LINEAR INDEPENDENCE OF INFINITE PRODUCTS GENERATED BY THE LUCAS NUMBERS

DANIEL DUVERNEY AND YOHEI TACHIYA

Dedicated to Professor Iekata Shiokawa on the occasion of his 80th birthday.

Abstract. The purpose of this paper is to give linear independence results for the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{q^n z}{q^{2n}+1}\right),$$

where q ($|q|>1$) and z are algebraic integers with suitable conditions. As an application, we derive that the ten numbers

$$1,\quad \sum_{n=1}^{\infty}\frac{1}{L_{2n}},\quad \prod_{n=1}^{\infty}\left(1\pm\frac{1}{L_{2n}}\right),\quad \prod_{n=1}^{\infty}\left(1\pm\frac{2}{L_{2n}}\right),\quad \prod_{n=1}^{\infty}\left(1\pm\frac{\Phi}{L_{2n}}\right),\quad \prod_{n=1}^{\infty}\left(1\pm\frac{\Phi^{-1}}{L_{2n}}\right)$$

are linearly independent over $\mathbb{Q}(\sqrt{5})$, where L_{2n} is the $2n$-th Lucas number and Φ is the golden ratio, and that

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}+a}\notin\mathbb{Q}(\sqrt{5})$$

for any $a=\pm1,\pm2,\pm\Phi,\pm\Phi^{-1}$.

1. Introduction and main results

Let q be a complex number with $|q|>1$. Then the infinite product

$$H_q(z):=\prod_{n=1}^{\infty}\left(1+\frac{q^n z}{q^{2n}+1}\right)$$

defines an entire function and its logarithmic differentiation furnishes

$$H_q'(z)=H_q(z)\sum_{n=1}^{\infty}\frac{q^n}{q^{2n}+q^n z+1} \tag{1.1}$$

for any complex number z. In particular,

$$H_q(0)=1,\qquad H_q'(0)=\sum_{n=1}^{\infty}\frac{q^n}{q^{2n}+1}. \tag{1.2}$$

In the case where q is a rational integer with $|q|\geq 2$, J.-P. Bézivin [5] proved that the numbers 1 and $H_q^{(\ell)}(z_j)$ $(1\leq j\leq m, 0\leq \ell\leq s)$ are linearly independent over $\mathbb{Q}$ when the rational numbers $z_1,z_2,\dots,z_m$ satisfy certain conditions. Moreover, using the formula (1.1), he showed that the number

$$\sum_{n=1}^{\infty}\frac{q^n}{q^{2n}+q^n a+1}$$

This work was supported by JSPS KAKENHI Grant Number JP18K03201.

is irrational when q is a rational integer with $|q| \geq 2$ and a is a rational number satisfying $(q^\ell + 1)^2 \neq a^2 q^\ell$ for every non-zero rational integer ℓ. Shortly later, P. Bundschuh and K. Väänänen [6] have succeeded in handling the case where q and $z_1, z_2, \ldots, z_m$ are algebraic numbers with suitable conditions, and moreover, obtained a quantitative refinement of Bézivin's result, that is a measure of linear independence for the numbers 1 and $H_q^{(\ell)}(z_j)$ for $1 \leq j \leq m$ and $0 \leq \ell \leq s$.

The purpose of this paper is to give another kind of a sufficient condition for q and the points $z_1, z_2, \ldots, z_m$ such that the numbers $H_q(z_j)$ and $H_q'(z_j)$ $(1 \leq i \leq m)$ are linearly independent over the number field $\mathbb{Q}(q)$. In order to state our main theorem, we need some notations. In what follows, for a complex number z, we define the sequence $\{\theta_n(z)\}_{n\geq 0}$ by

$$\theta_{n+2}(z) = z\theta_{n+1}(z) - \theta_n(z), \qquad n \geq 0, \tag{1.3}$$

with the initial values $\theta_0(z) = 1$ and $\theta_1(z) = z - 1$. For example, it is easy to check that

$$\theta_n(2\cos 2t) = \frac{\cos(2n+1)t}{\cos t}, \qquad n \geq 0$$

for any t with $0 \leq t < \pi/2$, and consequently

$$\begin{cases} \theta_n(0) = \{\overline{1, -1, -1, 1}\}, \\ \theta_n(1) = \{\overline{1, 0, -1, -1, 0, 1}\}, \\ \theta_n(2) = \{\overline{1}\}, \\ \theta_n(-1) = \{\overline{1, -2, 1}\}, \\ \theta_n(-2) = (-1)^n(2n+1), \end{cases} \tag{1.4}$$

where $\{\overline{a_1, a_2, \ldots, a_\ell}\}$ denotes a purely periodic sequence $a_1, a_2, \ldots, a_\ell, \ldots, a_n, \ldots$ with period ℓ. Note that the sequence $\theta_n(z)$ is closely connected to the sequence $V_n(z)$ of the Chebychev polynomials of the third kind, since

$$\theta_n(2z) = V_n(z) \quad (n \geq 0) \tag{1.5}$$

(cf. [7, Chapter 7], [18, Chapter 1]). However, it is more convenient to use $\theta_n(z)$ here because it is a monic polynomial for all $n \geq 0$. For an algebraic integer α, we define the *size* of α by

$$s(\alpha) := \log\max\{1, \overline{|\alpha|}\} = \begin{cases} 0 & \text{if } \alpha = 0, \\ \log\overline{|\alpha|} & \text{if } \alpha \neq 0 \end{cases}$$

(cf. [16, Chapter I]), where $\overline{|\alpha|}$ denotes the maximal modulus of the conjugates of α over $\mathbb{Q}$, namely the *house* of α.

Theorem 1.1. *Let q be an algebraic integer with modulus greater than 1, whose other conjugates, except itself and its complex conjugate, are of modulus less than 1. Let $z_1, z_2, \ldots, z_m$ be distinct algebraic integers in $\mathbb{Q}(q)$ and $\{\theta_n(z_j)\}_{n\geq 0}$ $(j = 1, 2, \ldots, m)$ sequences defined by* (1.3). *Suppose that $s(\theta_n(z_j)) = o(n)$ for every $j = 1, 2, \ldots, m$. Then the $2m$ numbers $H_q(z_j), H_q'(z_j)$ $(j = 1, 2, \ldots, m)$ are linearly independent over $\mathbb{Q}(q)$.*

Theorem 1.1 will be proved in Section 4.

