THE CONTINUED FRACTION PENDULUM

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Abstract. For an irrational number α let $\langle i\alpha\rangle$ denote the fractional part of $i\alpha$ where i is any integer. The three distance theorem states that any t points $\langle i\alpha\rangle$, $1 \le i \le t$, partition the unit interval into gaps of at most three distinct lengths. We know that the process of splitting gaps for increasing t swings like an escalating pendulum in the unit interval and we show that the margins are determined by the denominators of the convergents of the continued fraction representation of α.

Moreover, for a positive real number ξ the points $(\langle i\alpha\rangle, i/\xi)$ provide a strip of a lattice. The main result states that the smallest distance between lattice points is determined by a denominator of a principal convergent. Regarding this and the second smallest distance, lattices are classified into a landscape of phyllotactic patterns.

1. Introduction

We consider distances between regular points in the unit interval and between lattice points. Why? There are several answers:

- we better understand Liang's method for proving the three distance theorem independent of continued fractions,
- the metaphor of an escalating pendulum with residual lengths becoming infinitesimal small provides an enticing approach to continued fractions,
- the margins are closely related to the denominators of the convergents (see Lemma 3.5),
- the smallest distance between lattice points is closely related to the denominator of a principal convergent (see Theorem 4.2, which solves a problem proposal of the 19^{th} International Fibonacci Conference),
- the classification of lattices by pairs of numbers corresponding to the first and second smallest distance provides an interesting fractal structure (see Figure 1),
- we propose a method for calculating the transition from one pair to another.

Concerning the three distance theorem, we recommend [1] and [6]. Lattices as a model of phyllotaxis are described in [2], as well as in the encouraging introduction to Turing's collected works on morphogenesis [10]. Recent results are presented in [9] and [12].

Throughout this paper α is an irrational number. The fractional part of a real number x is denoted by $\langle x\rangle$, i.e., if $\lfloor x\rfloor$ denotes the largest integer not exceeding x, then $\langle x\rangle = x - \lfloor x\rfloor$. We implicitly use the fact that if x is not an integer, then $\langle -x\rangle = 1 - \langle x\rangle$.

If t is a positive integer, then $\langle t\alpha\rangle$ splits the unit interval $[0, 1)$ into a left part $[0, \langle t\alpha\rangle)$ and a right part $[\langle t\alpha\rangle, 1)$. Obviously, $\langle t\alpha\rangle$ is the length of the left part. Whereas, by the fact mentioned above, $\langle -t\alpha\rangle$ is the length of the right part.

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2. Previous Results

From [4] we know two things. One thing is how the three distance theorem evolves as a swinging process of refinement. The other thing relates the first thing to continued fractions.

2.1. The specified three distance theorem.

For a positive integer t, the points $\langle i\alpha\rangle$, $1 \leq i \leq t$, partition the unit interval $[0,1)$ into $t+1$ *gaps*, i.e., left-closed and right-open intervals between neighboring points. The *marginal numbers* l_t on the left and r_t on the right are defined by

$$\langle l_t\alpha\rangle = \min_{1\leq i\leq t}\langle i\alpha\rangle \text{ and } \langle r_t\alpha\rangle = \max_{1\leq i\leq t}\langle i\alpha\rangle, \text{respectively.}$$

As α is irrational, these numbers are unique. The *marginal lengths* λ_t and ϱ_t are defined by $\lambda_t = \langle l_t\alpha\rangle$ and $\varrho_t = 1 - \langle r_t\alpha\rangle$, respectively.

Definition 2.1. *Let α be irrational. For a positive integer t we define the* quadruple of marginal numbers and lengths $Q(t) = (l_t, r_t, \lambda_t, \varrho_t)$.

We collect some facts from [4] that we will often use in the remainder of the paper: [4, Lemma 1] asserts that $\lambda_t \neq \varrho_t$. The specified three distance theorem states that the lengths of the $t+1$ gaps are λ_t, ϱ_t, and if a third length occurs, then it is the sum $\lambda_t + \varrho_t$ of the marginal lengths. We can take s to be the number such that $s = l_t + r_t$. By [4, assertion (5)], we know that $s > t$. If t increases, then the partition of the unit interval is refined by splitting largest gaps. By [4, assertion (7)], s determines the next length foreshortening, more precisely, we write two sentences:

$$Q(u) = Q(t) \text{ for all } u \text{ such that } t \leq u < s, \tag{2.1}$$

$$Q(s) = \begin{cases} (l_t, s, \lambda_t, \varrho_t - \lambda_t) & \text{if } \varrho_t > \lambda_t, \\ (s, r_t, \lambda_t - \varrho_t, \varrho_t) & \text{else.} \end{cases} \tag{2.2}$$

In other words, the larger marginal gap will be foreshortened by the smaller one.

