MATRICES IN THE DETERMINANT HOSOYA TRIANGLE

MATTHEW BLAIR, RIGOBERTO FLÓREZ, ANTARA MUKHERJEE, AND JOSÉ L. RAMÍREZ

Abstract. The *determinant Hosoya triangle* is a triangle with determinants of two-by-two Fibonacci matrices as its entries. In this paper we give a combinatorial interpretation of this triangle and explore properties of square matrices embedded in the triangle (in particular, symmetric and persymmetric). Specifically, we explore the eigenvalues, eigenvectors, and characteristic polynomials of these matrices and provide closed formulas for the same in terms of Fibonacci and Lucas numbers.

1. Introduction

The *determinant Hosoya triangle*, denoted by $\mathcal{H}$, is a triangular array where its entries are determinants of two-by-two matrices with Fibonacci numbers as entries. For example, using Proposition 2.1 we have that the entry $H_{6,3}$ of $\mathcal{H}$ is given by

$$H_{6,3} = \begin{vmatrix} F_5 & F_4 \\ F_3 & F_4 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 2 & 3 \end{vmatrix} = 9.$$

The Table 1 shows the first rows of $\mathcal{H}$.

0						
1	1					
1	3	1				
2	4	4	2			
3	7	5	7	3		
5	11	9	9	11	5	
8	18	14	16	14	18	8

Table 1. Determinant Hosoya Triangle $\mathcal{H}$.

Several authors have been interested in the study of matrices with Fibonacci numbers (eigenvalues, eigenvectors, graphs). For example, in 2020 Ching et al. [5] took the matrices from this triangle mod 2 to obtain three infinite families of cographs with one of them integral. Blair et al. [4] study sequences and geometric properties of the triangle considered here. In 2018, Blair et al. [3] studied properties of matrices in the Hosoya triangle and hence bridged linear algebra with combinatorial triangles (other related papers are [17, 22]).

In this paper we discuss properties of matrices embedded in $\mathcal{H}$. A matrix in $\mathcal{H}$ is *slash* (*backslash*) if its columns are embedded in the slash (backslash) diagonals of $\mathcal{H}$, as depicted in Figures 1 and 5. We analyze these two types of matrices and use linear algebra and geometry to explore different patterns in this triangle.

We show that these matrices embedded in the determinant Hosoya triangle are of rank two and therefore can be written as sum of two products of vectors. Thus, a matrix within $\mathcal{H}$ has

Several of the main results in this paper were found by the first author while working on his undergraduate research project under the guidance of the second and third authors (who followed the guidelines given in [9]). The second and third authors were partially supported by The Citadel Foundation.

the form $\mathbf{u_1}^T\mathbf{v_1} + \mathbf{u_2}^T\mathbf{v_2}$, where $\mathbf{u_1}, \mathbf{v_1}, \mathbf{u_2}, \mathbf{v_2}$ are vectors. The entries of these vectors are consecutive Fibonacci numbers (located on the sides of the triangle as in Figure 1). Matrices with this behavior allow us to explore properties where the outcomes are again Fibonacci numbers. For example, these matrices have closed formulas for trace, eigenvalues, and eigenvectors primarily in terms of Fibonacci and Lucas numbers.

We also look at particular cases of these matrices one of which are families of symmetric matrices and persymmetric matrices —symmetric with respect to the antidiagonal. The closed formulas are also connected to the geometry of the triangle. For example, the trace of the symmetric matrices give rise to a hockey stick-like property found in the Hosoya triangle (see [7]). Some of these properties were also explored in the Hosoya triangle [3] (this is a triangular array defined recursively similar to the triangle here, but with different initial conditions, see Section 6).

The properties of the matrices and their proofs, mainly depend on the recursive nature of the entries of the triangle and some well-known linear algebra techniques.

It is well-known that the Fibonacci number F_n counts the number of tilings of a board of length $n-1$ with cells labelled 1 to $n-1$ from left to right with only squares and dominoes [1]. We use this idea to give a combinatorial interpretation of the entries of the determinant Hosoya triangle.

2. The determinant Hosoya triangle

In this section we give a recursive definition of the determinant Hosoya triangle, we present a closed formula for the entries of the triangle and investigate several properties of the triangle.

The determinant Hosoya triangle was originally discovered by Sloane [21, A108038]. The *determinant Hosoya sequence* $\{H(r,k)\}_{r,k>0}$ is defined using this double recursion.

$$H(r,k) = H(r-1,k) + H(r-2,k) \quad \text{and} \quad H(r,k) = H(r-1,k-1) + H(r-2,k-2) \quad (2.1)$$

with initial conditions $H(1,1) = 0, H(2,1) = H(2,2) = 1$, and $H(3,2) = 3$ where $r > 2$ and $1 \leq k \leq r$. For brevity, we write $H_{r,k}$ instead of $H(r,k)$ for the rest of the paper. This sequence gives rise to the determinant Hosoya triangle, denoted by $\mathcal{H}$, where the entry in position k (taken from left to right) of the rth row is equal to $H_{r,k}$. Note that the left hand-side of (2.1) gives of rise to slash diagonals and the right hand-side give rise to backslash diagonals of the triangle.

It is easy to see that every diagonal in the determinant Hosoya triangle is a generalized Fibonacci number. For instance, the fifth (slash or backslash) diagonal in Figure 1 is $3, 11, 14, 25, 39, 64, \ldots$ this sequence corresponds to the generalized Fibonacci number $G_n^{(5)} = G_{n-1}^{(5)} + G_{n-2}^{(5)}$, where $G_1^{(5)} = 3$ and $G_2^{(5)} = 11$. In this paper we use F_m and L_m to represent Fibonacci and Lucas numbers. In general, the entries of the mth diagonal of this triangle are given by the generalized Fibonacci number

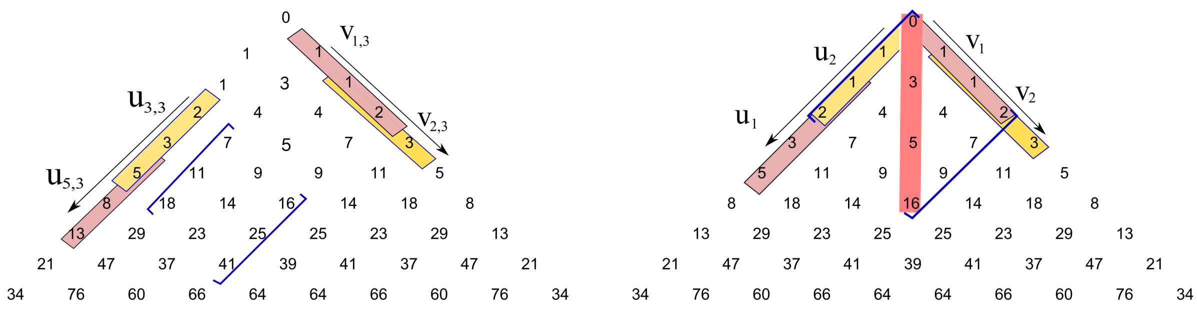
$$G_n^{(m)} = G_{n-1}^{(m)} + G_{n-2}^{(m)}, \quad \text{where} \quad G_1^{(m)} = F_{m-1} \quad \text{and} \quad G_2^{(m)} = L_m.$$

The following proposition gives an equivalent definition of the entries of the triangle using determinants of 2×2 matrices with Fibonacci numbers as entries. In this paper we use $|A|$ or $\det A$ to represent determinant of the matrix A.

Proposition 2.1. *If r, k are positive integers with $k \leq r$ with $H_{r,k}$ as defined in (2.1), then*

(1) $H_{r,k} = F_{k+1}F_{r-k+2} - F_kF_{r-k+1}$. *Thus,*

$$H_{r,k} = \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix}.$$



(a) (b)

FIGURE 1. (a) Slash matrix $S(2,4,3)$ (b) Symmetric matrix $S(4)$ in $\mathcal{H}$.

(2) $H_{r,k} = F_{k-1}F_{r-k+2} + F_kF_{r-k}$. *Thus,*

$$H_{r,k} = \begin{vmatrix} F_{r-k+2} & -F_{r-k} \\ F_k & F_{k-1} \end{vmatrix}.$$

(3) $H_{n,k} = L_kF_{n-k} + F_{k-1}F_{-k+n-1}$. *Thus,*

$$H_{r,k} = \begin{vmatrix} F_{n-k} & -F_{-k+n-1} \\ F_{k-1} & L_k \end{vmatrix}.$$

Proof. We prove Part (1) by mathematical induction, the proof of Part (2) follows from Part (1) substituting F_{k+1} by $F_k + F_{k+1}$ and simplifying, and Part (3) follows from generalized Fibonacci numbers (for details, see [4]).

Let $P(r,k)$ be the statement:

$$H_{r,k} = F_{k+1}F_{r-k+2} - F_{r-k+1}F_k \quad \text{for every } 1 \le k \le r.$$

We prove this statement by mathematical induction.

We first prove the statements $P(r,1)$ for $r \ge 1$ and $P(r,2)$ for $r \ge 2$ are true. To prove $P(r,1)$, we show that $H_{r,1} = F_{r-1}$ for every $r \ge 1$. From the recursive definition of $H_{r,1}$ it is easy to verify that $P(1,1)$, $P(2,1)$ are true. Let $n > 2$ be a fixed integer number, if we suppose that for $r = n$ the statements $P(n,1)$ and $P(n-1,1)$ are true, then by the recursive definition of $H_{r,1}$ we have that $P(n+1,1)$ is true. The statement $P(r,2)$ (i.e., $H_{r,2} = L_{r-1}$) can be proved using a similar argument.