Remark 1.2 (cf. [11])**.** *Let q be as in Theorem 1.1. Then the number q is called a Pisot number or a Pisot–Vijayaraghavan number, if q is a real positive number. The Pisot numbers of degree one are exactly the rational integers greater than one. Also, the number q is called a complex Pisot number, if q is non-real.*

The following Corollaries 1.3 and 1.4 are immediate consequences of Theorem 1.1. Note that (1.1), (1.2), and the five sequences given in (1.4) satisfy the assumptions in Theorem 1.1 when $z_1 = 0$, $z_2 = 1$, $z_3 = -1$, $z_4 = 2$, and $z_5 = -2$.

Corollary 1.3. *Let q be as in Theorem 1.1. Then the ten numbers*

$$1, \qquad H_q(\pm 1) = \prod_{n=1}^{\infty}\left(1 \pm \frac{q^n}{q^{2n}+1}\right), \qquad H_q(\pm 2) = \prod_{n=1}^{\infty}\left(1 \pm \frac{2q^n}{q^{2n}+1}\right),$$

$$H_q'(0) = \sum_{n=1}^{\infty}\frac{q^n}{q^{2n}+1}, \qquad H_q'(\pm 1), \qquad H_q'(\pm 2)$$

are linearly independent over $\mathbb{Q}(q)$.

Corollary 1.4. *Let q be as in Theorem 1.1. For any integer $a \in \{0, \pm 1, \pm 2\}$, the six numbers*

$$1, \qquad \frac{H_q'(a)}{H_q(a)} = \sum_{n=1}^{\infty}\frac{q^n}{q^{2n}+q^n a+1}, \qquad \frac{H_q(a+j)}{H_q(a)} = \prod_{n=1}^{\infty}\left(1 + \frac{jq^n}{q^{2n}+q^n a+1}\right),$$

$j = -a, -a \pm 1, -a \pm 2$ with $j \neq 0$, are linearly independent over $\mathbb{Q}(q)$.

We apply Corollary 1.4 to show linear independence of certain infinite products generated by binary recurrences. Let $A \geq 1$ be a rational integer and $\{R_n\}_{n\geq 0}$ the sequence defined by

$$R_{n+2} = AR_{n+1} + R_n, \qquad n \geq 0,$$

with the initial values $R_0 = 2$ and $R_1 = A$. Then we have a closed form formula $R_n = \alpha^n + \beta^n$ ($n \geq 0$), where α ($|\alpha| > 1$) and $\beta := -\alpha^{-1}$ are the roots of the characteristic polynomial $x^2 - Ax - 1$. Clearly, the numbers $q := \alpha^2, -\alpha^2 \in \mathbb{Q}(q) = \mathbb{Q}(\sqrt{A^2+4})$ satisfy the assumption in Theorem 1.1. Hence, noting that

$$q^n + q^{-n} = \begin{cases} R_{2n} & \text{if } q = \alpha^2, \\ (-1)^n R_{2n} & \text{if } q = -\alpha^2, \end{cases}$$

we find by Corollary 1.4

Corollary 1.5. *Let $\varepsilon = 1$ or -1. Then for any integer $a \in \{0, \pm 1, \pm 2\}$ the six numbers*

$$1, \qquad \sum_{n=1}^{\infty}\frac{1}{\varepsilon^n R_{2n}+a}, \qquad \prod_{n=1}^{\infty}\left(1 + \frac{j}{\varepsilon^n R_{2n}+a}\right), \tag{1.6}$$

$j = -a, -a \pm 1, -a \pm 2$ with $j \neq 0$, are linearly independent over $\mathbb{Q}(\sqrt{A^2+4})$. In particular,

$$\sum_{n=1}^{\infty}\frac{1}{R_{2n}+a}, \qquad \sum_{n=1}^{\infty}\frac{1}{(-1)^n R_{2n}+a} \notin \mathbb{Q}(\sqrt{A^2+4})$$

for any $a = 0, \pm 1, \pm 2$.

For example, substituting $\varepsilon = 1$ and $a = 0$ into (1.6), we obtain the linear independence over $\mathbb{Q}(\sqrt{A^2+4})$ of the six numbers

$$1, \qquad \sum_{n=1}^{\infty}\frac{1}{R_{2n}}, \qquad \prod_{n=1}^{\infty}\left(1 \pm \frac{1}{R_{2n}}\right), \qquad \prod_{n=1}^{\infty}\left(1 \pm \frac{2}{R_{2n}}\right). \tag{1.7}$$

Moreover, substituting $\varepsilon = -1$ and $a = 2$ into (1.6) and noting that $(-1)^n R_{2n} + 2 = (-1)^n R_n^2$ $(n \geq 0)$, we obtain among others the linear independence over $\mathbb{Q}(\sqrt{A^2+4})$ of the six numbers

$$1, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{R_n^2}, \qquad \prod_{n=1}^{\infty} \left(1 + \frac{j\,(-1)^{n-1}}{R_n^2}\right), \qquad j = 1, 2, 3, 4. \tag{1.8}$$

It should be noted that transcendence of numbers $\sum_{n=1}^{\infty} 1/R_{2n}$ and $\sum_{n=1}^{\infty}(-1)^{n-1}/R_n^2$ appeared in (1.7) and (1.8) have been obtained in [8, 10].

According to the choice of q in Theorem 1.1, we can choose more algebraic integers z's. For example, we obtain the following Corollaries 1.6 and 1.7, which will be proved in Section 4.

