# SOME HIGH DEGREE GENERALIZED FIBONACCI IDENTITIES

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Abstract. The Gelin-Cesáro identity states that for integers $n \geq 2$,

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1,$$

where $\{F_n\}$ is the Fibonacci sequence. Horadam generalized the Fibonacci sequence by defining the sequence $\{W_n\}$ where $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$ and $a$, $b$, $p$ and $q$ are integers and $q \neq 0$. Using this sequence, Melham and Shannon generalized the Gelin-Cesáro identity by proving that for integers $n \geq 2$,

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2} - W_n^4 = cq^{n-2}(p^2+q)W_n^2 + c^2q^{2n-3}p^2,$$

where $c = pab - qa^2 - b^2$. We will discover and prove some similar high degree generalized Fibonacci identities.

## 1. Introduction

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas sequences, respectively. Many authors have studied Fibonacci identities and generalized Fibonacci identities. For example, Gelin stated and Cesáro proved [2, p. 401] that for integers $n \geq 2$,

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1. \tag{1}$$

To generalize (1), we need the following definition due to Horadam [3, p. 161].

**Definition 1.** *Let $\{W_n\}$ be defined by $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$, where $a$, $b$, $p$, and $q$ are integers and $q \neq 0$. Let $c = pab - qa^2 - b^2$.*

Melham and Shannon [5] generalized (1) by proving that for integers $n \geq 2$,

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2} - W_n^4 = cq^{n-2}(p^2+q)W_n^2 + c^2q^{2n-3}p^2. \tag{2}$$

In this paper, we will generalize and prove some similar high degree generalized Fibonacci identities.

## 2. Generalization of the Melham and Shannon Identity

To generalize (2), we need the following definition.

**Definition 2.** *Let $\{U_n\}$ be defined by $U_0 = 0$, $U_1 = 1$, and $U_n = pU_{n-1} - qU_{n-2}$ for $n \geq 2$, where $p$ and $q$ are integers and $q \neq 0$.*

The sequence $\{U_n\}$ is the fundamental sequence of Lucas [4]. With this definition, we can state a generalization of the Melham and Shannon identity.

**Theorem 1.** *Let $r$ and $s$ be positive integers and $n \geq r+s$ be an integer. Then*

$$W_{n-r-s}W_{n-r}W_{n+r}W_{n+r+s} = W_n^4 + cq^{n-r-s}(q^sU_r^2 + U_{r+s}^2)W_n^2 + c^2q^{2n-2r-s}U_r^2U_{r+s}^2. \tag{3}$$

We note that when $r = 1$ and $s = 1$, (3) becomes (2).

The proof of (2) can be found in Melham and Shannon [5]. The proof of Theorem 1 is similar to the proof of (2), but with a few modifications. Before we begin the proof of Theorem 1, we require more definitions and a lemma from Melham and Shannon [5, pp. 82–83].

**Definition 3.** *Let* $\{Y_n\}$ *be defined by* $Y_0 = a_1$, $Y_1 = b_1$, *and* $Y_n = pY_{n-1} - qY_{n-2}$ *for* $n \geq 2$, *where* $a_1$, $b_1$, $p$, *and* $q$ *are integers and* $q \neq 0$.

**Definition 4.** *Let* $s$ *be a nonnegative integer. Let*

$$\Psi(s) = (pa_1b - qaa_1 - bb_1)U_s + (ab_1 - a_1b)U_{s+1}.$$

**Lemma 1.** *Let* $n$ *be a nonnegative integer and* $r$ *and* $s$ *be positive integers. Then*

$$W_nY_{n+r+s} - W_{n+r}Y_{n+s} = \Psi(s)q^nU_r. \tag{4}$$

Now we can begin the proof of Theorem 1.

*Proof.* In (4), replacing $n$ by $n - r$ and $s$ by $r$ gives

$$W_{n-r}Y_{n+r} - W_nY_n = \Psi(r)q^{n-r}U_r. \tag{5}$$

Replacing $r$ by $r + s$ in (5), we have

$$W_{n-r-s}Y_{n+r+s} - W_nY_n = \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{6}$$

Adding (5) and (6) gives

$$W_{n-r}Y_{n+r} + W_{n-r-s}Y_{n+r+s} = 2W_nY_n + \Psi(r)q^{n-r}U_r + \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{7}$$

Subtracting (6) from (5) gives

$$W_{n-r}Y_{n+r} - W_{n-r-s}Y_{n+r+s} = \Psi(r)q^{n-r}U_r - \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{8}$$