We now prove the statement for $P(r,k)$ for any $r \ge k$. Since $P(r,1)$ and $P(r,2)$ are true, we have proved the basis step. Suppose that $P(r-1,k-1)$ and that $P(r-2,k-2)$ are true for any $r \ge k$. Thus, $H_{r-2,k-2} = F_{k-2+1}F_{r-2-(k-2)+2} - F_{r-2-(k-2)+1}F_{k-2}$ and $H_{r-1,k-1} = F_{k-1+1}F_{r-1-(k-1)+2} - F_{r-1-(k-1)+1}F_{k-1}$. Since, $H_{r,k} = H_{r-1,k-1} + H_{r-2,k-2}$ we have that

$$H_{r,k} = F_kF_{r-k+2} - F_{r-k+1}F_{k-1} + F_{k-1}F_{r-k+2} - F_{r-k+1}F_{k-2} = F_{k+1}F_{r-k+2} - F_{r-k+1}F_k.$$

This completes the proof. □

The following proposition provides a closed formula for the sum of entries in each row of the determinant Hosoya triangle.

Proposition 2.2. *If r, k are positive integers with $k \le r$, then*

$$\sum_{k=1}^{r} H_{r,k} = \sum_{k=1}^{r} \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix} = (rL_{r+2} - 4F_r)/5 = ((7r-4)F_{r-1} + 4(r-1)F_{r-2})/5.$$

Proof. From Proposition 2.1 we have

$$\sum_{k=1}^{r} H_{r,k} = \sum_{k=1}^{r}(F_{k+1}F_{r-k+2} - F_{r-k+1}F_k) = \sum_{k=1}^{r}(F_kF_{r-k} + F_{k-1}F_{r-k+2}).$$

Using the identity for convolution of Fibonacci numbers, $\sum_{i=0}^{n} F_iF_{n-i} = (nL_n - F_n)/5$, we have

$$\begin{aligned}\sum_{k=1}^{r}(F_kF_{r-k} + F_{k-1}F_{r-k+2}) &= \sum_{k=1}^{r} F_kF_{r-k} + \sum_{k=1}^{r} F_{k-1}F_{r-k+2}\\ &= (rL_r - F_r)/5 + ((r+1)L_{r+1} - F_{r+1} - 5F_r)/5\\ &= (rL_{r+2} - 4F_r)/5.\end{aligned}$$

This proves the middle equality in the proposition. The right equality of the proposition holds using $L_{r+2} = F_{r+1} + F_{r+3}$ in the middle equality. □

3. A combinatorial interpretation and connections with other triangular arrays

In this section we give a combinatorial interpretation of the entries of the determinant Hosoya triangle and some connections with other triangles. Benjamin [2] asked the first author the following question after his presentation at the conference, *Is there any combinatorial interpretation for the entries of the determinant Hosoya triangle?* Here we give a combinatorial interpretation to address this question.

3.1. **A combinatorial interpretation.** It is well-known that the Fibonacci number F_{n+1} counts the number of tilings of a board of length n (n-board) with cells labelled 1 to n from left to right with only squares and dominoes (cf. [1]). Let $\mathcal{T}_n$ denote the set of all n-tilings, then $|\mathcal{T}_n| = F_{n+1}$, for all $n \geq 0$. For example, in Figure 2 we show the elements of $\mathcal{T}_4$. We use this interpretation to give a combinatorial interpretation of the determinant Hosoya sequence $H_{n,k}$.

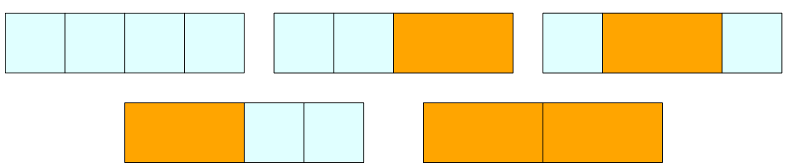


Figure 2. Different ways to tile a 4-board.

For $n, k \geqslant 0$, let $h_{n,k}$ denote the number of tilings in $\mathcal{T}_n$ such that in the cell k or $k+1$ or both there is a hole, and we can not put any tile over the hole. For example, $h_{4,1} = 7$, the relevant tilings are in Figure 3.

Proposition 3.1. *For* $0 \leq k \leq n$*, we have that* $h_{n,k} = F_kF_{n-k+2} + F_{k+1}F_{n-k}$*. Therefore,* $H_{n+1,k+1} = h_{n,k}$*.*

Proof. For any tiling T in $\mathcal{T}_n$, $(n \geq 0)$ there are three cases: there is either a hole in the kth cell or there is a hole in the $(k+1)$th cell or there are holes in both cells. In the first case, the n-board is divided into a $(k-1)$-board (it could be empty) and a $(n-k)$-board. Then there are F_kF_{n-k+1} possibilities. In the second case, the n-board is divided into a k-board and a

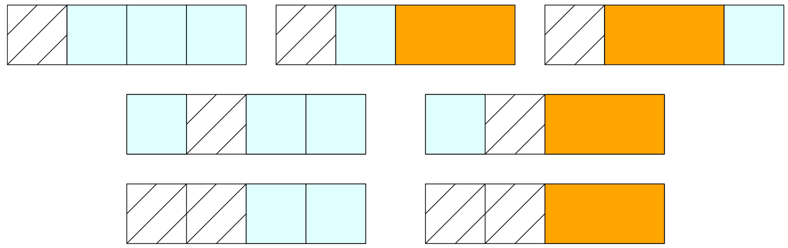


FIGURE 3. Different ways to tile a 4-board with holes.

$(n-k-1)$-board, so there are $F_{k+1}F_{n-k}$ possibilities. Finally, in the third case, we can use a similar argument, and then we obtain F_kF_{n-k} options. Altogether, we have

$$h_{n,k} = F_kF_{n-k+1} + F_{k+1}F_{n-k} + F_kF_{n-k} = F_kF_{n-k+2} + F_{k+1}F_{n-k}.$$ □

Notice that from the combinatorial interpretation we obtain the following corollary.

Corollary 3.2. *The sequence $h_{n,k}$ satisfies that*

- $h_{n,k} = h_{n,n-k}$, *therefore, the determinant Hosoya triangle is symmetric.*
- $h_{n,k} = h_{n-1,k} + h_{n-2,k}$, *for $n \geq 2$ and $0 \leq k \leq n$.*
- $h_{n,k} = h_{n-1,k-1} + h_{n-2,k-2}$, *for $n \geq 2$ and $2 \leq k \leq n$.*

Corollary 3.3. *The bivariate generating function of the sequence $h_{n,k}$ is given by*

$$\sum_{n,k\geq 0} h_{n,k}x^ny^k = \frac{x + xy + x^2y}{(1-x-x^2)(1-xy-x^2y^2)}.$$

Proof. From the Proposition 3.1 we have

$$\begin{aligned}\sum_{n,k\geq 0} h_{n,k}x^ny^k &= x + y + \sum_{n\geq 2} x^n \sum_{k=0}^{n} h_{n,k}y^k \\ &= x + y + 3x^2y + \sum_{n\geq 2} x^n \sum_{k=0}^{n} \left(F_kF_{n-k+2} + F_{k+1}F_{n-k}\right) y^k.\end{aligned}$$

The last sum can be evaluate using the software *Mathematica*®. In this case we obtain that this last sum is given by

$$\frac{x^2(1 + x + 3y - x^2y + y^2 - 2x^2y^2 - x^3y^2 + xy^3 - x^2y^3 - x^3y^3)}{(1-x-x^2)(1-xy-x^2y^2)}.$$

Simplifying we obtain the conclusion of the corollary. □

By taking $y = 1$ in the last corollary we obtain the generating function for the row sum of the determinant Hosoya triangle.

Corollary 3.4. *The generating function of the row sum is given by*

$$\sum_{n\geq 0}\sum_{k=0}^{n} H_{n+1,k+1}x^n = \sum_{n\geq 0}\sum_{k=0}^{n} h_{n,k}x^n = \frac{x(2+x)}{(1-x-x^2)^2}.$$

The first few values of the last generating function are

$$0, \quad 2, \quad 5, \quad 12, \quad 25, \quad 50, \quad 96, \quad 180, \quad 331, \quad 600, \quad 1075, \ldots$$

Note that this sequence is given by the closed formula from Proposition 2.2.

We now give the first of two potential problems that arise from this combinatorial interpretation.

Problem 1. If we think of the determinant Hosoya triangle as an infinite matrix, we could find a fractal structure associated to this matrix. For example, if we evaluated (using *Mathematica*®) their entries mod 3 and mod 11 we obtain two interesting patterns (see Figure 4). For example, studying these patterns (with these matrices) may yield interesting results.

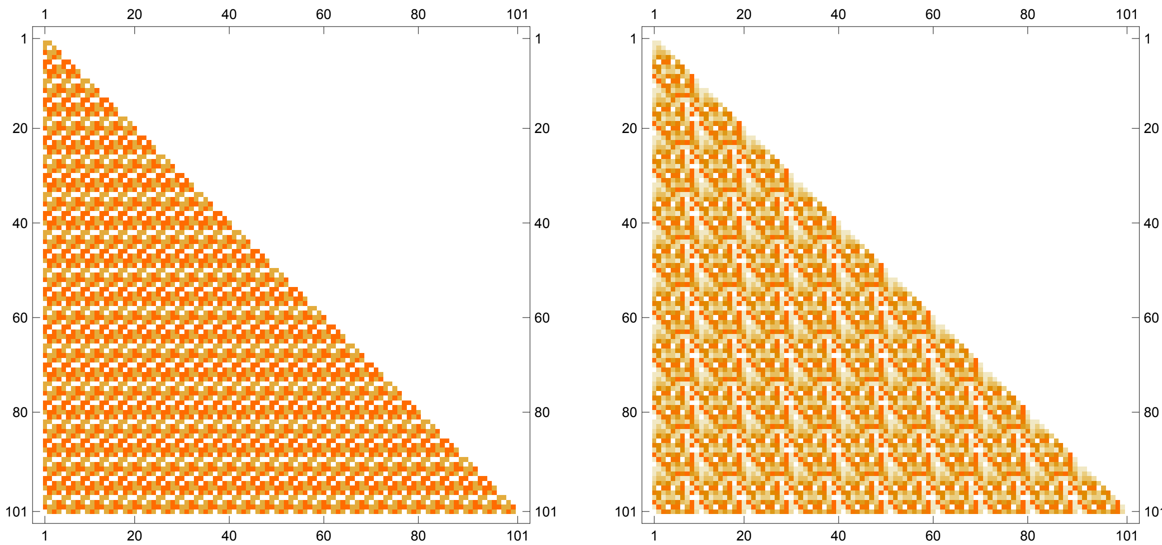


FIGURE 4. The determinant Hosoya triangle mod 3 and mod 11.

