(x + 2a_3 \mathcal{E} + a_3)^n = \sum_{k \geq l \geq 0} q_{k,l} \left[x + 2(a_2 - a_1)\mathcal{B} + 2(a_3 - a_2)\mathcal{B}' + a_1 \mathcal{E}^{(l)} + 2(a_2 - a_1)\mathcal{U}^{(l)} \right.$$

$$\left. + 2a_1 \mathcal{U}'^{(k-l)} + 2(a_2 - a_1)\mathcal{B}'^{(k-l)} + r_{k,l} \right]^n.$$

Applying (2.2) yields the result. $\qquad\square$

4. BESSEL PROCESS IN $\mathbb{R}^3$

4.1. Introduction.
We consider now a Bessel process in $\mathbb{R}^3$. Using similar notations $\phi_{a \to b}(z)$ and $\phi_{a \to b | c}(z)$ as in the previous section and considering sites labelled a, b, and c with $a < b < c$, that are now concentric spheres of radii a, b, and c, we will need the following formulas from [1, pp. 463–464]:

$$\phi_{a \to b}(z) = \frac{b \sinh(aw)}{a \sinh(bw)}, \tag{4.1}$$

$$\phi_{b \to c | a}(z) = \frac{c \sinh((b - a)w)}{b \sinh((c - a)w)}, \tag{4.2}$$

$$\phi_{b \to a | c}(z) = \frac{a \sinh((c - b)w)}{c \sinh((c - a)w)}. \tag{4.3}$$

Remark 4.1. *One can easily check that*

$$\phi_{a_1 \to 0 | a_2}(z) = \frac{0 \cdot \sinh((a_2 - a_1)w)}{a_2 \sinh((a_2 - 0)w)} = 0,$$

so that the first possible loop is now between a_1 and a_2, unlike in the 1-dimensional case, where the first possible loop was between the origin 0 and the first site a_1.

4.2. The case $N = 3$.
In the case of three concentric spheres of arbitrary radii, we deduce the following expression of a Bernoulli polynomial of degree n in terms of higher-order Bernoulli polynomials of the same degree.

Theorem 4.2. *For $0 < a_1 < a_2 < a_3$ and a positive integer n, we have*

$$B_n \left(\frac{x + a_3}{2a_3} \right) = \left(\frac{a_2}{a_3} \right)^n \sum_{k \geq 0} p_k B_n^{(k+1)} \left[\frac{x}{2a_2} + \beta_k + \frac{a_2 - a_1}{a_2} \mathcal{U} + \frac{a_3 - a_2}{a_2} \mathcal{U}^{(k)} + \frac{a_1}{a_2} \mathcal{U}'^{(k)} \right.$$

$$\left. + \frac{a_3 - a_1}{a_2} \mathcal{B}^{(k+1)} \right]^n,$$

with

$$p_k = (1 - \alpha)\alpha^k, \ \ \beta_k = \frac{a_3 + 2ka_2 - 2ka_1}{2a_3},$$

and

$$\alpha = \frac{(a_3 - a_2)a_1}{(a_3 - a_1)a_2}.$$

The special case of uniformly spaced radii, where $a_i = i$ for $i = 1, 2, 3$, is as follows.

Corollary 4.3. *For positive integer n, we have*

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6} + \frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6} + \frac{1}{2}\right)\right] = \frac{3}{4}\sum_{k \geq 0}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x + 3 + 2k}{2}\right). \qquad (4.4)$$

The specialization of this identity to the case $x = 0$ and n odd is as follows.