Corollary 1.6. *Let $\{L_n\}_{n\geq 0}$ be the sequence of the Lucas numbers defined by*

$$L_{n+2} = L_{n+1} + L_n, \qquad n \geq 0,$$

with $L_0 = 2$ and $L_1 = 1$. Then the ten numbers

$$\begin{gathered} 1, \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n}}, \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{1}{L_{2n}}\right), \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{2}{L_{2n}}\right), \\ \prod_{n=1}^{\infty} \left(1 \pm \frac{\Phi}{L_{2n}}\right), \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{\Phi^{-1}}{L_{2n}}\right) \end{gathered} \tag{1.9}$$

are linearly independent over $\mathbb{Q}(\sqrt{5})$, where $\Phi := (1+\sqrt{5})/2$ is the golden ratio. Moreover,

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n} + a} \notin \mathbb{Q}(\sqrt{5})$$

for any $a = \pm 1, \pm 2, \pm\Phi, \pm\Phi^{-1}$.

It has been proved in [2, 4] that, for any integer $m \geq 1$,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_{2m}} = \frac{m}{\sqrt{5}F_{2m}} + \begin{cases} \dfrac{1}{2L_m^2} & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{10F_m^2} & \text{if } m \text{ is odd.} \end{cases}$$

In particular,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + 3} = \frac{2\sqrt{5}+1}{10}, \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 7} = \frac{12\sqrt{5}+5}{90},$$

and therefore the second part of Corollary 1.6 is not valid when $a = L_{2m}$, in particular for $a = 3$. For a survey paper on infinite series containing Fibonacci and Lucas numbers, see [9].

Corollary 1.7. *Let $\{Q_n\}_{n\geq 0}$ be the sequence of the Pell-Lucas numbers defined by*

$$Q_{n+2} = 2Q_{n+1} + Q_n, \qquad n \geq 0,$$

with $Q_0 = 2$ and $Q_1 = 2$. Then the eight numbers

$$1, \qquad \sum_{n=1}^{\infty} \frac{1}{Q_{2n}}, \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{1}{Q_{2n}}\right), \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{2}{Q_{2n}}\right), \qquad \prod_{n=1}^{\infty} \left(1 \pm \frac{\sqrt{2}}{Q_{2n}}\right) \tag{1.10}$$

are linearly independent over $\mathbb{Q}(\sqrt{2})$*, and moreover,*

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n}+a} \notin \mathbb{Q}(\sqrt{2})$$

for any $a = \pm 1, \pm 2, \pm\sqrt{2}$.

Remark 1.8. *For a given algebraic number* q*, there are only finitely many points* $z_1, z_2, \ldots, z_m$ *satisfying the assumptions in Theorem 1.1 and all these points are real (see Lemma 3.5 in Section 3). In this sense, our assumptions are much stricter than those in* [5, 6]*. On the other hand, it should be noted that the points* $z_1, z_2, \ldots, z_m$ *in Theorem 1.1 are effectively computable and include* $z = 0, \pm 1, \pm 2$ *independently of the choice of* q*, whereas* $z = \pm 2$ *are excluded points in* [5, 6]*. Moreover, the results in* [5] *are valid only if* q *is a rational integer. In* [6]*, they are valid also when* q *satisfies the hypotheses of our Theorem 1.1. However, it does not seem easy to apply them to a specifically given set of values of* $H_q(z)$ *for* $z \neq \pm 2$ *when* q *is not a rational number. This could be the matter for further study.*

Our paper is organized as follows. In Section 2, we will present some known arithmetical results and closed form expressions for the infinite products involving Fibonacci and Lucas numbers. In Section 3, we give an expression of $H_q(z)$ by means of the sequence $\{\theta_n(z)\}_{n\geq 0}$, using a remarkable formula which connects the infinite product $H_q(z)$ to the Tschakaloff function (see Lemma 3.1 below). This formula was used in [5, Théorème 1] and [6, Theorem 3] together with a criterion of linear independence for the values of the Tschakaloff function. In our work, we use an elementary criterion of irrationality (Lemma 4.1), together with an expression of $H_q(z)$ obtained in Section 3. Theorem 1.1 and Corollaries 1.6, 1.7 will be shown in Section 4.

2. Some known results and closed forms

Mahler's method is one of the few successful methods for approaching to transcendence and algebraic independence of infinite products involving Fibonacci and Lucas numbers (cf. [20]). For example, the second author [21] derived that for any positive integer j the infinite product

$$\gamma_j := \prod_{n=1}^{\infty}\left(1 + \frac{j}{L_{2^n}}\right)$$

is transcendental, except for only one algebraic case $\gamma_2 = \sqrt{5}$. This result was extended in [15] to algebraic independence over $\mathbb{Q}$ of the numbers $\gamma_1, \gamma_3, \ldots, \gamma_m$ for any integer $m \geq 3$. In [17], the algebraic independence over $\mathbb{Q}$ of the two numbers

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{F_{2^n}}\right), \qquad \prod_{n=1}^{\infty}\left(1 + \frac{1}{L_{2^n}}\right)$$

is proved, where $\{F_n\}_{n\geq 0}$ is the sequence of the Fibonacci numbers. Note that in the above cases the subscripts of the Lucas numbers form a geometric progression, while in our Corollary 1.6 they form an arithmetic progression. On the other hand, the algebraic case for γ_2 results from the use of *telescoping* infinite products, and this can also be used in our situation. Indeed, let q be any complex number with $|q| > 1$ and m be a positive integer. Noting that

$$1 + \frac{q^m - q^{-m}}{q^n - q^{-n}} = \frac{1 + q^{-n+m}}{1 + q^{-n}} \cdot \frac{1 - q^{-n-m}}{1 - q^{-n}},$$

we have a telescoping infinite product and get

$$\prod_{n=1}^{\infty}\left(1+\frac{q^m-q^{-m}}{q^n-q^{-n}}\right)=\frac{\prod_{n=0}^{m-1}(1+q^n)}{\prod_{n=1}^{m}(1-q^{-n})}. \tag{2.1}$$

Similarly we see that

$$\prod_{n=m+1}^{\infty}\left(1-\frac{q^m-q^{-m}}{q^n-q^{-n}}\right)=\frac{\prod_{n=1}^{m}(1-q^{-n})}{\prod_{n=1}^{m}(1+q^{-n-m})}. \tag{2.2}$$

In particular, substituting $m=1$ and $q=\Phi^2$ into (2.1) and (2.2), we obtain

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}}\right)=2\Phi, \qquad \prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}}\right)=\frac{\Phi}{3}. \tag{2.3}$$