3.2. **Connections with other triangular arrays.** Here we provide two connections. However, there are more connections, see for example [3, 6–8, 10, 11, 14–16, 20].

Hosoya triangle. The entries of the Hosoya triangle are given by $H'_{m,n} = F_n F_{m-n+1}$ (see Section 6 or [14]). As a consequence of Proposition 2.1 we have a relation between the entries of the determinant Hosoya triangle and the Hosoya triangle. Thus, $H_{r,k} = H'_{r+2,k+1} - H'_{r,k}$ for $1 \leq k \leq r$.

Fibonomial triangle. Vajda [23] shows that the entries of the Fibonomial triangle are based on the identity

$$F_{k+n} = F_{k-1}F_n + F_k F_{n+1}.$$

See A010048 or [23, (8)]. The entries of the determinant Hosoya triangle are obtained by replacing —on the right-hand side of above identity— the first n by $(r-k+2)$ and the second n by $(r-k-1)$ where $1 \leq k \leq r$. Therefore, for $1 \leq k \leq r$ the entries of the determinant Hosoya triangle are given by

$$H_{r,k} = F_{k-1}F_{r-k+2} + F_k F_{r-k}. \tag{3.1}$$

We observe that fixing k and varying r, gives rise to the sequence $\{H_{r,k}\}_{r\in\mathbb{Z}_{>0}}$ comprising of generalized Fibonacci numbers.

4. Rank-2 matrices in the determinant Hosoya triangle

We recall that a matrix in $\mathcal{H}$ is slash (backslash) if its columns are embedded in the slash (backslash) diagonals of $\mathcal{H}$. In this section we study the properties of the slash matrices embedded in the determinant Hosoya triangle $\mathcal{H}$. We begin by showing that these matrices can be expressed as a sum of products of two vectors (see Figure 1). We then discuss the eigenvalues and eigenvectors of the matrices.

4.1. Slash matrices in the determinant Hosoya triangle.

In this section we use $S(m,n,t)$ to represent a member of a family of rank-2 matrices within the triangle. We use three cases to analyze this type of matrices; in the first case we consider $S(m,n,t)$ with $m \le n$, in the second we analyze the case $m > n$, and the third is the case in which $m = n = 1$, a special case of the symmetric matrix $S(n,n,t)$.

We denote a $t \times t$ slash matrix in $\mathcal{H}$ by $S(m,n,t)$. If m, n, and t are positive integers with $m \le n$ and $r_i = (m+n-1)+i$ for $0 \le i \le t-1$, then we define, formally, the *slash matrix* of this form,

$$S(m,n,t) = \begin{bmatrix} H_{r_0,m} & H_{r_1,m+1} & H_{r_2,m+2} & \cdots & H_{r_{t-1},m+t-1} \\ H_{r_1,m} & H_{r_2,m+1} & H_{r_3,m+2} & \cdots & H_{r_t,m+t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r_{t-1},m} & H_{r_t,m+1} & H_{r_{t+1},m+2} & \cdots & H_{r_{2(t-1)},m+t-1} \end{bmatrix}. \tag{4.1}$$

For example, in Figure 1(a) the matrix is $S(2,4,3)$ and the first entry of the matrix is $H_{2+4-1,2} = 7$. Note that the coordinates (m,n) denote the intersection of the mth slash diagonal with the nth backslash diagonal in $\mathcal{H}$ and this ordered pair is the location of the $(1,1)$ entry of the matrix $S(m,n,t)$. We now define some vectors needed several times in this paper.

$$u_{r,t} := [F_r, F_{r+1}, \ldots, F_{r+t-1}], \quad v_{l,t} := [F_l, F_{l+1}, \ldots, F_{l+t-1}] \quad \text{with} \quad l, r \ge 0. \tag{4.2}$$

We use $A(l,r,t)$ to denote the $t \times t$ rank one matrix $u_{r,t}^T \cdot v_{l,t}$. We recall that $S(m,n,t)$ is defined for $m \le n$.

Proposition 4.1. *If $u_{n,t}$ and $v_{m,t}$ are as defined in (4.2), then*

$$S(m,n,t) = A(m-1,n+1,t) + A(m,n-1,t) = u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t}.$$

Proof. Let i, j be fixed positive integers such that $0 \le i, j \le t-1$ and $r_i = (m+n-1)+i$. From Proposition 2.1 Part (2), we have that the entry (i,j) of matrix $S(m,n,t)$ is given by

$$H_{r_{i+j},m+j} = F_{n+i+1}F_{m-j-1} + F_{m+j}F_{n+i-1}.$$

Similarly, the entry (i,j) of the matrix $u_{n+1,t} \cdot v_{m-1,t} + u_{n-1,t} \cdot v_{m,t}$ is given by $F_{n+i+1}F_{m-j-1} + F_{m+j}F_{n+i-1}$. This completes the proof. $\square$

For example, in Figure 1(a) we see that if

$$u_{5,3} = [F_5, F_6, F_7], \quad v_{1,3} = [F_1, F_2, F_3], \quad u_{3,3} = [F_3, F_4, F_5], \quad \textit{and} \quad v_{2,3} = [F_2, F_3, F_4],$$

then the slash matrix $S(2,4,3)$ is given as follows

$$u_{5,3}^T \cdot v_{1,3} + u_{3,3}^T \cdot v_{2,3} = \begin{bmatrix} F_5 \\ F_6 \\ F_7 \end{bmatrix} \cdot [F_1, F_2, F_3] + \begin{bmatrix} F_3 \\ F_4 \\ F_5 \end{bmatrix} \cdot [F_2, F_3, F_4] = \begin{bmatrix} 7 & 9 & 16 \\ 11 & 14 & 25 \\ 18 & 23 & 41 \end{bmatrix}.$$

Proposition 4.2. *If $S(m,n,t)$ is the slash matrix in $\mathcal{H}$ given in (4.1), where $m, n,$ and t are positive integers, then*

$$\operatorname{tr}(S(m,n,t)) = \begin{cases} \left(L_{m+n+2t} - L_{m+n}\right)/5, & \text{if } t \text{ is even;} \\ \left(L_{m+n+2t} - L_{m+n} - (-1)^{n+t}(L_{m-n-2} + L_{m-n+1})\right)/5, & \text{if } t \text{ is odd.} \end{cases}$$

Proof. From Proposition 4.1 we conclude that

$$\operatorname{tr}(S(m,n,t)) = \operatorname{tr}(A(m-1,n+1,t)) + \operatorname{tr}(A(m,n-1,t)).$$

This and Proposition 6.1 Part (c) (see Appendix on Page 50) prove the proposition. □

Next we show that all (slash) matrices of the form $u_1^T v_1 + u_2^T v_2$ embedded in $\mathcal{H}$ are of rank two. The vectors in the following lemma are formed by consecutive Fibonacci numbers (located on the sides of the triangle as in Figure 1).

Lemma 4.3. *The matrix $S(m,n,t)$ as given in (4.1) has rank 2.*

Proof. Clearly $S(m,n,t) \neq \mathbf{0}$. Since $u_1^T v_1$ and $u_2^T v_2$ are rank-one matrices, $\operatorname{rank}(S(m,n,t)) \leq 2$. In addition, since $u_1 \neq \alpha u_2$ and $v_1 \neq \beta v_2$, $\operatorname{rank}(S(m,n,t)) \neq 1$. This completes the proof. □

We now find the eigenvalues of the slash matrices. Before we introduce this result, we recall that the only non-zero eigenvalue of the slash matrix $A(m,n,t)$ in the Hosoya triangle is given by $\lambda = \operatorname{tr}(A(m,n,t))$ in Proposition 6.1 (c). The proofs of the following theorem and Proposition 4.6 are based on well-known linear algebra techniques. We adapted these techniques to fit our objective in this paper, (see for example [12, 13, 19, 24]).

Theorem 4.4. *If $S(m,n,t)$ is the slash matrix in $\mathcal{H}$ given in (4.1), with $1 < m \leq n$, where $A = \left(L_{m+n} - L_{m+n+2t} + (-1)^{n+t}(L_{m-n-2} + L_{m-n+1})\right)/5$ and $B = (L_{m+n+2t} - L_{m+n})/5$, then these hold*

(a) the characteristic polynomial of $S(m,n,t)$ is given by

$$P(x) = \begin{cases} -x^{t-2}(x^2 + Ax + (-1)^{m+n+t+1}(1 - F_t^2)), & \text{if } t \text{ is odd;} \\ x^{t-2}(x^2 + Bx + (-1)^{m+n+t+1}F_t^2), & \text{if } t \text{ is even.} \end{cases}$$

(b) The eigenvalues of $S(m,n,t)$ are $\lambda_0 = 0$ with multiplicity $(t-2)$, and for $i \in \{1,2\}$

$$\lambda_i = \begin{cases} \dfrac{-A \pm \sqrt{A^2 + 4(-1)^{m+n+t}(1 - F_t^2)}}{2}, & \text{if } t \text{ is odd;} \\ \dfrac{-B \pm \sqrt{B^2 + 4(-1)^{m+n+t}F_t^2}}{2}, & \text{if } t \text{ is even.} \end{cases}$$

Proof. Let $p(x) = (-1)^t(x^t + c_{t-1}x^{t-1} + c_{t-2}x^{t-2} + \cdots + c_1 x + c_0)$ be the characteristic polynomial of $S' := S(m,n,t)$. From Lemma 4.3 we know that S' is of rank 2. Therefore, $x^{t-2}(x^2 + c_{t-1}x + c_{t-2}) = 0$. Let x_1 and x_2 be the non-zero eigenvalues of S'. So, the eigenvalues of S' are $x_0 = 0$ and

$$x_1 = \frac{-c_{t-1} + \sqrt{(c_{t-1})^2 - 4c_{t-2}}}{2} \quad \text{and} \quad x_2 = \frac{-c_{t-1} - \sqrt{(c_{t-1})^2 - 4c_{t-2}}}{2}. \tag{4.3}$$