2.2. The escalating pendulum.

Let $n \geq 1$. Given positive real numbers μ_{n-2} and μ_{n-1} such that $\mu_{n-2} > \mu_{n-1}$, we can always divide an interval of length μ_{n-2} into smaller intervals of length μ_{n-1} and get a positive integer quotient b_n and a remainder of length μ_n such that $\mu_{n-1} > \mu_n$. Starting from $\mu_{-1} = 1$ and $\mu_0 = \langle\alpha\rangle$, positive integers b_n and *residual lengths* μ_n are defined recursively:

$$b_n = \lfloor \mu_{n-2}/\mu_{n-1} \rfloor \text{ and } \mu_n = \mu_{n-2} - b_n\mu_{n-1}.$$

If we add $b_0 = \lfloor\alpha\rfloor$ and if $[b_0, b_1, b_2, \ldots]$ denotes an infinite continued fraction (cf. [11]), then [4, Proposition 1] tells us that $\alpha = [b_0, b_1, b_2, \ldots]$. Hence, the above definition provides an enticing approach to continued fractions. Especially, we could call b_n a partial quotient.

Revisiting the unit interval, we know that μ_0 is the length of the interval $[0, \langle\alpha\rangle)$ on the left and that μ_1 is the length of the interval $[b_1\langle\alpha\rangle, 1)$ on the right, since $\mu_1 = 1 - b_1\langle\alpha\rangle$. Now let $n \geq 2$ be even. Then μ_n is the remainder on the left of $[0, \mu_{n-2})$ divided from the right into smaller intervals of length μ_{n-1}. Similarly, if $n \geq 3$ is odd, then μ_n is the remainder on the right of $[1 - \mu_{n-2}, 1)$ divided from the left into smaller intervals of length μ_{n-1}. ThatÂt's what we call "the continued fraction pendulum."

3. New Results

The complication is to connect the marginal lengths of $Q(t)$ and the residual lengths of the pendulum! Well, the ideas of principal and intermediate convergents will help, when they are applied to residual lengths.

3.1. **Principal and intermediate residual lengths.**

Recall that an infinite continued fraction representation $[b_0, b_1, b_2, \ldots]$ for an irrational number α is useful because its initial segments provide rational approximations to the number. These rational numbers are called the principal convergents of the continued fraction. Starting from $q_{-1} = 0$ and $q_0 = 1$, the denominators of the principal convergents are defined recursively:

$$q_n = q_{n-2} + b_n q_{n-1} \text{ for } n \geq 1.$$

Of course, $q_1 = b_1$. As b_1 like any subsequent partial quotient is positive, it follows $q_0 \leq q_1$. Note that $q_1 < q_2 < q_3 < \ldots$ The denominators of the intermediate convergents are defined as follows:

$$q_{n,i} = q_{n-2} + i q_{n-1} \text{ for positive integers } n \text{ and } i < b_n.$$

The denominators of the principal convergents could be included symbolically by means of $q_n = q_{n,b_n}$ for $n \geq 1$. For the proper intermediate case that $1 \leq i < b_n$, we are sure that $q_{n,i} < q_{n,i+1}$. Our terminology is shortened to "q_n or $q_{n,i}$ is a denominator" with the words "of a principal or intermediate convergent" implicitly understood. Similarly, we say that "q_n is a principal denominator". In the sequel, the symbols q_n and $q_{n,i}$ always denote denominators of a given irrational number α.

By the way, the denominators are independent of b_0 or, equivalently, they only depend on the fractional part $\langle \alpha \rangle$. Furthermore, the partial quotients and denominators of $\langle \alpha \rangle$ and $\langle -\alpha \rangle$ are almost the same but shifted by one.