Hence, in the first part of Corollary 1.6 we cannot replace the Lucas numbers by the Fibonacci numbers, since the two numbers (2.3) are clearly linearly dependent over $\mathbb{Q}$. Note that the identities (2.3) yield

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}^2}\right)=\frac{1}{3}(\Phi+1)$$

and the well-known formula

$$\prod_{n=2}^{\infty}\frac{F_{2n}+1}{F_{2n}-1}=3$$

([19], see also [13, p. 49, 363]). Now we return to (2.1). Replacing m by $2m$ yields

$$\prod_{n=1}^{\infty}\left(1+\frac{q^{2m}-q^{-2m}}{q^n-q^{-n}}\right)=\frac{\prod_{n=0}^{2m-1}(1+q^n)}{\prod_{n=1}^{2m}(1-q^{-n})}. \tag{2.4}$$

On the other hand, replacing q by q^2 in (2.1) yields

$$\prod_{n=1}^{\infty}\left(1+\frac{q^{2m}-q^{-2m}}{q^{2n}-q^{-2n}}\right)=\frac{\prod_{n=0}^{m-1}(1+q^{2n})}{\prod_{n=1}^{m}(1-q^{-2n})}. \tag{2.5}$$

Dividing (2.4) by (2.5), we obtain

$$\prod_{n=1}^{\infty}\left(1+\frac{q^{2m}-q^{-2m}}{q^{2n-1}-q^{-2n+1}}\right)=\frac{\prod_{n=0}^{2m-1}(1+q^n)\prod_{n=1}^{m}(1-q^{-2n})}{\prod_{n=1}^{2m}(1-q^{-n})\prod_{n=0}^{m-1}(1+q^{2n})}.$$

In particular, for $m=1$ and $q=\pm\Phi$, we get

$$\prod_{n=1}^{\infty}\left(1+\frac{\sqrt{5}}{L_{2n-1}}\right)=3\Phi+2, \qquad \prod_{n=1}^{\infty}\left(1-\frac{\sqrt{5}}{L_{2n-1}}\right)=\Phi-2, \tag{2.6}$$

and the identities (2.6) yield

$$\prod_{n=1}^{\infty}\left(1-\frac{5}{L_{2n-1}^2}\right)=-\Phi-1, \qquad \prod_{n=1}^{\infty}\frac{L_{2n-1}+\sqrt{5}}{L_{2n-1}-\sqrt{5}}=-8\Phi-5,$$

where the latter identity is obtained in [1]. Our last case of telescoping infinite product is given by

$$\prod_{n=1}^{\infty}\left(1\pm i\frac{q^m-q^{-m}}{q^n+q^{-n}}\right)=\frac{\prod_{n=0}^{m-1}(1\pm iq^n)}{\prod_{n=1}^{m}(1\mp iq^{-n})}. \tag{2.7}$$

In particular, for $m=1$ and $q=\Phi^2$, we find that

$$\prod_{n=1}^{\infty}\left(1\pm\frac{i\sqrt{5}}{L_{2n}}\right)=\frac{\Phi^2(1\pm i)}{\Phi^2\mp i},\qquad \prod_{n=1}^{\infty}\left(1+\frac{5}{L_{2n}^2}\right)=\frac{2}{3}(\Phi+1). \tag{2.8}$$

3. Lemmas

The proof of Theorem 1.1 depends on a remarkable connection between the function $H_q(z)$ and the Tschakaloff function

$$T_q(z):=\sum_{n=0}^{\infty}q^{-n(n+1)/2}z^n,\qquad |q|>1. \tag{3.1}$$

This connection is due to Bezivin [5, Lemma 2 (a)] and is given by Lemma 3.1 below. For the convenience of the reader, we will recall the proof of this lemma, which is based on Jacobi's triple product identity (cf. [3, Theorem 14.6]):

$$\sum_{n=-\infty}^{\infty}z^nq^{-n^2}=\prod_{n=1}^{\infty}\left(1-q^{-2n}\right)\left(1+zq^{-2n+1}\right)\left(1+z^{-1}q^{-2n+1}\right), \tag{3.2}$$

provided that $|q|>1$ and $z\neq 0$.

Lemma 3.1 ([5, Lemma 2 (a)])**.** *Let q be a complex number with $|q|>1$ and w a non-zero complex number. Then we have*

$$(1+w)H_q(w+w^{-1})=\Lambda_q\left(wT_q(w)+T_q(w^{-1})\right), \tag{3.3}$$

where

$$\Lambda_q:=\prod_{n=1}^{\infty}\frac{q^{3n}}{(q^n-1)(q^{2n}+1)}.$$

Proof. Let q and w be as in Lemma 3.1. Replacing first z by wq^{-1}, and second q by $q^{\frac{1}{2}}$ in Jacobi's triple product identity yields

$$(1+w)\prod_{n=1}^{\infty}\left(1-q^{-n}\right)\left(1+wq^{-n}\right)\left(1+w^{-1}q^{-n}\right)=wT_q(w)+T_q(w^{-1}). \tag{3.4}$$

On the other hand, since

$$1+\frac{\left(w+w^{-1}\right)q^n}{q^{2n}+1}=\frac{(q^n+w)\left(q^n+w^{-1}\right)}{q^{2n}+1}=\frac{q^{2n}}{q^{2n}+1}\left(1+wq^{-n}\right)\left(1+w^{-1}q^{-n}\right),$$

multiplying for n from 1 to infinity yields

$$H_q\left(w+w^{-1}\right)=\prod_{n=1}^{\infty}\frac{q^{2n}}{q^{2n}+1}\left(1+wq^{-n}\right)\left(1+w^{-1}q^{-n}\right). \tag{3.5}$$

Lemma 3.1 follows from (3.4) and (3.5). □

Lemma 3.2. *For any complex number z, we have*

$$H_q(z)=\Lambda_q\sum_{n=0}^{\infty}\theta_n(z)q^{-n(n+1)/2}, \tag{3.6}$$

where $\{\theta_n(z)\}_{n\geq 0}$ is the sequence defined by (1.3).