Square both sides of (7), and similarly for (8). Then subtract the latter of the equations resulting from this from the former to obtain

$$\begin{aligned}&4W_{n-r-s}W_{n-r}Y_{n+r}Y_{n+r+s}\\ &= 4W_n^2Y_n^2 + 4q^{n-r-s}(q^s\Psi(r)U_r + \Psi(r+s)U_{r+s})W_nY_n + 4\Psi(r)\Psi(r+s)q^{2n-2r-s}U_rU_{r+s}.\end{aligned} \tag{9}$$

Divide both sides of the equation by 4. Now, if $(a_1, b_1) = (a, b)$, then $\{W_n\} = \{Y_n\}$, $\Psi(r) = cU_r$, and $\Psi(r+s) = cU_{r+s}$. Substituting these quantities in (9), we see that (9) becomes (3). This is what we wanted to prove. □

Note that if $r = 1$ and $s = 1$, then (3) becomes (2). Also, note that if $a = 0$, $b = 1$, $p = 1$, and $q = -1$, then $\{W_n\} = \{F_n\}$. Thus from (3), we have the following identity for Fibonacci numbers.

$$F_{n-r-s}F_{n-r}F_{n+r}F_{n+r+s} = F_n^4 + (-1)^{n-r-s-1}((-1)^sF_r^2 + F_{r+s}^2)F_n^2 + (-1)^sF_r^2F_{r+s}^2.$$

## 3. A Generalized Sixth Degree Identity

Next, we wish to prove the following theorem.

**Theorem 2.** *Let* $r$ *and* $s$ *be positive integers and* $n \geq r + s$ *be an integer. Then*

$$\begin{aligned}&3W_{n-r-s}W_{n-r}^2W_{n+r}^2W_{n+r+s} + W_{n-r-s}^3W_{n+r+s}^3\\ &= 4W_n^6 + 6cq^{n-r-s}(q^sU_r^2 + U_{r+s}^2)W_n^4 + 3c^2q^{2n-2r-2s}(q^{2s}U_r^4 + 2q^sU_r^2U_{r+s}^2 + U_{r+s}^4)W_n^2\\ &\quad + c^3q^{3n-3r-3s}(3q^{2s}U_r^4U_{r+s}^2 + U_{r+s}^6).\end{aligned} \tag{10}$$

Again, the proof of Theorem 2 is similar to the proof of (2), but with a few modifications.

*Proof.* We start the proof of Theorem 2 as we started the proof of Theorem 1. Instead of squaring, we cube both sides of (7), and similarly for (8). The latter of the resulting equations is then subtracted from the former to obtain

$$\begin{aligned}
&6W_{n-r}^2Y_{n+r}^2W_{n-r-s}Y_{n+r+s}+2W_{n-r-s}^3Y_{n+r+s}^3 \\
&=8W_n^3Y_n^3+12W_n^2Y_n^2\Psi(r)q^{n-r}U_r+12W_n^2Y_n^2\Psi(r+s)q^{n-r-s}U_{r+s} \\
&+6W_nY_n\Psi(r)^2q^{2n-2r}U_r^2+12W_nY_n\Psi(r)q^{2n-2r-s}\Psi(r+s)U_rU_{r+s} \\
&+6W_nY_n\Psi(r+s)^2q^{2n-2r-2s}U_{r+s}^2+6q^{3n-3r-s}\Psi(r)^2\Psi(r+s)U_r^2U_{r+s} \\
&+2q^{3n-3r-3s}\Psi(r+s)^3U_{r+s}^3.
\end{aligned} \tag{11}$$

Divide both sides of the equation by 2. Again, if $(a_1,b_1)=(a,b)$, then $\{W_n\}=\{Y_n\}$, $\Psi(r)=cU_r$, and $\Psi(r+s)=cU_{r+s}$. Substituting these quantities in (11), we see that (11) becomes (10). This is what we wanted to prove. □

Again, if $a=0$, $b=1$, $p=1$, and $q=-1$, then $\{W_n\}=\{F_n\}$. Thus from (10), we have the following identity for Fibonacci numbers.

$$\begin{aligned}
&3F_{n-r-s}F_{n-r}^2F_{n+r}^2F_{n+r+s}+F_{n-r-s}^3F_{n+r+s}^3 \\
&=4F_n^6+6(-1)^{n-r-s-1}((-1)^sF_r^2+F_{r+s}^2)F_n^4+3(F_r^4+2(-1)^sF_r^2F_{r+s}^2+F_{r+s}^4)F_n^2 \\
&\quad+(-1)^{n-r-s-1}(3F_r^4F_{r+s}^2+F_{r+s}^6).
\end{aligned}$$

## 4. A Generalized 2$k$th Degree Identity

Next, we wish to prove the following theorem which generalizes the previous theorems.