Corollary 4.4. *The even-index Bernoulli number B_{2m} can be expressed as a linear combination of higher-order Euler polynomials as*

$$B_{2m} = \frac{m}{(1 - 2^{1-2m})(3^{2m} - 1)}\sum_{k \geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k + \frac{3}{2}\right).$$

Proof. Take $x = 0$ and $n = 2m - 1$ in (4.4). By Entries 24.4.27 and 24.4.32 of [8], the left-hand side becomes

$$\frac{3^{2m}}{2m}\left[B_{2m}\left(\frac{5}{6}\right) - B_{2m}\left(\frac{1}{2}\right)\right] = \frac{3^{2m}}{2m}\left[\frac{1}{2}(1 - 2^{1-2m})(1 - 3^{1-2m})B_{2m} + (1 - 2^{1-2m})B_{2m}\right]$$

$$= \frac{3^{2m}}{2m}(1 - 2^{1-2m})B_{2m}\left(\frac{1 - 3^{1-2m}}{2} + 1\right)$$

$$= \frac{3}{4m}(1 - 2^{1-2m})(3^{2m} - 1)B_{2m},$$

while the right-hand side is

$$\sum_{k \geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k + \frac{3}{2}\right).$$

After simplification, we obtain the desired result. $\qquad \square$

Proof of Theorem 4.2. Remarking that $\lim_{x \to 0}[\sinh(xt)/x] = t$, from (4.1)–(4.3), we have

$$\phi_{a_0 \to a_1}(z) = \frac{a_1 w}{\sinh(a_1 w)}, \quad \phi_{a_0 \to a_3}(z) = \frac{a_3 w}{\sinh(a_3 w)}, \quad \phi_{a_1 \to a_2|a_0}(z) = \frac{a_2}{a_1} \cdot \frac{\sinh(a_1 w)}{\sinh(a_2 w)},$$

$$\phi_{a_2 \to a_1|a_3}(z) = \frac{a_1}{a_2} \cdot \frac{\sinh((a_3 - a_2)w)}{\sinh((a_3 - a_1)w)}, \quad \phi_{a_2 \to a_3|a_1}(z) = \frac{a_3}{a_2} \cdot \frac{\sinh((a_2 - a_1)w)}{\sinh((a_3 - a_1)w)},$$

so that

$$\phi_{a_0 \to a_3}(z) = \sum_{k \geq 0}\phi_{a_0 \to a_1}(z)\phi_{a_1 \to a_2|a_0}(z)\left(\phi_{a_2 \to a_1|a_3}(z)\phi_{a_1 \to a_2|a_0}(z)\right)^k\phi_{a_2 \to a_3|a_1}(z)$$

$$= \left(\frac{a_1 w}{\sinh(a_1 w)}\right)\left(\frac{a_2 \sinh(a_1 w)}{a_1 \sinh(a_2 w)}\right)\left(\frac{a_3 \sinh((a_2 - a_1)w)}{a_2 \sinh((a_3 - a_1)w)}\right)$$

$$\times \sum_{k \geq 0}\left(\frac{\sinh((a_3 - a_2)w)}{\sinh((a_3 - a_1)w)}\frac{\sinh(a_1 w)}{\sinh(a_2 w)}\right)^k$$

$$=a_3 w \sinh((a_2 - a_1)w) \sum_{k \geq 0} \frac{\sinh^k((a_3 - a_2)w) \sinh^k(a_1 w)}{\sinh^{k+1}((a_3 - a_1)w) \sinh^{k+1}(a_2 w)}.$$

Then, we obtain

$$e^{a_3 w (2\mathcal{B}+1)} = e^{w(a_2 - a_1)(2\mathcal{U}-1)} \sum_{k \geq 0} \left\{ r_k e^{w\left[(a_3 - a_2)(2\mathcal{U}^{(k)} - k) + a_1(2\mathcal{U}'^{(k)} - k)\right]} \right. \tag{4.5}$$

$$\left. \times e^{w\left[(a_3 - a_1)(2\mathcal{B}^{(k+1)} + k + 1) + a_2(2\mathcal{B}'^{(k+1)} + k + 1)\right]} \right\}$$

where

$$r_k = \frac{a_3(a_2 - a_1)(a_3 - a_2)^k a_1^k}{(a_3 - a_1)^{k+1} a_2^{k+1}}.$$

In analogy to the proof of Theorem 3.1, we multiply both sides of (4.5) by e^{wx} and look at the coefficients of w^n. For simplicity, we let

$$s_k = a_3 + 2k a_2 - 2k a_1,$$

so that

$$(2a_3)^n B_n \left(\frac{x + a_3}{2a_3} \right) = \sum_{k \geq 0} r_k \left[x + 2a_3 \mathcal{B} + a_3 + s_k + 2(a_2 - a_1)\mathcal{U} + 2(a_3 - a_2)\mathcal{U}^{(k)} \right.$$

$$\left. + 2a_1 \mathcal{U}'^{(k)} + 2(a_3 - a_1)\mathcal{B}^{(k+1)} + 2a_2 \mathcal{B}'^{(k+1)} \right]^n.$$