Proof. Let z be any complex number and w be one of the roots of the polynomial $x^2 - zx + 1$. Then we have

$$\theta_n(z) = \frac{1}{w^n} \sum_{i=0}^{2n} (-w)^i, \quad n \geq 0, \tag{3.7}$$

since $z = w + w^{-1}$ and the sequence in the above right-hand side satisfies the recurrence relation (1.3) with the same initial values as $\theta_n(z)$. Assume that $z \neq -2$. Then $w \neq -1$ and

$$\theta_n(z) = \frac{1}{w^n} \sum_{i=0}^{2n} (-w)^i = \frac{1 + w^{2n+1}}{(1+w)w^n}, \qquad n \geq 0, \tag{3.8}$$

so that by (3.1), (3.3), and (3.8)

$$H_q(z) = H_q(w + w^{-1}) = \Lambda_q \sum_{n=0}^{\infty} \frac{1 + w^{2n+1}}{(1+w)w^n} q^{-n(n+1)/2} = \Lambda_q \sum_{n=0}^{\infty} \theta_n(z) q^{-n(n+1)/2}. \tag{3.9}$$

The equality (3.9) also holds for $z = -2$, since the series $\sum_{n=0}^{\infty} \theta_n(z) q^{-n(n+1)/2}$ converges uniformly on any compact subset of $\mathbb{C}$ and thus defines an entire function. The proof of Lemma 3.2 is completed. $\square$

Remark 3.3. *Although we won't use it here, it is interesting to note that*

$$H_q(2z) = \Lambda_q \sum_{n=0}^{\infty} V_n(z) q^{-n(n+1)/2}$$

by using the sequence $V_n(z)$ of the Chebychev polynomials of the third kind in (3.6) together with (1.5). Moreover, let $U_n(z)$ be the sequence of the Chebychev polynomials of the second kind. As $V_n(z) = U_n(z) - U_{n-1}(z)$ for $n \geq 1$ (cf. [7, p. 87, (7.1.2)]*), we obtain also*

$$H_q(2z) = \Lambda_q \sum_{n=0}^{\infty} U_n(z) \left(1 - q^{-n}\right) q^{-n(n+1)/2}.$$

Finally, let $f_n(z)$ be the sequence of the Fibonacci polynomials. We know by [13, p. 396]*, that $U_n(z) = i^n f_{n+1}(-2iz)$ for $n \geq 0$. Hence,*

$$H_q(iz) = \Lambda_q \sum_{n=0}^{\infty} i^n f_{n+1}(z) \left(1 - q^{-n}\right) q^{-n(n+1)/2}.$$

In particular, for $z = 1$, we obtain the identity

$$\prod_{n=1}^{\infty} \left(1 + \frac{iq^n}{q^{2n} + 1}\right) = \Lambda_q \sum_{n=0}^{\infty} i^n F_{n+1} \left(1 - q^{-n}\right) q^{-n(n+1)/2}.$$

For a generalization of Chebychev and Fibonacci polynomials, see [12].

Next, we investigate linear independence of the sequences $\{\theta_n(z_j)\}_{n\geq 0}$ and $\{\theta'_n(z_j)\}_{n\geq 0}$ for distinct complex numbers $z_1, z_2, \ldots, z_m$. To see this, we consider the generating functions of the polynomial sequences $\{\theta_n(z)\}_{n\geq 0}$ and $\{\theta'_n(z)\}_{n\geq 0}$, defined by

$$f(z,t) := \sum_{n=0}^{\infty} \theta_n(z) t^n, \qquad g(z,t) = \frac{\partial}{\partial z} f(z,t) = \sum_{n=0}^{\infty} \theta'_n(z) t^n. \tag{3.10}$$

Then we can write by using the recurrence relation (1.3)

$$\begin{aligned} f(z,t) &= \theta_0(z) + \theta_1(z)t + \sum_{n=2}^{\infty} \theta_n(z)t^n = 1 + (z-1)\,t + \sum_{n=0}^{\infty} \theta_{n+2}(z)t^{n+2} \\ &= 1 + (z-1)\,t + tz\sum_{n=0}^{\infty} \theta_{n+1}(z)t^{n+1} - t^2 \sum_{n=0}^{\infty} \theta_n(z)t^n \\ &= 1 + (z-1)\,t + tz\,(f(z,t)-1) - t^2 f(z,t). \end{aligned}$$

Therefore, we obtain

$$f(z,t) = \frac{1-t}{t^2 - zt + 1}, \qquad g\,(z,t) = \frac{t\,(1-t)}{(t^2 - zt + 1)^2}. \tag{3.11}$$

These generating functions could have been deduced also, evidently, from the well-known generating function of the sequence $V_n(z)$ (cf. [7, 18]). We observe that $f(z,t)$ and $g(z,t)$ given in (3.11) are irreducible rational functions, except if $z = 2$, in which cases

$$f(2,t) = \frac{1}{1-t}, \qquad g\,(2,t) = \frac{t}{(1-t)^3}. \tag{3.12}$$

In what follows, the sequences $a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(m)}$ of complex numbers are said to be linearly dependent over the field $\mathbb{C}$, if there exist $x_1, x_2, \ldots, x_m \in \mathbb{C}$, not all zero, such that the sequence

$$x_1 a_n^{(1)} + x_2 a_n^{(2)} + \cdots + x_m a_n^{(m)}, \qquad n \geq 0,$$

is identically zero from some point on. If such x's do not exist, then the sequences are said to be linearly independent over $\mathbb{C}$.