Since $\operatorname{rank}(S') = 2$, from linear algebra we know that c_{t-1} is the trace of S' (the sum of all eigenvalues of S') and that c_{t-2} is the product of the two non-zero eigenvalues of S'. Thus,

$$c_{t-1} = -(x_1 + x_2) \quad \text{and} \quad c_{t-2} = x_1 x_2.$$

We now find c_{t-1} and c_{t-2}. From Proposition 4.1 and (4.2) we know that

$$S' = u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t}. \tag{4.4}$$

It is well known that

$$\begin{aligned}
\mu_1 &:= v_{m-1,t} \cdot u_{n+1,t}^T && \text{is the non-zero eigenvalue of } A(m-1,n+1,t) = u_{n+1,t}^T \cdot v_{m-1,t};\\
\mu_2 &:= v_{m,t} \cdot u_{n-1,t}^T && \text{is the non-zero eigenvalue of } A(m,n-1,t) = u_{n-1,t}^T \cdot v_{m,t};\\
\mu_{12} &:= v_{m,t} \cdot u_{n+1,t}^T && \text{is the non-zero eigenvalue of } A(m,n+1,t) = u_{n+1,t}^T v_{m,t}; \qquad (4.5)\\
\mu_{21} &:= v_{m-1,t} \cdot u_{n-1,t}^T && \text{is the non-zero eigenvalue of } A(m-1,n-1,t) = u_{n-1,t}^T \cdot v_{m-1,t}.
\end{aligned}$$

We divide the rest of the proof into two claims.
Claim 1: $\operatorname{tr}(S') = \mu_1 + \mu_2$. This is clear, because $S' = u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t}$.
Claim 2: $\operatorname{tr}((S')^2) = \mu_1^2 + 2\mu_{12}\mu_{21} + \mu_2^2$.
Proof of **Claim 2.** From (4.4) we have that

$$\begin{aligned}
(S')^2 = u_{n+1,t}^T \cdot (v_{m-1,t} \cdot u_{n+1,t}^T) \cdot v_{m-1,t} + u_{n-1,t}^T \cdot (v_{m,t} \cdot u_{n+1,t}^T) \cdot v_{m-1,t} + \\
u_{n+1,t}^T \cdot (v_{m-1,t} \cdot u_{n-1,t}^T) \cdot v_{m,t} + u_{n-1,t}^T \cdot (v_{m,t} \cdot u_{n-1,t}^T) \cdot v_{m,t}.
\end{aligned}$$

Thus,

$$(S')^2 = u_{n+1,t}^T(\mu_1)v_{m-1,t} + u_{n-1,t}^T(\mu_{12})v_{m-1,t} + u_{n+1,t}^T(\mu_{21})v_{m,t} + u_{n-1,t}^T(\mu_2)v_{m,t}.$$

Therefore,

$$\begin{aligned}
\operatorname{tr}(S')^2 &= \mu_1 \operatorname{tr}(u_{n+1,t}^T v_{m-1,t}) + \mu_{12} \operatorname{tr}(u_{n-1,t}^T v_{m-1,t}) + \mu_{21} \operatorname{tr}(u_{n+1,t}^T v_{m,t}) + \mu_2 \operatorname{tr}(u_{n-1,t}^T v_{m,t})\\
&= \mu_1(v_{m-1,t}u_{n+1,t}^T) + \mu_{12}(v_{m-1,t}u_{n-1,t}^T) + \mu_{21}(v_{m,t}u_{n+1,t}^T) + \mu_2(v_{m,t}u_{n-1,t}^T)\\
&= \mu_1^2 + 2\mu_{12}\mu_{21} + \mu_2^2.
\end{aligned}$$

This completes the proof of Claim 2.

Let us now recall a well-known property from linear algebra, $[\operatorname{tr}(S')]^2 - 2c_{t-2} = \operatorname{tr}(S')^2$. Therefore, using this fact and Claims 1 and 2 we have that

$$\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 - 2c_{t-2} = \mu_1^2 + 2\mu_{12}\mu_{21} + \mu_2^2.$$

Simplifying, we obtain $2\mu_{12}\mu_{21} = 2\mu_1\mu_2 - 2c_{t-2}$. So, $c_{t-2} = \mu_1\mu_2 - \mu_{12}\mu_{21}$.

The equations in (4.5) and the closed formula given in Proposition 6.1 Part (c) (see Appendix on Page 50) imply that for t odd this holds

$$c_{t-2} = \mu_1\mu_2 - \mu_{12}\mu_{21} = (-1)^{m+n+t+1}(1 - F_t^2),$$

and for t even this holds

$$c_{t-2} = \mu_1\mu_2 - \mu_{12}\mu_{21} = (-1)^{m+n+t+1}(F_t^2).$$

In addition, from Claim 1 we have $c_{t-1} = \mu_1 + \mu_2$. This gives that c_{t-1} equals $-A$ if t is odd and equals $-B$ if t is even, where A and B are as given in the statement of this theorem. This completes the proof. □

We now define

$$\mathbf{w}_1 = \begin{bmatrix} -F_1 \\ -F_2 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -F_2 \\ -F_3 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -F_3 \\ -F_4 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{w}_{t-2} = \begin{bmatrix} -F_{t-2} \\ -F_{t-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.6}$$

It is easy to see that $\{\mathbf{w}_j : j = 1, 2, \ldots, t-2\}$ is linearly independent. From (4.1) we know that $S(m, n, t) = u^T_{n+1,t} \cdot v_{m-1,t} + u^T_{n-1,t} \cdot v_{m,t}$, and from (4.2) also know that $v_{m-1,t} = [F_{m-1}, F_m, \ldots, F_{m-1+t-1}]$ and $v_{m,t} = [F_m, F_{m+1}, \ldots, F_{m+t-1}]$. Note that $v_{m-i,t}.\mathbf{w}_j = 0$ for $i \in \{0, 1\}$ and $j = 1, \ldots, t-2$. Thus, $\mathbf{w}_j$ is orthogonal to $v_{m-i,t}$. This is a geometric proof of the following proposition.

Proposition 4.5. *If $S(m, n, t)$ is as given in (4.1), then $\{\mathbf{w}_j : j \in \{1, 2, \ldots, t-2\}\}$ is the set of the eigenvectors associated to the eigenvalue $\lambda_0 = 0$.*

Proof. We first observe that each entry of the matrix $S(m, n, t)$ can be written as a generalized Fibonacci number of the form $G_{ij} = F_{j-1}G_{i1} + F_jG_{i2}$ for $3 \le i, j \le t$, with initial conditions $G_{i1} = H_{i,1}$ and $G_{i2} = H_{i,2}$. From this observation it is easy to see that each row of $S(m, n, t)$ is a linear combination of the first two rows. Therefore, we can reduce the rest of the $(t-2)$ rows using the following row operations: first, for the i-th row denoted by R_i, $R_i \to R_i - F_{i-2}R_1$, where $i > 2$. Next, for the same i, we apply the row operation, $R_i \to R_i - F_{i-1}R_2$ to obtain the matrix

$$S(m, n, t) = \begin{bmatrix} H_{1,1} & H_{2,1} & H_{3,1} & \cdots & H_{t,1} \\ H_{2,2} & H_{3,2} & H_{4,2} & \cdots & H_{t+1,2} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Using the definition of the entries of the triangle from Proposition 2.1, we have, $H_{j,1} = F_2F_{j+1} - F_jF_1 = F_{j-1}$ and $H_{j,2} = L_j$. Then applying the row operations $R_2 \leftrightarrow R_1$ (interchanging R_1 with R_2) and $R_1 \to R_1 - 3R_2$ on the non-zero rows of $S(m, n, t)$, and finally using the identity $F_1F_j + F_3F_{j-1} = L_j$ and applying $R_1 \to R_2 - R_1$ twice

$$S(m, n, t) = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 & \cdots & F_{t-1} \\ L_1 & L_2 & L_3 & L_4 & \cdots & L_t \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 & \cdots & F_{t-1} \\ F_2 & F_2 & F_3 & F_4 & \cdots & F_t \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We now solve the system of equations $S(m, n, t)\mathbf{x} = \mathbf{0}$ (where $\mathbf{x} = [x_1, x_2, \ldots, x_t]^T$ and $\mathbf{0}$ is the zero vector), to obtain the eigenvectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{t-2}$ given in (4.6). □

For the following proposition we use the same notation as in Theorem 4.4. For example, $\mu_1 = v_{m-1,t} \cdot u^T_{n+1,t}$, and $\mu_{12} = v_{m,t} \cdot u^T_{n+1,t}$ are as given in (4.5), λ_j for $j \in \{1, 2\}$ is as given in Theorem 4.4 Part b and $S(m, n, t)$ as given in (4.1).

Proposition 4.6. *Let* $\mu_1 = v_{m-1,t} \cdot u_{n+1,t}^T$ *and* $\mu_{21} = v_{m-1,t} \cdot u_{n-1,t}^T$*. If* $j \in \{1,2\}$*, then* $\mathbf{x}_j = \mu_{21} u_{n+1,t}^T + (\lambda_j - \mu_1) u_{n-1,t}^T$ *is an eigenvector of* $S(m,n,t)$ *associated to* λ_j*.*

Proof. Let $S' := S(m,n,t) = u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t}$. It is easy to see that $u_{n+1,t}$ and $u_{n-1,t}$ are not in span($\{\mathbf{w}_j : j \in \{1,2,\ldots,t-2\}\}$) (the orthogonal subspace to the vectors $v_{m-1,t}$, $v_{m,t}$, see (4.6)). Therefore, $\mathbb{R}^t = \text{span}(\{u_{n+1,t}, u_{n-1,t}\}) \cup \text{span}(\{\mathbf{w}_j : j \in \{1,2,\ldots,t-2\}\})$.