**Theorem 3.** *Let $r$ and $s$ be positive integers, $k\ge 2$ be an integer, and $n\ge r+s$ be an integer. Then*

$$\begin{aligned}
&2\sum_{i\ge1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1} \\
&=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}c^iq^{in-ir-is}(q^sU_r^2+U_{r+s}^2)^i \\
&+2c^kq^{kn-kr-ks}\sum_{i\ge1}\binom{k}{2i-1}q^{(k+1-2i)s}U_r^{2(k+1-2i)}U_{r+s}^{2(2i-1)}.
\end{aligned} \tag{12}$$

Again, the proof of Theorem 3 is similar to the proof of (2), but with a few modification.

*Proof.* We start the proof of Theorem 3 as we started the proof of Theorem 1. But, this time, we consider (7) and (8) and let $(a_1,b_1)=(a,b)$. Then $\{W_n\}=\{Y_n\}$, $\Psi(r)=cU_r$, and $\Psi(r+s)=cU_{r+s}$. Substituting these quantities in (7) and (8), we obtain

$$W_{n-r}W_{n+r}+W_{n-r-s}W_{n+r+s}=2W_n^2+cq^{n-r}U_r^2+cq^{n-r-s}U_{r+s}^2 \tag{13}$$

and

$$W_{n-r}W_{n+r}-W_{n-r-s}W_{n+r+s}=cq^{n-r}U_r^2-cq^{n-r-s}U_{r+s}^2. \tag{14}$$

Instead of squaring, we raise both sides of (13) to the $k$th power, and similarly for (14). The latter of the resulting equations is then subtracted from the former to obtain

$$\begin{aligned}&(W_{n-r}W_{n+r}+W_{n-r-s}W_{n+r+s})^k-(W_{n-r}W_{n+r}-W_{n-r-s}W_{n+r+s})^k\\&=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}(cq^{n-r}U_r^2+cq^{n-r-s}U_{r+s}^2)^i\\&+(cq^{n-r}U_r^2+cq^{n-r-s}U_{r+s}^2)^k-(cq^{n-r}U_r^2-cq^{n-r-s}U_{r+s}^2)^k.\end{aligned}$$

Expanding the products on both sides of the equation and collecting and canceling terms gives

$$\begin{aligned}&2\sum_{i\geq 1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1}\\&=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}c^iq^{in-ir-is}(q^sU_r^2+U_{r+s}^2)^i\\&+2c^k\sum_{i\geq 1}\binom{k}{2i-1}(q^{n-r}U_r^2)^{k+1-2i}(q^{n-r-s}U_{r+s}^2)^{2i-1}.\end{aligned}$$

Simplifying some more, we obtain

$$\begin{aligned}&2\sum_{i\geq 1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1}\\&=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}c^iq^{in-ir-is}(q^sU_r^2+U_{r+s}^2)^i\\&+2c^kq^{kn-kr-ks}\sum_{i\geq 1}\binom{k}{2i-1}q^{(k+1-2i)s}U_r^{2(k+1-2i)}U_{r+s}^{2(2i-1)}.\end{aligned}$$

This is what we wanted to prove. □

Again, if $a=0$, $b=1$, $p=1$, and $q=-1$, then $\{W_n\}=\{F_n\}$. Thus from (12), we have the following identity for Fibonacci numbers.

$$\begin{aligned}&2\sum_{i\geq 1}\binom{k}{2i-1}(F_{n-r}F_{n+r})^{k+1-2i}(F_{n-r-s}F_{n+r+s})^{2i-1}\\&=\sum_{i=0}^{k-1}\binom{k}{i}(2F_n^2)^{k-i}(-1)^{in-ir-is+i}((-1)^sF_r^2+F_{r+s}^2)^i\\&+2(-1)^k\sum_{i\geq 1}\binom{k}{2i-1}(-1)^{kn-kr-ks}(-1)^{(k+1-2i)s}F_r^{2(k+1-2i)}F_{r+s}^{2(2i-1)}.\end{aligned}$$

## 5. A Generalization of a Fourth Degree Fibonacci Identity

Next, we give another fourth degree Fibonacci identity from [1, p. 46]. If $n$ is a nonnegative integer, then

$$F_nF_{n+4}^3-F_{n+2}^3F_{n+6}=(-1)^{n+1}F_{n+3}L_{n+3}. \tag{15}$$

To state a generalization to (15), we need a definition due to Rabinowitz [6, p. 166].