Lemma 3.4. *For any distinct complex numbers $z_1, z_2, \ldots, z_m$, the $2m$ sequences $\{\theta_n(z_j)\}_{n\geq 0}$, $\{\theta'_n(z_j)\}_{n\geq 0}$ $(j = 1, 2, \ldots, m)$ are linearly independent over $\mathbb{C}$.*

Proof. Let $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ be complex numbers and suppose that there exists a positive integer N such that

$$\sum_{j=1}^{m} x_j \theta_n(z_j) + \sum_{j=1}^{m} y_j \theta'_n(z_j) = 0 \quad (n \geq N)\,.$$

Then the rational function

$$F(t) := \sum_{j=1}^{m} x_j f(z_j, t) + \sum_{j=1}^{m} y_j g(z_j, t) \tag{3.13}$$

is a polynomial. First assume that one of the z_j is equal to 2. Without loss of generality, we may assume that $z_m = 2$. Then we have by (3.11) and (3.12)

$$F(t) = \frac{x_m}{1-t} + \frac{y_m t}{(1-t)^3} + \sum_{j=1}^{m-1} \left(\frac{x_j\,(1-t)}{t^2 - z_j t + 1} + \frac{y_j t\,(1-t)}{(t^2 - z_j t + 1)^2} \right).$$

Since $F(t)$ is a polynomial, we have $\lim_{t\to 1}\,(t-1)^3\,F(t) = 0$, which proves that $y_m = 0$. Hence, we have $\lim_{t\to 1}\,(t-1)\,F(t) = 0$, and therefore $x_m = 0$. So in (3.13) we may assume that $z_j \neq 2$ $(j = 1, 2, \ldots, m)$ and have

$$F(t) = \sum_{j=1}^{m} \left(\frac{x_j\,(1-t)}{t^2 - z_j t + 1} + \frac{y_j t\,(1-t)}{(t^2 - z_j t + 1)^2} \right). \tag{3.14}$$

For each $j = 1, 2, \ldots, m$, let $w_j \neq 0$ be one of the roots of the polynomial $x^2 - z_j x + 1 = 0$. Then $w_j \neq 1$, since $z_j \neq 2$. Moreover, $w_1, w_2, \ldots, w_m$ are distinct, since so are $z_1, z_2, \ldots, z_m$. Hence, by (3.14)

$$\lim_{t \to w_j} \left(t^2 - z_j t + 1\right)^2 F(t) = 0 = y_j w_j \left(1 - w_j\right),$$

and therefore $y_j = 0 \ (j = 1, 2, \ldots, m)$. Similarly, we can deduce $x_j = 0 \ (j = 1, 2, \ldots, m)$, and the proof of Lemma 3.4 is complete. □

The following lemma allows to find all numbers z satisfying the conditions in Theorem 1.1.

Lemma 3.5. *Let z be an algebraic integer of degree d and $\{\theta_n(z)\}_{n\geq 0}$ be the sequence defined by (1.3). Then $\theta_n(z)$ and $\theta'_n(z)$ are algebraic integers in the field $\mathbb{Q}(z)$ for every $n \geq 0$. Moreover, the following are equivalent.*
(i) *$s(\theta_n(z)) = o(n)$.*
(ii) *$s(\theta'_n(z)) = o(n)$.*
(iii) *There exists a primitive ℓ-th root of unity ζ such that $z = \zeta + \zeta^{-1}$ and $\varphi(\ell) \leq 2d$, where $\varphi(n)$ denotes the Euler's totient function.*

Proof. The first assertion is clear, since by definition (1.3) $\theta_n(z)$ is an integer polynomial for every $n \geq 0$, and so is $\theta'_n(z)$. Now we prove that (i)⇒(iii). Let $\zeta \neq 0$ satisfy $z = \zeta + \zeta^{-1}$. Then ζ is an algebraic integer, since ζ is a root of the monic polynomial $x^2 - zx + 1 = 0$ with algebraic integer coefficients. Moreover, we have $\theta_n(z) \in \mathbb{Z}[z] \subset \mathbb{Q}(\zeta)$. Let σ be any $\mathbb{Q}$-embedding of $\mathbb{Q}(\zeta)$ into $\mathbb{C}$. Assume that $|\sigma(\zeta)| \neq 1$ and we define $c := \max\{|\sigma(\zeta)|, |\sigma(\zeta)|^{-1}\} > 1$. Then by (3.8) and the assumption (i)

$$c^{n/2} < \left|\frac{\sigma(\zeta)^{n+1} + \sigma(\zeta)^{-n}}{1 + \sigma(\zeta)}\right| = |\sigma(\theta_n(z))| \leq e^{s(\theta_n(z))} = e^{o(n)}$$

for large integer n, a contradiction. Thus, $|\sigma(\zeta)| = 1$ for all conjugates $\sigma(\zeta)$ of ζ, and hence, ζ is a root of unity ([14], cf. [22, Lemma 1.6]). Let ℓ be a positive integer such that ζ is a primitive ℓ-th root of unity. Then we obtain a tower of field extensions $\mathbb{Q} \subset \mathbb{Q}(z) \subset \mathbb{Q}(\zeta)$, and consequently

$$[\mathbb{Q}(\zeta) : \mathbb{Q}] = [\mathbb{Q}(\zeta) : \mathbb{Q}(z)] \cdot [\mathbb{Q}(z) : \mathbb{Q}], \tag{3.15}$$

where $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(\ell)$ (cf. [22, Theorem 2.5]) and $[\mathbb{Q}(z) : \mathbb{Q}] = d$. Moreover, $[\mathbb{Q}(\zeta) : \mathbb{Q}(z)] \leq 2$, since ζ is a root of the polynomial $x^2 - zx + 1$ over $\mathbb{Q}(z)$. Thus, we obtain by (3.15) that $\varphi(\ell) \leq 2d$, which proves that (i)⇒(iii).

Next, we show that (iii)⇒(ii). Assume that there exists a root of unity ζ such that $z = \zeta + \zeta^{-1}$. If $z \neq \pm 2$, then $\zeta \neq \zeta^{-1}$, and hence, by (3.11) there exist numbers A, B, C, D in $\mathbb{Q}(\zeta)$ such that

$$\sum_{n=0}^{\infty} \theta'_n(z) t^n = \frac{t(1-t)}{(1-\zeta t)^2 (1-\zeta^{-1}t)^2} = \frac{A}{1-\zeta t} + \frac{B}{(1-\zeta t)^2} + \frac{C}{1-\zeta^{-1}t} + \frac{D}{(1-\zeta^{-1}t)^2}.$$

Thus, we have $\theta'_n(z) \in \mathbb{Q}(\zeta)$ for every $n \geq 0$ and $s(\theta'_n(z)) = O(\log n) = o(n)$, since $|\sigma(\zeta)| = 1$ for any $\mathbb{Q}$-embedding σ of $\mathbb{Q}(\zeta)$ into $\mathbb{C}$. Similarly, we obtain $s(\theta'_n(\pm 2)) = o(n)$ using the expressions

$$\sum_{n=0}^{\infty} \theta'_n(-2) t^n = \frac{t(1-t)}{(t+1)^4}, \qquad \sum_{n=0}^{\infty} \theta'_n(2) t^n = \frac{t}{(1-t)^3},$$

which follow from (3.11) and (3.12), respectively. Thus, (iii)⇒(ii) is proved.