If $\mathbf{x}_j$ is the eigenvector associated to λ_j for $j \in \{1,2\}$, then there are constants α_j and β_j such that $\mathbf{x}_j = \alpha_j u_{n+1,t}^T + \beta_j u_{n-1,t}^T$. Since $\lambda_j \mathbf{x}_j = S' \mathbf{x}_j$, we have that (with the notation given in (4.5))

$$\begin{aligned}
\lambda_j(\alpha_j u_{n+1,t}^T + \beta_j u_{n-1,t}^T) &= (u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t})\mathbf{x}_j \\
&= (u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t})(\alpha_j u_{n+1,t}^T + \beta_j u_{n-1,t}^T) \\
&= \alpha_j u_{n+1,t}^T \mu_1 + \alpha_j u_{n-1,t}^T \mu_{12} + \beta_j u_{n+1,t}^T \mu_{21} + \beta_j u_{n-1,t}^T \mu_2.
\end{aligned}$$

Since

$$\lambda_j \alpha_j u_{n+1,t}^T + \lambda_j \beta_j u_{n-1,t}^T = (\alpha_j \mu_1 + \beta_j \mu_{21}) u_{n+1,t}^T + (\alpha_j \mu_{12} + \beta_j \mu_2) u_{n-1,t}^T,$$

we have that $\alpha_j = \mu_{21}$ and $\beta_j = \lambda_j - \mu_1$ (or equivalently, $\alpha_j = \lambda_j - \mu_2$ and $\beta_j = \mu_{12}$). Therefore, $\mathbf{x}_j = \mu_{21} u_{n+1,t}^T + (\lambda_j - \mu_1) u_{n-1,t}^T$. □

We note here that for $1 < m, t \leq n$, the slash matrices $S(m,n,t)$ are diagonalizable. In fact, if we define the matrix of eigenvectors $T = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_{t-2} \end{bmatrix}$ and V is the $t \times t$ diagonal matrix with the diagonal elements λ_1, λ_2, and 0 given by the eigenpairs $(\lambda_j, \mathbf{x}_j)$, with $j \in \{1,2\}$ and $(0, \mathbf{w}_l)$ with $l \in \{1,2,\ldots,t-2\}$, then $S(m,n,t) = TVT^{-1}$.

4.2. **Slash matrices and their transposes in the determinant Hosoya triangle.** In this section we analyze $S(m,n,t)$ with $1 \leq n < m$.

Proposition 4.7. *This identity holds in* $\mathcal{H}$*,* $S(m,n,t)^T = S(n,m,t)$*.*

Proof. The ith column of $S(m,n,t)$ is on the $(m+i)$th slash diagonal of the triangle, where the first entry of every column is on the nth backslash diagonal of the triangle. On the other hand, the ith row of $S(n,m,t)$ is on the $(m+i)$th backslash diagonal of the triangle, where the first entry of every row is on the nth slash diagonal of the triangle. Since the determinant Hosoya triangle is symmetric with respect to its median, we have that the columns of $S(m,n,t)$ are equal to the rows of $S(n,m,t)$. □

Corollary 4.8. *If* $S(m,n,t) = u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t}$*, then* $S(n,m,t) = v_{m-1,t}^T u_{n+1,t} + v_{m,t}^T u_{n-1,t}$*.*

Proof. Since $S(m,n,t)^T = S(n,m,t)$, we have that

$$S(m,n,t)^T = (u_{n+1,t}^T \cdot v_{m-1,t} + u_{n-1,t}^T \cdot v_{m,t})^T = v_{m-1,t}^T u_{n+1,t} + v_{m,t}^T u_{n-1,t} = S(n,m,t). \quad \square$$

Using this corollary and the techniques in Subsection 4.1 we can find the eigenvectors and eigenvalues of $S(m,n,t)$ when $m > n$. Since $S(m,n,t)$ is the transpose of $S(n,m,t)$, they both share eigenvalues and characteristic polynomial, in this case Proposition 5.4 works for both.

4.3. **Symmetric matrices of rank two in the determinant Hosoya triangle.** The matrix $S(m,n,t)$ is symmetric if $m=n$. Fixing $m=n=1$, we obtain a special case of this symmetric matrix, $S(1,1,t)$. Its first row and its first column are formed with the first entries of the right-hand side border and the left-hand side border of the triangle, respectively. For simplicity, we use $S(t):=S(1,1,t)$, where t is the size of the matrix. For example, $S(4)$ is depicted in Figure 1(b) on page 36. In this section we discuss properties of $S(t)$ within $\mathcal{H}$. The results here are corollaries of the previous results. We believe that these particular results help in getting a better understanding of the symmetric matrices embedded in the determinant Hosoya triangle.

The matrix $S(t)$ is formally stated as follows

$$S(t)=\begin{bmatrix} H_{1,1} & H_{2,2} & H_{3,3} & \cdots & H_{t,t} \\ H_{2,1} & H_{3,2} & H_{4,3} & \cdots & H_{t+1,t} \\ H_{3,1} & H_{4,2} & H_{5,3} & \cdots & H_{t+2,t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{t,1} & H_{t+1,2} & H_{t+2,3} & \cdots & H_{2t-1,t} \end{bmatrix}. \tag{4.7}$$

Since $S(t)^T=S(t)$, we use $S(t)$ and $S(t)^T$ interchangeably in the results below as needed. From Corollary 3.2 (see also Table 1 or Figure 1) we have that $H_{r,k}=H_{r,r-k}$ (the triangle is symmetric with respect to the median).

From Proposition 4.2 we have the following corollary. Another proof for this corollary follows expanding the indicated sum and using a telescoping sum.

Corollary 4.9. *For a positive integer $t\geq 2$, the trace of the matrix $S(t)$ is given by*

$$\operatorname{tr}(S(t))=\sum_{i=1}^{n} H_{2i-1,i}=\sum_{i=1}^{t}\begin{vmatrix} F_{i+1} & F_i \\ F_i & F_{i+1}\end{vmatrix}=F_{t+1}^2-1.$$

Using the geometry of the determinant Hosoya triangle we can easily see that for any $t\in\mathbb{N}$,

$$\operatorname{tr}(S(t))=F_{t+1}^2-1=H_{2t,t}-1.$$

This geometric property is similar to hockey stick properties present, for example, in the Pascal triangle or the Hosoya triangle (see [7]). In addition, we also see that

$$\operatorname{tr}(S(t))=\begin{cases}\displaystyle\sum_{i=1}^{t-1} F_{i+1}F_{i+2}, & \text{if } t \text{ is odd;}\\ \displaystyle\sum_{i=1}^{t-1} F_{i+1}F_{i+2}+1, & \text{if } t \text{ is even.}\end{cases}$$

The following is a corollary of Theorem 4.4 that provides the eigenvalues of the $t\times t$ symmetric matrices $S(t)$ embedded in the determinant Hosoya traingle. These matrices have two non-zero, (distinct) eigenvalues and the rest of the $(t-2)$ eigenvalues are all zero.

The eigenvectors of the symmetric matrix $S(t)$, associated with the eigenvalue $\lambda_0=0$, are the same vectors as given in the statement of Proposition 4.5. The corollary provides the eigenvectors of the symmetric matrix $S(t)$ associated with the non-zero eigenvalues.

Corollary 4.10. *If $S(t)$ is as given in (4.7), then these hold*

(1) the eigenvalues of $S(t)$ are $\lambda_0=0$ with multiplicity $(t-2)$, and

$$\lambda_1=\frac{c_{t-1}+\sqrt{(c_{t-1})^2-4c_{t-2}}}{2}, \quad \textit{and} \quad \lambda_2=\frac{c_{t-1}-\sqrt{(c_{t-1})^2-4c_{t-2}}}{2},$$

where c_{t-1} and c_{t-2}, for $k \in \mathbb{N}$ are given by

$$c_{t-1} = \begin{cases} F_tF_{t+2} - 2, & \text{if } t = 2k+1; \\ F_tF_{t+2}, & \text{if } t = 2k; \end{cases} \quad \text{and} \quad c_{t-2} = \begin{cases} 1 - F_t^2, & \text{if } t = 2k+1; \\ -F_t^2, & \text{if } t = 2k. \end{cases}$$

(2) If $j \in \{1,2\}$, then $\mathbf{x}_j = \mu_{21} u_{2,t}^T + (\lambda_j - \mu_1) u_{0,t}^T$ is an eigenvector of $S(t)$ associated to λ_j, where $\mu_{21} = F_tF_{t-1}$ and

$$\mu_1 = \begin{cases} F_tF_{t+1}, & \text{if } t \text{ is even ;} \\ F_tF_{t+1} - 1, & \text{if } t \text{ is odd.} \end{cases}$$

5. Backslash matrices of rank two in the determinant Hosoya triangle

A matrix embedded in the determinant Hosoya triangle is *backslash* if its columns are in the backslash diagonals of the triangle. In this section we study this type of matrix. These matrices can be expressed as a sum of products of two vectors (see Figure 5). We begin this section giving some results from [3]. Let $u_{l,t} := [F_l, F_{l+1}, \dots, F_{l+t-1}]$ (as given in (4.2)) and $w_{r,t} := [F_r, F_{r-1}, \dots, F_{r-t+1}]$ and let $B(m,n,t) := u_{m,t}^T \cdot w_{n,t}$. The matrix $B(m,n,t)$ is a type of backslash matrix that is embedded in the Hosoya triangle $\mathcal{H}'$, see Appendix on page 50.