Note that $q_{n,1} < \ldots < q_{n,b_n} < q_{n+1,1}$ increases stepwise by q_{n-1}. Let t be a positive integer. As $q_{1,1} = 1$, it turns out that

$$q_{n,i} \leq t < q_{n,i} + q_{n-1} \text{ for unique positive integers } n \text{ and } i \leq b_n. \tag{3.1}$$

The induced segmentation of positive integers is helpful in the proof of Lemma 3.4.

Here we define the *intermediate residual lengths*:

$$\mu_{n,i} = \mu_{n-2} - i\mu_{n-1} \text{ for positive integers } n \text{ and } i < b_n.$$

A residual length μ_n could be included symbolically by means of $\mu_n = \mu_{n,b_n}$. From now on, μ_n is called a *principal residual length* and the term "residual length" will cover both principal and intermediate residual lengths. We see that $\mu_{n-2} > \mu_{n,1} > \ldots > \mu_{n,b_n}$ decreases stepwise by μ_{n-1}. At the same time, if $b_n > 1$ then $\mu_{n,b_n-1} > \mu_{n-1}$. We conclude that there is a decreasing sequences of $b_n - 1$ proper intermediate residual lengths between μ_{n-2} and μ_{n-1} for $n \geq 1$.

Definition 3.1. *Let α be irrational and $[b_0, b_1, b_2, \ldots]$ its continued fraction representation. For positive integers n and $i \leq b_n$ we define the* quadruple $Q_{n,i}$ of denominators and residual lengths *via* $Q_{n,i} = \begin{cases} (q_{n-1}, q_{n,i}, \mu_{n-1}, \mu_{n,i}) & \text{if } n \text{ is odd,} \\ (q_{n,i}, q_{n-1}, \mu_{n,i}, \mu_{n-1}) & \text{else.} \end{cases}$

We can easily see that both the sets of denominators $\{q_{n-1}, q_{n,i}\}$ and residual lengths $\{\mu_{n-1}, \mu_{n,i}\}$ are the same whether or not n is odd.

3.2. **Quadruple equations.**

By means of the next lemmas we want to show that for $t \geq 1$ the marginal numbers and lengths of $Q(t)$ correspond to a quadruple of denominators and residual lengths.

Lemma 3.2. *Let α be irrational and $[b_0, b_1, b_2, \ldots]$ its continued fraction representation. If n is a positive integer such that $Q(q_{n,1}) = Q_{n,1}$ then*

$$Q(q_{n,i}) = Q_{n,i} \text{ for all } i \text{ such that } 1 \leq i \leq b_n.$$

Proof. Let n be a positive integer such that $Q(q_{n,1}) = Q_{n,1}$. The consequent is shown by induction on i, case $i = 1$ being given. For the induction step, assume $Q(q_{n,i}) = Q_{n,i}$ and let $i + 1 \leq b_n$. Thus, $i < b_n$ and, hence, $\mu_{n,i} > \mu_{n-1}$. Put $z = q_{n,i}$ and our assumption states

$$(l_z, r_z, \lambda_z, \varrho_z) = \begin{cases} (q_{n-1}, q_{n,i}, \mu_{n-1}, \mu_{n,i}) & \text{if } n \text{ is odd,} \\ (q_{n,i}, q_{n-1}, \mu_{n,i}, \mu_{n-1}) & \text{else.} \end{cases}$$

Put $s = l_z + r_z$ and note that $s = q_{n,i} + q_{n-1}$. By (2.2),

$$Q(s) = \begin{cases} (q_{n-1}, s, \mu_{n-1}, \mu_{n,i} - \mu_{n-1}) & \text{if } n \text{ is odd,} \\ (s, q_{n-1}, \mu_{n,i} - \mu_{n-1}, \mu_{n-1}) & \text{else.} \end{cases}$$

Since $i < b_n$, we also have $q_{n,i} + q_{n-1} = q_{n,i+1}$ and $\mu_{n,i} - \mu_{n-1} = \mu_{n,i+1}$. Therefore,

$$Q(s) = \begin{cases} (q_{n-1}, q_{n,i+1}, \mu_{n-1}, \mu_{n,i+1}) & \text{if } n \text{ is odd,} \\ (q_{n,i+1}, q_{n-1}, \mu_{n,i+1}, \mu_{n-1}) & \text{else.} \end{cases}$$