Differentiating both sides of (1.3) with respect to z, we have

$$\theta_n(z) = \theta'_{n+1}(z) - z\theta'_n(z) + \theta'_{n-1}(z), \qquad n \geq 1,$$

and hence, the assertion (ii)⇒(i) follows immediately. Therefore, the proof of Lemma 3.5 is completed. □

4. Proofs of Theorem 1.1 and Corollaries 1.6, 1.7

We first show the following lemma.

Lemma 4.1. *Let q be as in Theorem 1.1 and $\{\theta_n\}_{n\geq 0}$ a sequence of algebraic integers in $\mathbb{Q}(q)$ satisfying $s(\theta_n) = o(n)$. Suppose that*

$$\sum_{n=0}^{\infty} \theta_n q^{-\frac{n(n+1)}{2}} = 0. \tag{4.1}$$

Then $\theta_n = 0$ for every large n.

Proof. Suppose that (4.1) holds. Then for any positive integer k we have

$$L_k := -\sum_{n=0}^{k} \theta_n q^{\frac{k(k+1)}{2}-\frac{n(n+1)}{2}} = \sum_{n=k+1}^{\infty} \theta_n q^{\frac{k(k+1)}{2}-\frac{n(n+1)}{2}} = \sum_{m=1}^{\infty} \theta_{m+k} q^{-\frac{m(m+1)}{2}-mk}. \tag{4.2}$$

In what follows, $C_1, C_2, \dots$ denote positive constants independent of k. Let d be the degree of q over $\mathbb{Q}$. Choose $\varepsilon > 0$ such that $e^{d\varepsilon} < |q|$, which is possible since $|q| > 1$. By the assumption $s(\theta_n) = o(n)$ we have

$$\overline{|\theta_n|} \leq C_1 e^{\varepsilon n}, \qquad n \geq 0. \tag{4.3}$$

Therefore by (4.2) and (4.3)

$$|L_k| \leq C_1 \sum_{m=1}^{\infty} \frac{e^{\varepsilon(m+k)}}{|q|^{mk}} \leq C_1 e^{\varepsilon k} \sum_{m=1}^{\infty} \frac{e^{\varepsilon m}}{|q|^{mk}} \leq C_1 \frac{e^{\varepsilon k}}{|q|^k} \sum_{m=1}^{\infty} \frac{e^{\varepsilon m}}{|q|^{m-1}},$$

so that

$$|L_k| \leq C_2 \left(e^{\varepsilon}|q|^{-1}\right)^k. \tag{4.4}$$

Now, let $\sigma_1, \sigma_2, \dots, \sigma_\ell$ be all $\mathbb{Q}$-embeddings of $\mathbb{Q}(q)$ into $\mathbb{C}$, except the identity map and the complex conjugation. Note that $\ell = d-1$ if $\mathbb{Q}(q)$ is totally real, and $\ell = d-2$ otherwise. Define

$$L_{k,j} := \sigma_j(L_k) = -\sum_{n=0}^{k} \sigma_j(\theta_n)\, \sigma_j(q)^{\frac{k(k+1)}{2}-\frac{n(n+1)}{2}}, \qquad j = 1, 2, \dots, \ell.$$

Since $|\sigma_j(q)| < 1$ for $j = 1, 2, \dots, \ell$, we have by (4.3)

$$|L_{k,j}| \leq \sum_{n=0}^{k} \overline{|\theta_n|} \leq C_1 \sum_{n=0}^{k} e^{\varepsilon n} \leq C_3 e^{\varepsilon k}, \qquad j = 1, 2, \dots, \ell. \tag{4.5}$$

Clearly, L_k is an algebraic integer in $\mathbb{Q}(q)$, and hence, the norm of L_k over $\mathbb{Q}$

$$N_{\mathbb{Q}(q)/\mathbb{Q}}(L_k) := L_k M_k \prod_{j=1}^{\ell} L_{k,j}$$

is a rational integer, where $M_k := 1$ if $\mathbb{Q}(q)$ is totally real, and M_k is the complex conjugate of L_k otherwise. By (4.4) we have $|M_k| \leq \max\{1, |L_k|\} = 1$ for every large k, and so by (4.4) and (4.5)

$$\left|N_{\mathbb{Q}(q)/\mathbb{Q}}(L_k)\right| \leq C_4 (e^{d\varepsilon}|q|^{-1})^k$$

for every large k. Since $e^{d\varepsilon}|q|^{-1} < 1$ by our choice of ε, $\lim_{k\to\infty} N_{\mathbb{Q}(q)/\mathbb{Q}}(L_k) = 0$ and thus $L_k = 0$ for every large k. Hence, we obtain $\theta_k = 0$ for every large k using the equalities

$$L_k = -\sum_{n=1}^{k-1} \theta_n q^{\frac{k(k+1)}{2} - \frac{n(n+1)}{2}} - \theta_k = q^k L_{k-1} - \theta_k, \qquad k \geq 1.$$