Proposition 5.1 ([3]). *If m, n, t are fixed positive integers with $t \le n$, then $B(m,n,t)$ has these properties,*

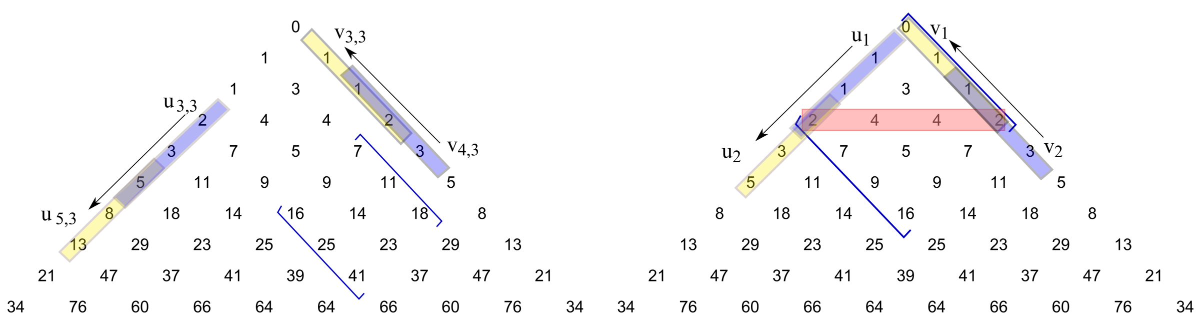
(1) the eigenvalues of $B(m,n,t)$ are $\lambda_{b1} = \operatorname{tr}(B(m,n,t))$ and $\lambda_{b2} = 0$ with algebraic multiplicity 1 and $(t-1)$, respectively,

(2) the trace of $B(m,n,t)$ is given by

$$\operatorname{tr}(B(m,n,t)) = \sum_{i=0}^{t-1} F_{m+i}F_{n-i} = \frac{1}{5}(tL_{m+n} + (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}).$$

If $m, n,$ and t are positive integers with $m, t \le n$, $s = t-1$, and $r_i = (m+n-1)+i$ for $0 \le i \le s$, then the backslash matrix is given by

$$K(m,n,t) := \begin{bmatrix} H_{r_0,m} & H_{r_0-1,m} & H_{r_0-2,m} & \cdots & H_{r_0-s,m} \\ H_{r_1,m+1} & H_{r_1-1,m+1} & H_{r_1-2,m+1} & \cdots & H_{r_1-s,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r_s,m+s} & H_{r_s-1,m+s} & H_{r_s-2,m+s} & \cdots & H_{r_s-s,m+s} \end{bmatrix}. \tag{5.1}$$



(a) (b)

FIGURE 5. (a) Backslash matrix $K(4,4,3)$ (b) Persymmetric matrix $K(4)$ in $\mathcal{H}$.

Proposition 5.2. *If* $u_{m,t} = [F_m, F_{m+1}, \dots, F_{m+t-1}]$ *and* $w_{n,t} = [F_n, F_{n-1}, \dots, F_{n-t+1}]$*, then*

$$\begin{aligned} K(m,n,t) &= B(m+1, n-1, t) + B(m-1, n, t) \\ &= u_{m+1,t}^T \cdot w_{n-1,t} + u_{m-1,t}^T \cdot w_{n,t}. \end{aligned}$$

The proof of this proposition is similar to the proof of Proposition 4.1.

Proposition 5.3. *If* $m, n,$ *and* t *are positive integers, then*

$$\operatorname{tr}(K(m,n,t)) = \left(tL_{m+n+1} + 2(-1)^{n-t-1}F_{m-n+2t-1} + 2(-1)^m F_{n-m+1}\right)/5.$$

Proof. The proposition is an application of Proposition 5.2 and Proposition 5.1 Part (2). □

For the rest of the section of use ζ to represent the eigenvalues of the matrices $K(\cdot,\cdot,\cdot)$.

Proposition 5.4. *Let* $m, n,$ *and* t *be positive integers. If* $t \neq 5$ *and*

$$C = \left(tL_{m+n+1} + (-1)^{n-t-1}2F_{m-n+2t-1} + 2(-1)^m F_{n-m+1}\right)/5, \quad \textit{and}$$
$$D = (-1)^{m+n+t+1}(L_{2t} + (-1)^t(5t^2 - 2))/25,$$

then these hold

(a) the characteristic polynomial of $K(m,n,t)$ *is given by*

$$P(x) = (-1)^t x^{t-2}(x^2 - Cx + D).$$

(b) The eigenvalues of $K(m,n,t)$ *are* $\zeta_0 = 0$ *with multiplicity* $(t-2)$*, and*

$$\zeta_i = \frac{C \pm \sqrt{C^2 - 4D}}{2}, \ \textit{for } i = 1, 2.$$

Proof. We prove Part (a), the proof of Part (b) follows using the quadratic formula on part Part (a). This proof follows similar steps to the proof of Theorem 4.4. Let

$$p(x) = (-1)^t(x^t + c_{t-1}x^{t-1} + c_{t-2}x^{t-2} + \cdots + c_1 x + c_0)$$

be the characteristic polynomial of $K' := K(m,n,t)$. Similar to Lemma 4.3, from Proposition 5.2 we can deduce that K' is of rank 2. This implies that $x^{t-2}(x^2 + c_{t-1}x + c_{t-2}) = 0$. Let x_1 and x_2 be the non-zero eigenvalues of K'. So, the eigenvalues of K' are $x_0 = 0$ and

$$x_1 = \frac{-c_{t-1} + \sqrt{(c_{t-1})^2 - 4c_{t-2}}}{2} \quad \text{and} \quad x_2 = \frac{-c_{t-1} - \sqrt{(c_{t-1})^2 - 4c_{t-2}}}{2}. \tag{5.2}$$

Since $\operatorname{rank}(K') = 2$, from linear algebra we know that c_{t-1} is the trace of K' (the sum of all eigenvalues of K') and that c_{t-2} is the product of the two non-zero eigenvalues of K'. Thus,

$$c_{t-1} = -(x_1 + x_2) \quad \text{and} \quad c_{t-2} = x_1 x_2.$$

We know that

$\nu_1 = \operatorname{tr}(B(m+1, n-1, t)) = w_{n-1,t} \cdot u_{m+1,t}^T$ is the non-zero eigenvalue of $B(m+1, n-1, t)$;

$\nu_2 = \operatorname{tr}(B(m-1, n, t)) = w_{n,t} \cdot u_{m-1,t}^T$ is the non-zero eigenvalue of $(B(m-1, n, t)$;

$\nu_{12} = \operatorname{tr}(B(m+1, n, t)) = w_{n,t} \cdot u_{m+1,t}^T$ is the non-zero eigenvalue of $B(m+1, n, t)$; and

$\nu_{21} = \operatorname{tr}(B(m-1, n-1, t)) = w_{n-1,t} \cdot u_{m-1,t}^T$ is the non-zero eigenvalue of $B(m-1, n-1, t)$.

It is easy to see that $\operatorname{tr}(K') = \nu_1 + \nu_2$. Using an identical procedure as used in the proof of Claim 2 in the proof of Theorem 4.4, we have that $\operatorname{tr}((K')^2) = \nu_1^2 + 2\nu_{12}\nu_{21} + \nu_2^2$. These and the fact that $[\operatorname{tr}(K')]^2 - 2c_{t-2} = \operatorname{tr}(S')^2$ (this is known in linear algebra) imply that

$$\nu_1^2 + 2\nu_1\nu_2 + \nu_2^2 - 2c_{t-2} = \nu_1^2 + 2\nu_{12}\nu_{21} + \nu_2^2.$$

Simplifying, we obtain $2\nu_{12}\nu_{21} = 2\nu_1\nu_2 - 2c_{t-2}$. So, $c_{t-2} = \nu_1\nu_2 - \nu_{12}\nu_{21}$. From Proposition 5.2 we conclude that

$$c_{t-1} = -(\mathrm{tr}(B(m+1, n-1, t)) + \mathrm{tr}(B(m-1, n, t))) = -(\nu_1 + \nu_2).$$

We use identities on Lucas and Fibonacci numbers to obtain c_{t-1} or

$$C = \left(tL_{m+n+1} + 2(-1)^{n-t-1}F_{m-n+2t-1} + 2(-1)^m F_{n-m+1}\right)/5.$$

Finally, since $c_{t-2} = (\nu_1\nu_2 - \nu_{12}\nu_{21})$, we use identities to obtain c_{t-2} or

$$D = (-1)^{m+n+t+1}(L_{2t} + (-1)^t(5t^2 - 2))/25.$$

This complete the proof of Part (a). □

The following three theorems use the same notation and results found in Proposition 5.4. In particular, the value C from Proposition 5.4 simplifies to $C = 2(-1)^m L_{n-m-4} + L_{n+m+1}$ and $D = 0$ when $t = 5$. The proof of the following proposition is similar to the proof of Proposition 5.4. Recall that $K(m, n, t)$ is defined for $m, t \leq n$.

Proposition 5.5. *Let m, n be positive integers with $n \geq 5$. If $t = 5$,*

$$C = 2(-1)^m L_{n-m-4} + L_{n+m+1}, \text{ and } D = 0,$$

then these hold

(a) the characteristic polynomial of $K(m, n, 5)$ is given by

$$P(x) = -x^{t-1}(x - C).$$

(b) The eigenvalues of $K(m, n, 5)$ are $\zeta_0 = 0$ with multiplicity $(t-2)$, $\zeta_1 = C$, and $\zeta_2 = 0$.

We find eigenvectors associated to $\zeta = 0$. It is easy to see that $\{\mathbf{y}_j : j = 1, 2, \ldots, t-2\}$ is linearly independent. From Proposition (5.2) we know that $K(m, n, t) = u^T_{m+1,t} \cdot w_{n-1,t} + u^T_{m-1,t} \cdot w_{n,t}$. Note that if $w_{n,t} := [F_n, F_{n-1}, \ldots, F_{n-t+1}]$, then we have that $w_{n-i,t}.\mathbf{y}_j = 0$ for $i \in \{0, 1\}$ and $j = 1, \ldots, t-2$. Thus, $\mathbf{y}_j$ is orthogonal to $w_{m-i,t}$. This is basically the proof of the following proposition.

Proposition 5.6. *If $K(m, n, t)$ is as given in (5.1), then the eigenvectors associated to the eigenvalue $\zeta_0 = 0$ are given by*

$$\mathbf{y_1} = \begin{bmatrix} -F_1 \\ F_2 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{y_2} = \begin{bmatrix} F_2 \\ -F_3 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{y_3} = \begin{bmatrix} -F_3 \\ F_4 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{y_{t-2}} = \begin{bmatrix} (-1)^n F_{t-2} \\ (-1)^{n-1} F_{t-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Proposition 5.7. *Let $\nu_1 = w_{n-1,t} \cdot u^T_{m+1,t}$, and $\mu_{21} = v_{n-1,t} \cdot u^T_{m-1,t}$. If $j \in \{1, 2\}$, then $\mathbf{x}_j = \nu_{21} u^T_{m+1,t} + (\zeta_j - \nu_1) u^T_{m-1,t}$ is an eigenvector of $K(m, n, t)$ associated to ζ_j.*

Proof. This proof is similar to the proof of Proposition 4.6. Here we replace v by w, μ by ν, and $S' = S(n, m, t)$ by $K' := K(m, n, t) = u^T_{m+1,t} \cdot w_{n-1,t} + u^T_{m-1,t} \cdot w_{n,t}$. □

We note here that for $1 < m, t \leq n$, the backslash matrices $K(m, n, t)$ for $t \neq 5$ are diagonalizable. In fact, if we define the matrix of eigenvectors $U = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{t-2} \end{bmatrix}$ and W is the $t \times t$ diagonal matrix with the diagonal elements as the eigenvalues of $K(m, n, t)$, or ζ_1, ζ_2, and 0 given by the eigenpairs $(\zeta_j, \mathbf{x}_j)$, with $j \in \{1, 2\}$ and $(0, \mathbf{y}_l)$ with $l \in \{1, 2, \ldots, t-2\}$, then $K(m, n, t) = UWU^{-1}$.