The special case $a_i = i$, with

$$\beta_k = s_k \big|_{a_i = i} = 3 + 2k \quad \text{and} \quad \rho_k = r_k \big|_{a_i = i} = \frac{3}{4} \left(\frac{1}{4} \right)^k,$$

produces

$$6^n B_n \left(\frac{x + 3}{6} \right) = \sum_{k \geq 0} \rho_k \left(x + 3 + 2k + 2\mathcal{U} + 2\mathcal{U}^{(k)} + 2\mathcal{U}'^{(k)} + 4\mathcal{B}^{(k+1)} + 4\mathcal{B}'^{(k+1)} \right)^n$$

$$= \sum_{k \geq 0} \rho_k \left(x + 3 + 2k + 2\mathcal{U}^{(2k+1)} + 4\mathcal{B}^{(2k+2)} \right)^n$$

$$= \sum_{k \geq 0} \rho_k \left(x + 3 + 2k + 2\mathcal{U}^{(2k+1)} + 2\mathcal{B}^{(2k+2)} + 2\mathcal{E}^{(2k+2)} \right)^n$$

$$= \sum_{k \geq 0} \rho_k \left(x + 3 + 2k + 2\mathcal{B} + 2\mathcal{E}^{(2k+2)} \right)^n$$

$$= \sum_{k \geq 0} \rho_k 2^n E_n^{(2k+2)} \left(\frac{x + 3 + 2k}{2} + \mathcal{B} \right),$$

namely,

$$3^n B_n \left(\frac{x}{6} + \frac{1}{2} \right) = \sum_{k \geq 0} \rho_k E_n^{(2k+2)} \left(\frac{x + 3 + 2k}{2} + \mathcal{B} \right).$$

Replacing x by $x + 2\mathcal{U}$ completes the proof. $\qquad \square$

4.3. **The case** $N = 4$. We have now two possible loops among the sites with radii a_2, a_3 and a_4. Following the notation in Subsection 2.2, we have

$$I_{a_1,a_2} = \phi_{a_1 \to a_2 | a_0}(z) \phi_{a_2 \to a_1 | a_3}(z) = \frac{\sinh(a_1 w) \sinh((a_3 - a_2)w)}{\sinh(a_2 w) \sinh((a_3 - a_1)w)}$$

and similarly,

$$I_{a_2,a_3} = \phi_{a_2 \to a_3 | a_1}(z) \phi_{a_3 \to a_2 | a_4}(z) = \frac{\sinh((a_2 - a_1)w) \sinh((a_4 - a_3)w)}{\sinh((a_3 - a_1)w) \sinh((a_4 - a_2)w)}.$$

In order to obtain a simple expression for $I_{a_1,a_2} + I_{a_2,a_3}$, we further assume that the levels are uniformly distributed, i.e., $a_i = i$, for $i = 0, 1, 2, 3, 4$, so that

$$I_{a_1,a_2} + I_{a_2,a_3} = I_{1,2} + I_{2,3} = \frac{2 \sinh^2(w)}{\sinh^2(2w)} = \frac{\text{sech}^2(w)}{2}.$$

This produces the following identity.