The proof of Lemma 4.1 is completed. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m \in \mathbb{Q}(q)$ and suppose that

$$\sum_{j=1}^{m} \left(x_j H_q(z_j) + y_j H_q'(z_j)\right) = 0. \tag{4.6}$$

Without loss of generality, we may assume that $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ are algebraic integers in $\mathbb{Q}(q)$. Then by (4.6) and Lemma 3.2 with $\Lambda_q \neq 0$, we have

$$\sum_{n=0}^{\infty} \theta_n q^{-n(n+1)/2} = 0,$$

where $\theta_n := \sum_{j=1}^{m} \left(x_j \theta_n(z_j) + y_j \theta_n'(z_j)\right)$. By Lemma 3.5, $\{\theta_n(z_j)\}_{n\geq 0}$ and $\{\theta_n'(z_j)\}_{n\geq 0}$ are sequences of algebraic integers in $\mathbb{Q}(q)$ for every $j = 1, 2, \ldots, m$, and so is $\{\theta_n\}_{n\geq 0}$. Moreover, by Lemma 3.5 (i)⇒(ii) our assumptions $s(\theta_n(z_j)) = o(n)$ for every $j = 1, 2, \ldots, m$ imply that so are $\theta_n'(z_j)$. Hence, we have $s(\theta_n) = o(n)$ and thus the sequence $\{\theta_n\}_{n\geq 0}$ satisfies the assumptions in Lemma 4.1, and consequently

$$\sum_{j=1}^{m} \left(x_j \theta_n(z_j) + y_j \theta_n'(z_j)\right) = 0$$

holds for every large n. Therefore, by Lemma 3.4 we have $x_j = y_j = 0$ for every $j = 1, 2, \ldots, m$, and the proof of Theorem 1.1 is completed. □

Proofs of Corollaries 1.6 and 1.7. For an algebraic number field $\mathbb{K}$ of degree $d \leq 2$, we define $S_{\mathbb{K}}$ by the set of algebraic integers z in $\mathbb{K}$ satisfying $s(\theta_n(z)) = o(n)$. By Lemma 3.5 (i)⇒(iii) such z can be written as $z = \zeta + \zeta^{-1}$, where ζ is a primitive ℓth root of unity with $\varphi(\ell) \leq 4$. Thus, we obtain $\ell = 1, 2, 3, 4, 5, 6, 8, 10, 12$, and hence,

$$S_{\mathbb{K}} = \left\{0, \pm 1, \pm 2, \pm\sqrt{2}, \pm\sqrt{3}, \pm\Phi, \pm\Phi^{-1}\right\} \cap \mathbb{K}. \tag{4.7}$$

In particular, if $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, then we have

$$S_{\mathbb{Q}(\sqrt{5})} = \left\{0, \pm 1, \pm 2, \pm\Phi, \pm\Phi^{-1}\right\}.$$

Applying Theorem 1.1 with $q := \Phi^2$ and the numbers in $S_{\mathbb{Q}(\sqrt{5})}$, we can deduce Corollary 1.6 similar to Corollary 1.5. Corollary 1.7 follows by considering the set

$$S_{\mathbb{Q}(\sqrt{2})} = \left\{0, \pm 1, \pm 2, \pm\sqrt{2}\right\}.$$

Acknowledgements

The authors would like to express their sincere gratitude to Professors Jean-Paul Bézivin, Keijo Väänänen, and Michel Waldschmidt for their help while writing this paper.

References

[1] K. Adegoke, Some infinite product identities involving Fibonacci and Lucas numbers, *Fibonacci Quart.* **55** (2017), 343–351.
[2] G. Almkvist, A solution to a tantalizing problem, *Fibonacci Quart.* **24** (1986), 316–322.
[3] T. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.
[4] R. P. Backstrom, On reciprocal series related to Fibonacci numbers with subscripts in arithmetic progression, *Fibonacci Quart.* **19** (1981), 14–21.
[5] J.-P. Bézivin, Irrationalité de certaines sommes de séries, *Manuscripta Math.* **126** (2008), 41–47.
[6] P. Bundschuh and K. Väänänen, Quantitative linear independence of an infinite product and its derivatives, *Manuscripta Math.* **129** (2009), 423–436.
[7] B. G. S. Doman, *The Classical Orthogonal Polynomials*, World Scientific, 2016.
[8] D. Duverney, Ke. Nishioka, Ku. Nishioka, and I. Shiokawa, Transcendence of Jacobi's theta series and related results, *Number theory (Eger, 1996)*, 157–168, *de Gruyter, Berlin*, 1998.
[9] D. Duverney and I. Shiokawa, On series involving Fibonacci and Lucas numbers I, *DARF* 2007/2008, AIP conference proceedings, Melville, New York, 2008.
[10] C. Elsner, S. Shimomura, and I. Shiokawa, Algebraic relations for reciprocal sums of Fibonacci numbers, *Acta Arith.* **130** (2007), 37–60.
[11] D. Garth, Complex Pisot numbers of small modulus, *C. R. Math. Acad. Sci. Paris* **336** (2003), 967–970.
[12] A. F. Horadam, A synthesis of certain polynomial sequences, in *Applications of Fibonacci Numbers*, Vol. 6, 215–229, Kluwer Academic Publishers, 1996.
[13] T. Koshy, *Fibonacci and Lucas Numbers With Applications*, Vol. 2, John Wiley & Sons, Hoboken, NJ, 2019.
[14] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten, *J. Reine Angew. Math.* **53** (1857), 173–175.
[15] T. Kurosawa, Y. Tachiya, and T. Tanaka, Algebraic independence of infinite products generated by Fibonacci numbers, *Tsukuba J. Math.* **34** (2010), 255–264.
[16] S. Lang, *Introduction to Transcendental Number Theory*, Addison-Wesley, 1966.
[17] F. Luca and Y. Tachiya, Algebraic independence of infinite products generated by Fibonacci and Lucas numbers, *Hokkaido Math. J.* **43** (2014), 1–20.
[18] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall, 2003.
[19] R. S. Melham and A. G. Shannon, Inverse trigonometric and hyperbolic summation formulas involving generalized Fibonacci numbers, *Fibonacci Quart.* **33** (1995), 32–40.
[20] K. Nishioka, *Mahler Functions and Transcendence*, Springer, 1996.
[21] Y. Tachiya, Transcendence of certain infinite products, *J. Number Theory* **125** (2007), 182–200.
[22] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Graduate Texts in Mathematics **83**, Springer, 1997.

MSC2020: 11J72, 11B39

Bâtiment A1, 110 rue du Chevalier Français, 59000 Lille, France
E-mail address: daniel.duverney@orange.fr

Hirosaki University, Graduate School of Science and Technology, Hirosaki 036-8561, Japan
E-mail address: tachiya@hirosaki-u.ac.jp