Since $D = (-1)^{m+n+t+1}(L_{2t}+(-1)^t(5t^2-2))/25 = 0$ when $t = 5$, we have another eigenvalue equal to zero (see Proposition 5.5 Part (b)). In this Proposition we analyze the case $t = 5$. Proposition 5.8 Part (2) is the same as given in Proposition 5.7. But we restate it here for the case $t = 5$.

Proposition 5.8. *If $K(m, n, 5)$ is as given in (5.1), then*

(1) the eigenvector associated to the eigenvalue $\zeta_2 = 0$ is given by $\mathbf{z}_2^T = [0, 0, 0, 0, 0]$.
(2) the eigenvector associated to the eigenvalue $\zeta_1 = 2(-1)^m L_{n-m-4} + L_{n+m+1}$ is the third column of the matrix $K(m, n, 5)$. Thus,

$$\mathbf{z}_1 = \begin{bmatrix} H_{m+n-3,m} \\ H_{m+n-2,m+1} \\ H_{m+n-1,m+2} \\ H_{m+n,m+3} \\ H_{m+n+1,m+4} \end{bmatrix}.$$

The matrix $K(m, n, 5)$ has rank 2 (easy to check, similar to Lemma 4.3), but it is not diagonalizable since it has only four linearly independent eigenvectors.

5.1. **Persymmetric matrices in the determinant Hosoya triangle.** In this section we discuss a special case for the backslash matrices obtained by taking $m = 1$ and $n = t \neq 5$. These matrices are symmetric about the antidiagonal and are called persymmetric matrices. If $u_{m,t} = [F_m, F_{m+1}, \ldots, F_{m+t-1}]$ and $w_{n,t} = [F_n, F_{n-1}, \ldots, F_{n-t+1}]$, then

$$K(1, n, n) = u_{2,n}^T \cdot w_{n-1,n} + u_{0,n}^T \cdot w_{n,n}.$$

For simplicity we use $K(n)$ for $K(1, n, n)$ with $n \neq 5$. We see the example of the 4×4 persymmetric matrix $K(4)$ in Figure 5 Part (b) on Page 46.

From Proposition 5.3, we know that the trace of the persymmetric matrix $K(n)$ is given by the following formula

$$\operatorname{tr}(K(n)) = (nL_{n+2} - 4F_n)/5 = ((7n - 4)F_{n-1} + 4(n - 1)F_{n-2})/5.$$

We note here that the trace of the persymmetric matrix $K(n)$ for $n > 1$, is the sum of entries of the nth row of the determinant Hosoya triangle.

The following is a corollary of Proposition 5.4 on page 47, this corollary provides the eigenvalues of the persymmetric matrices.

Corollary 5.9. *If $n \neq 5$ and $j \in \{1, 2\}$, then the eigenvalues of $K(n)$ are given by $\zeta_0 = 0$, with multiplicity $(n - 2)$, and*

$$\zeta_j = \frac{-4F_n + nL_{n+2} \pm \sqrt{(4F_n - nL_{n+2})^2 - 4\left(L_{2n} + (-1)^n\left(5n^2 - 2\right)\right)}}{10}.$$

The eigenvectors of the $n \times n$ persymmetric matrix $K(n)$, associated with an eigenvalue $\zeta_0 = 0$, are the same vectors as given in the statement of Proposition 5.6. The following is a corollary of Proposition 5.7 that provides the eigenvectors of the persymmetric matrix $K(n)$ associated with the non-zero eigenvalues.

Corollary 5.10. *If* $\nu_1 = (nF_{n+2} + (n-3)F_n)/5$ *and* $\nu_{21} = (nL_{n-1} - 2F_n)/5$, *then* $\mathbf{x}_j = \nu_{21} u^T_{n+1,t} + (\zeta_j - \nu_1) u^T_{n-1,t}$ *is an eigenvector of* $K(1,n,t)$ *associated to* ζ_j, *for* $j \in \{1,2\}$.

6. Appendix. The Hosoya triangle

The *Hosoya triangle*, denoted by $\mathcal{H}'$, is a triangular array where the entry in position k (taken from left to right) of the rth row is equal to $H'_{r,k} := F_k F_{r-k+1}$, where $1 \le k \le r$ (see Table 2 or [3,7,10,11,14–16]). If we take $H'(1,1) = H'(2,1) = H'(2,2) = H'(3,2) = 1$ instead of H in (2.1) we have a recursive definition of the Hosoya triangle.

1									
1	1								
2	1	2							
3	2	2	3						
5	3	4	3	5					
8	5	6	6	5	8				
13	8	10	9	10	8	13			
21	13	16	15	15	16	13	21		
34	21	26	24	25	24	26	21	34	
55	34	42	39	40	40	39	42	34	55

Table 2. Hosoya triangle.

If $r_i = (m+n-1) + i$ for $i = 0, 1, 2, \ldots, (t-1)$ then the matrix $A(m,n,t)$ embedded in the Hosoya triangle is given by

$$A(m,n,t) = \begin{bmatrix} H'_{r_0,m} & H'_{r_1,m+1} & H'_{r_2,m+2} & \cdots & H'_{r_{t-1},m+t-1} \\ H'_{r_1,m} & H'_{r_2,m+1} & H'_{r_3,m+2} & \cdots & H'_{r_t,m+t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H'_{r_{t-1},m} & H'_{r_t,m+1} & H'_{r_{t+1},m+2} & \cdots & H'_{r_{2(t-1)},m+t-1} \end{bmatrix}.$$

If $u_{n,t} = [F_n, F_{n+1}, \ldots, F_{n+t-1}]$ and $v_{m,t} = [F_m, F_{m+1}, \ldots, F_{m+t-1}]$, then $A(m,n,t) := u^T_{n,t} \cdot v_{m,t}$.

We recall that if $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then Binet formula for Fibonacci numbers is given by $F_n = (\alpha^n - \beta^n)/(\alpha - \beta) = (\alpha^n - \beta^n)/\sqrt{5}$, while the Binet formula for the Lucas numbers is given by $L_n = \alpha^n + \beta^n$.

Proposition 6.1. *For* $A(m,n,t)$ *these hold*

(a) $A(m,n,t)$ *is of rank 1.*
(b) The characteristic equation of $A(m,n,t)$ *is given by* $x^{t-1}(x - \operatorname{tr}(A(m,n,t))) = 0$.
(c) The non-zero eigenvalue of $A(m,n,t)$ *is given by* $\lambda_{a1} = \operatorname{tr}(A(m,n,t)) = v_{m,t} \cdot u^T_{n,t}$. *Thus,*

$$\lambda_{a1} = \begin{cases} (L_{m+n+2t-1} - L_{m+n-1})/5, & \text{if } t \text{ is even;} \\ \left(L_{m+n+2t-1} - L_{m+n-1} - (-1)^{n+t-1} L_{n-m}\right)/5, & \text{if } t \text{ is odd.} \end{cases}$$

(d) The vector $u_{n,t}$ *is the eigenvector associated with* λ_{a1}.
(e) The eigenvalue $\lambda_{a2} = 0$ *has algebraic multiplicity* $(t-1)$.

(f) The matrix $A(m,n,t)$ is diagonalizable and the eigenvectors of B' are given by,

$$\mathbf{u} = \begin{bmatrix} F_n \\ F_{n+1} \\ F_{n+2} \\ \vdots \\ F_{n+t-1} \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -F_{m+1} \\ F_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{v_{t-1}} = \begin{bmatrix} -F_{m+t-1} \\ 0 \\ 0 \\ \vdots \\ F_m \end{bmatrix}.$$

Proof. The proofs of parts (a), (b), (d), and (e) are straightforward applications of linear algebra, so we omit them (for a similar formal proof see [3]).

Proof of Part (c). To prove this part we first show that

$$\operatorname{tr}(A(m,n,t)) = \begin{cases} (L_{m+n+2t-1} - L_{m+n-1})/5, & \text{if } t \text{ is even;} \\ \left(L_{m+n+2t-1} - L_{m+n-1} - (-1)^{n+t-1} L_{n-m}\right)/5, & \text{if } t \text{ is odd.} \end{cases}$$

We observe that $\operatorname{tr}(A(m,n,t)) = \sum_{i=0}^{t-1} H'_{r_{2i},m+i}$ where $r_i = (m+n-1)+i$. By the definition of each entry of the Hosoya triangle, $H'_{r,k} := F_k F_{r-k+1}$, we have that the $\operatorname{tr}(A(m,n,t))$ equals $\sum_{i=0}^{t-1} F_{m+i}F_{n+i}$. Using the Binet formula for F_{m+i} and F_{n+i} (seen above), we obtain

$$\begin{aligned} \operatorname{tr}(A(m,n,t)) &= \sum_{i=0}^{t-1} \left(L_{m+n+2i} - (-1)^{n+i} L_{m-n}\right)/5 \\ &= \frac{1}{5}\sum_{i=0}^{t-1} L_{m+n+2i} - \frac{L_{m-n}}{5}\sum_{i=0}^{t-1}(-1)^{n+i}. \end{aligned} \tag{6.1}$$