Theorem 4.5. *For any positive integer n, we have*

$$\frac{B_{n+1}\left(\frac{x+4}{8}\right) - B_{n+1}\left(\frac{x}{8}\right)}{n+1} = \frac{1}{2^{2n+1}} \sum_{k \geq 0} \frac{1}{2^{k+1}} E_n^{(2k+2)}\left(\frac{x}{2} + k\right). \tag{4.6}$$

Proof. From

$$\phi_{0 \to 4}(z) = \phi_{0 \to 1}(z) \phi_{1 \to 2 | 0}(z) \phi_{2 \to 3 | 1}(z) \phi_{3 \to 4 | 2}(z) \sum_{k \geq 0} (I_{1,2} + I_{2,3})^k,$$

namely,

$$\frac{4w}{\sinh(4w)} = \frac{w}{\sinh w} \cdot \frac{2 \sinh w}{\sinh(2w)} \cdot \frac{3 \sinh w}{2 \sinh(2w)} \cdot \frac{4 \sinh w}{3 \sinh(2w)} \sum_{k \geq 0} \frac{\text{sech}^{2k}(w)}{2^k}$$

$$= \frac{w}{\sinh(2w)} \sum_{k \geq 0} \frac{\text{sech}^{2k+2}(w)}{2^k},$$

we deduce that

$$e^{4w(2\mathcal{B}+1)} = \frac{1}{2} e^{2w(2\mathcal{B}'+1)} \sum_{k \geq 0} \frac{1}{2^k} e^{w\left(2\mathcal{E}^{(2k+2)} + 2k + 2\right)}.$$

After multiplying by e^{xw}, applying the change of variable $x \mapsto x - 4$, and identifying the coefficients of w^n on both sides, we have

$$(x + 8\mathcal{B})^n = \sum_{k \geq 0} \frac{\left(x + 4\mathcal{B}' + 2\mathcal{E}^{(2k+2)} + 2k\right)^n}{2^{k+1}}.$$

Now we apply the substitution $x \mapsto x + 4\mathcal{U}$ to obtain for the left-hand side,

$$
\begin{aligned}
(x + 8\mathcal{B} + 4\mathcal{U})^n &= \int_0^1 (x + 8\mathcal{B} + 4u)^n \, du \\
&= \frac{(x + 8\mathcal{B} + 4)^{n+1} - (x + 8\mathcal{B})^{n+1}}{4(n+1)} \\
&= \frac{8^{n+1}}{4(n+1)} \left(\left(\mathcal{B} + \frac{x+4}{8} \right)^{n+1} - \left(\mathcal{B} + \frac{x}{8} \right)^{n+1} \right) \\
&= \frac{2^{3n+1} \left(B_{n+1} \left(\frac{x+4}{8} \right) - B_{n+1} \left(\frac{x}{8} \right) \right)}{n+1},
\end{aligned}
$$

while for the right-hand side,

$$
\sum_{k \geq 0} \frac{\left(x + 2\mathcal{E}^{(2k+2)} + 2k \right)^n}{2^{k+1}} = 2^n \sum_{k \geq 0} \frac{\left(\mathcal{E}^{(2k+2)} + \frac{x+2k}{2} \right)^n}{2^{k+1}} = 2^n \sum_{k \geq 0} \frac{E_n^{(2k+2)} \left(\frac{x}{2} + k \right)}{2^{k+1}}.
$$

Further simplification completes the proof. $\qquad\square$

Remark 4.6. *Note the analogy between* (4.6) *and* (3.6), *whereas these identities are obtained from two different setups.*

5. Conclusion

We have shown how the setup of random processes allows us to obtain nontrivial identities among higher-order Bernoulli and Euler polynomials. The underlying principle of this approach is that these special functions appear naturally in the generating functions of the hitting times of these random processes.

Several remarks are in order at this point:

- the identities obtained from this approach are not of the usual, convolutional type — see, for example [2], for such identities. Rather, they are connection-type identities between the usual Bernoulli and Euler polynomials and their higher-order counterparts;

- these identites inherently involve a mixture of higher-order Bernoulli and Euler polynomials;

- the interest of this approach is that each term in such a decomposition can be related to a physical object, namely one loop in a trajectory of a random process;

- this work should be considered as only a first approach to a more general project in which the richness of the possible setups for random walks is expected to generate a number of non-trivial identities about more general special functions.

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