**Definition 5.** *Let $n$ be an integer. Then*

$$X_n = W_{n+1} - qW_{n-1}.$$

The sequence $\{X_n\}$ may be considered to be a companion sequence to $\{W_n\}$, in the same sense that the Lucas sequence is the companion of the Fibonacci sequence. This sequence will be useful in stating our next theorem.

**Theorem 4.** *Let $n$ be a nonnegative integer. Then*

$$W_n W_{n+4}^3 - W_{n+2}^3 W_{n+6} = cp^3 q^n W_{n+3} X_{n+3}. \tag{16}$$

*Proof.* Let $n$ be a nonnegative integer. Let $x = W_n$ and $y = W_{n+1}$. Then, after some substitutions and collecting terms, we have

$$\begin{aligned}
W_n &= x \\
W_{n+1} &= y \\
W_{n+2} &= py - qx \\
W_{n+3} &= (p^2 - q)y - pqx \\
W_{n+4} &= (p^3 - 2pq)y + (-p^2q + q^2)x \\
W_{n+5} &= (p^4 - 3p^2q + q^2)y + (-p^3q + 2pq^2)x \\
W_{n+6} &= (p^5 - 4p^3q + 3pq^2)y + (-p^4q + 3p^2q^2 - q^3)x \\
X_{n+3} &= (p^3 - 3pq)y + (-p^2q + 2q^2)x.
\end{aligned}$$

We need one more quantity, $cq^n$. From Horadam [3, p. 171, eq. (4.3)], we have that

$$cq^n = W_n W_{n+2} - W_{n+1}^2 = x(py - qx) - y^2 = -qx^2 + pxy - y^2.$$

After substitutions and some algebraic manipulations, the left side of (16) simplifies to

$$\begin{aligned}
&(-p^6q^3 + 2p^4q^4)x^4 + (3p^7q^2 - 8p^5q^3 + 2p^3q^4)x^3y \\
&+ (-3p^8q + 9p^6q^2 - 3p^4q^3)x^2y^2 + (p^9 - 2p^7q - 3p^5q^2 + 2p^3q^3)xy^3 \\
&+ (-p^8 + 4p^6q - 3p^4q^2)y^4.
\end{aligned}$$

It can easily be shown that the right side of (16) also simplifies to this algebraic expression. Therefore, the left side and right side of (16) are equal. This completes the proof of the theorem. □

## 6. A Generalization of a Fifth Degree Fibonacci Identity

Finally, we present a fifth degree Fibonacci identity from [1, p. 46]. If $n$ is a nonnegative integer, then

$$F_n^2 F_{n+5}^3 - F_{n+1}^3 F_{n+6}^2 = (-1)^{n+1} L_{n+3}^3. \tag{17}$$

A generalization of (17) is presented in the following theorem.

**Theorem 5.** *Let $n$ be a nonnegative integer. Then*

$$W_n^2 W_{n+5}^3 - W_{n+1}^3 W_{n+6}^2 = cq^n X_{n+3}((2p^3 - 3pq)W_{n+3}^2 + (p^7 - 2p^5q + p^3q^2)cq^n). \tag{18}$$

*Proof.* Let $n$ be a nonnegative integer. Let $x = W_n$ and $y = W_{n+1}$. We require all the quantities from the proof of Theorem 4. After substitutions and some algebraic manipulations, the left side of (18) simplifies to

$$\begin{aligned}
&(-p^9q^3 + 6p^7q^4 - 12p^5q^5 + 8p^3q^6)x^5 \\
&+ (3p^{10}q^2 - 21p^8q^3 + 51p^6q^4 - 48p^4q^5 + 12p^2q^6)x^4y \\
&+ (-3p^{11}q + 24p^9q^2 - 69p^7q^3 + 84p^5q^4 - 39p^3q^5 + 6pq^6)x^3y^2 \\
&+ (p^{12} - 9p^{10}q + 29p^8q^2 - 39p^6q^3 + 19p^4q^4 - 3p^2q^5)x^2y^3 \\
&+ (2p^9q - 14p^7q^2 + 32p^5q^3 - 26p^3q^4 + 6pq^5)xy^4 \\
&+ (-p^{10} + 8p^8q - 22p^6q^2 + 24p^4q^3 - 9p^2q^4)y^5.
\end{aligned}$$

It can easily be shown that the right side of (18) also simplifies to this algebraic expression. Therefore, the left side and right side of (18) are equal. This completes the proof of the theorem. □

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