If t is even, then $\sum_{i=0}^{t-1}(-1)^{n+i} = 0$ and using the Binet formula for Lucas numbers and (6.1) we have that

$$\operatorname{tr}(A(m,n,t)) = \frac{1}{5}\sum_{i=0}^{t-1} L_{m+n+2i} = \frac{1}{5}\sum_{i=0}^{t-1}(\alpha^{m+n+2i} + \beta^{m+n+2i}).$$

Next, we observe that $\sum_{i=0}^{t-1} \alpha^{m+n+2i} = \alpha^{m+n}\sum_{i=0}^{t-1}\alpha^{2i} = \alpha^{m+n}(\alpha^{2t}/(\alpha^2-1))$ and similarly, $\sum_{i=0}^{t-1} \beta^{m+n+2i} = \beta^{m+n}(\beta^{2t}/(\beta^2-1))$. This implies that $\operatorname{tr}(A(m,n,t))$ is equal to

$$\left((\alpha^{m+n+2t} - \alpha^{m+n})(\beta^2-1) + (\beta^{m+n+2t} - \beta^{m+n})(\alpha^2-1)\right)/(5(\alpha^2-1)(\beta^2-1)).$$

Simplifying the numerator, using the identity $(\alpha^2-1)(\beta^2-1) = -1$, and the fact that $L_k = \alpha^k + \beta^k$ we see that

$$\operatorname{tr}(A(m,n,t)) = -\left(L_{m+n} - L_{m+n+2t} - L_{m+n-2} + L_{m+n+2t-2}\right)/5 = \left(L_{m+n+2t-1} - L_{m+n-1}\right)/5.$$

If t is odd, then $\sum_{i=0}^{t-1}(-1)^{n+i} = (-1)^{n+t-1}$. Therefore, from (6.1), the trace of $A(m,n,t)$ equals

$$\frac{1}{5}\sum_{i=0}^{t-1} L_{m+n+2i} + (-1)^{n+t}\frac{L_{m-n}}{5} = \left(L_{m+n+2t-1} - L_{m+n-1} + (-1)^{n+t} L_{m-n}\right)/5.$$

Next, by part (b) we know that the matrix $A(m,n,t)$ has only one non-zero eigenvalue λ_{a1}, we have that $\lambda_{a1} = \text{tr}(A(m,n,t))$. This completes the proof.

Proof of Part (f). We give a geometric proof for this part. We know from linear algebra that if a matrix A is given by $A = \mathbf{u} \cdot \mathbf{v}^T$, then A is diagonalizabe if and only if $\mathbf{v}^T \cdot \mathbf{u} \neq 0$ if and only if $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. The orthogonal complement of $\mathbf{v}$ is a basis of the null space. In particular, if $W = \{\mathbf{u}, \mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_{t-1}}\}$ is the set of eigenvectors of $A(m,n,t)$ (as seen in the statement of this part), then $\mathbf{v_i}$ is orthogonal to $v_{m,t}$ for all i. □

We now give a second potential problems to work (the first problem was given on page 39).

Problem 2.

Given a prime p, the p-adic valuation of $n \in \mathbb{N}$, denoted by $\nu_p(n)$, is the highest power of p that divides n. Given a sequence of positive integers $(a_n)_{n\geq 0}$ a description of the sequence of valuations $\nu_p(a_n)$ often presents interesting questions. For example, Lengyel [18], among other things, determined this expression for the 2-adic valuation for the Fibonacci numbers:

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2, \pmod 3; \\ 1, & \text{if } n \equiv 3, \pmod 6; \\ 3, & \text{if } n \equiv 6, \pmod{12}; \\ \nu_2(n) + 4, & \text{if } n \equiv 0, \pmod{12}. \end{cases}$$

In Figure 6 part (a), we show the 2-adic valuation for the first 150 values of $\nu_2(F_n)$. From Lengyel's result we conclude that $\nu_2(H'_{r,k}) = \nu_2(F_k) + \nu_2(F_{r-k+2})$. In Figure 6 part (b), we show the 2-adic valuation for $\nu_2(H'_{r,k})$ for $0 \leq r, k \leq 20$.

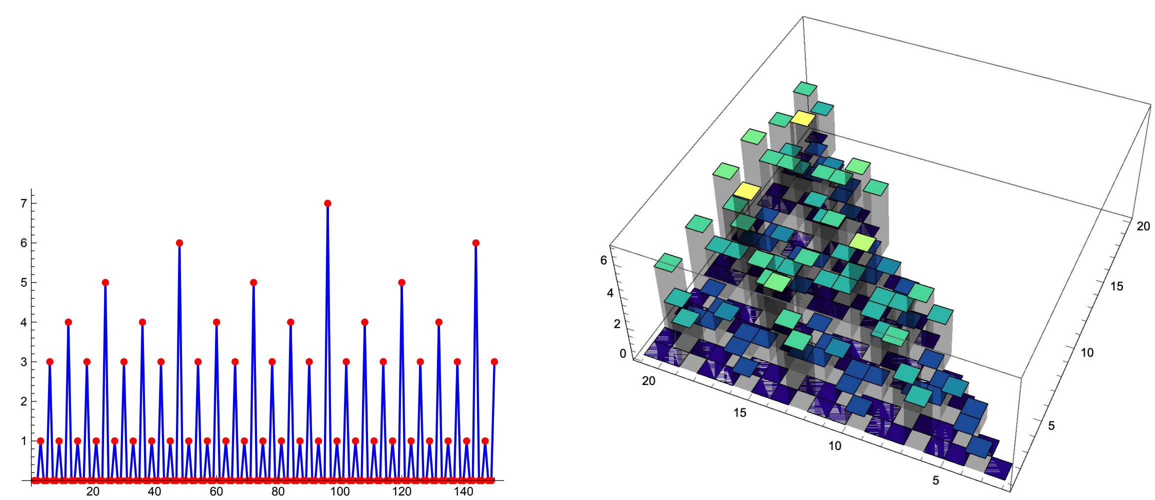


FIGURE 6. (a) The 2-adic valuation of F_n (b) The 2-adic valuation of $H'_{r,k}$.

A natural question is how to describe the 2-adic valuation of the determinant Hosoya triangle and its associated sequences. For example, for $h_n = \sum_{k=1}^{n} H_{n,k}$, the Figure 7 shows the first 150 values of $\nu_2(h_n)$.

7. Acknowledgement

The second and third authors were partially supported by The Citadel Foundation. The authors would like to thank Dr. Mei Chen for the numerous insightful discussions they had with her on various topics from linear algebra, these discussions helped improve the paper.

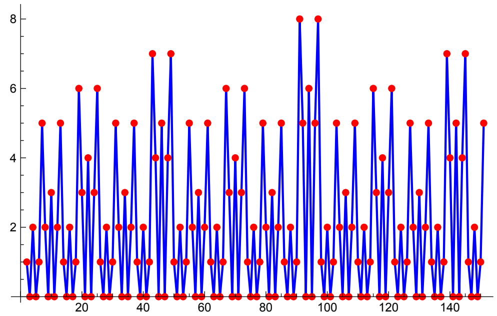


FIGURE 7. The 2-adic valuation of h_n.

REFERENCES

[1] A. Benjamin, J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, The Dolciani Mathematical Expositions, 27, Mathematical Association of America, 2003.

[2] A. T. Benjamin, private communication, 21 July 2020.

[3] M. Blair, R. Flórez, and A. Mukherjee, Matrices in the Hosoya triangle, *Fibonacci Quart.* **57** (2019), no. 5, 15–28.

[4] M. Blair, R. Flórez, and A. Mukherjee, Geometric patterns in the determinant Hosoya triangle. Submitted.

[5] H-Y. Ching, R. Flórez, and A. Mukherjee, Families of integral cographs within a triangular arrays. Submitted. https://arxiv.org/abs/2009.02770

[6] É. Czabarka, R. Flórez, and L. Junes, A discrete convolution on the generalized Hosoya triangle, *J. Integer Seq.* **18** (2018), Article 15.1.6.

[7] R. Flórez, R. Higuita, and A. Mukherjee, Geometric interpretations in the Hosoya triangle of some Fibonacci identities. Submitted, https://arxiv.org/abs/1804.02481v1.

[8] R. Flórez, R. Higuita, and A. Mukherjee, Star of David and other patterns in the Hosoya-like polynomials triangles. *J. Integer Seq.* **21** (2018), Article 18.4.6.

[9] R. Flórez and A. Mukherjee, Solving open problems with students as a first research experience, *Teach. Math. Its Appl.* **10** (2017), 1–7.

[10] R. Flórez, R. Higuita, and L. Junes, GCD property of the generalized star of David in the generalized Hosoya triangle, *J. Integer Seq.*, **17** (2014), Article 14.3.6, 17 pp.

[11] R. Flórez and L. Junes, GCD properties in Hosoya's triangle, *Fibonacci Quart.* **50** (2012), 163–174.

[12] G. H. Golub, and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, fourth edition, 2013.

[13] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[14] H. Hosoya, Fibonacci triangle, *Fibonacci Quart.* **14** (1976), no. 3, 173–178.

[15] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, 2001.

[16] T. Koshy, *Triangular Arrays with Applications*, Oxford University Press, 2011.

[17] G. Lee, J. Kim, and S. Lee, Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices, *Fibonacci Quart.* **40** (2002), no. 3, 203–211.

[18] T. Lengyel, The order of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **33** (1995), no. 3,234–239.

[19] https://math.stackexchange.com/

[20] J. L. Ramírez, The Pascal rhombus and the generalized grand Motzkin paths, *Fibonacci Quart.* **54** (2016), no. 2, 99–104.

[21] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

[22] P. Stanimirović, J. Nikolov, and I. Stanimirović, Generalizations of Fibonacci and Lucas matrices, *Discrete Appl. Math.* **156** (2008), 2606–2619.

[23] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section, Theory and Applications* John Wiley, 1989.

[24] F. Zhang, *Matrix Theory. Basic Results and Techniques*, second edition, Universitext, Springer, 2011.

2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B83, 05A15.

DEPARTMENT OF MATHEMATICAL SCIENCES, THE CITADEL
E-mail address: blairm@citadel.edu,rigo.florez@citadel.edu,mukherjeea1@citadel.edu

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
E-mail address: jlramirezr@unal.edu.co