The latter proves $Q(q_{n,i+1}) = Q_{n,i+1}$ and completes the induction step. □

Lemma 3.3. *Let α be irrational and $[b_0, b_1, b_2, \ldots]$ its continued fraction representation. For positive integers n and $i \leq b_n$, we have $Q(q_{n,i}) = Q_{n,i}$.*

Proof. By Lemma 3.2, it suffices to show $Q(q_{n,1}) = Q_{n,1}$. The latter is shown by induction on $n \geq 1$. For $n = 1$, on the one hand, $Q(q_{1,1}) = (1, 1, \langle\alpha\rangle, 1 - \langle\alpha\rangle)$, since $q_{1,1} = 1$. On the other hand, $Q_{1,1} = (q_0, q_{1,1}, \mu_0, \mu_{1,1})$, as 1 is odd. Thus, $Q(q_{1,1}) = Q_{1,1}$. For the induction step, we assume $Q(q_{n,1}) = Q_{n,1}$. By Lemma 3.2, $Q(q_n) = Q_{n,b_n}$, i.e., for $z = q_n$ it holds

$$(l_z, r_z, \lambda_z, \varrho_z) = \begin{cases} (q_{n-1}, q_n, \mu_{n-1}, \mu_n) & \text{if } n \text{ is odd,} \\ (q_n, q_{n-1}, \mu_n, \mu_{n-1}) & \text{else.} \end{cases}$$

Put $s = l_z + r_z$ and note that $s = q_n + q_{n-1}$ and $\mu_{n-1} > \mu_n$. By (2.2),

$$Q(s) = \begin{cases} (s, q_n, \mu_{n-1} - \mu_n, \mu_n) & \text{if } n \text{ is odd,} \\ (q_n, s, \mu_n, \mu_{n-1} - \mu_n) & \text{else.} \end{cases}$$

By $q_n + q_{n-1} = q_{n+1,1}$ and $\mu_{n-1} - \mu_n = \mu_{n+1,1}$, it follows

$$Q(s) = \begin{cases} (q_{n+1,1}, q_n, \mu_{n+1,1}, \mu_n) & \text{if } n+1 \text{ is even,} \\ (q_n, q_{n+1,1}, \mu_n, \mu_{n+1,1}) & \text{else.} \end{cases}$$

The latter yields $Q(q_{n+1,1}) = Q_{n+1,1}$. So the induction step is completed and $Q(q_{n,1}) = Q_{n,1}$ holds for all $n \geq 1$. □

Lemma 3.4. *Let α be irrational and $[b_0, b_1, b_2, \ldots]$ its continued fraction representation. For positive integers t, n, and $i \leq b_n$, we have $Q(t) = Q_{n,i}$ if and only if $q_{n,i} \leq t < q_{n,i} + q_{n-1}$.*

Proof. For the "if"-part let t, n, and $i \leq b_n$ be positive integers such that $q_{n,i} \leq t < q_{n,i} + q_{n-1}$. By Lemma 3.3, we get $Q(q_{n,i}) = Q_{n,i}$. Put $z = q_{n,i}$ so that $Q(z) = Q_{n,i}$, i.e.,

$$(l_z, r_z, \lambda_z, \varrho_z) = \begin{cases} (q_{n-1}, q_{n,i}, \mu_{n-1}, \mu_{n,i}) & \text{if } n \text{ is odd,} \\ (q_{n,i}, q_{n-1}, \mu_{n,i}, \mu_{n-1}) & \text{else.} \end{cases}$$

Put $s = l_z + r_z$ and note that $s = q_{n,i} + q_{n-1}$. Since $z \le t < s$, by (2.1), $Q(t) = Q(z)$, so $Q(t) = Q_{n,i}$ as desired.

For the "only if"-part let $Q(t) = Q_{n,i}$. By (3.1), $q_{m,j} \le t < q_{m,j} + q_{m-1}$ for unique positive integers m and $j \le b_m$. By the "if"-part, we get $Q(t) = Q_{m,j}$, so we get $Q_{n,i} = Q_{m,j}$. So $\{q_{n-1}, q_{n,i}\} = \{q_{m-1}, q_{m,j}\}$ and $\{\mu_{n-1}, \mu_{n,i}\} = \{\mu_{m-1}, \mu_{m,j}\}$. A moment's thought shows that each of the latter equations implies $n = m$ and $i = j$. Finally, we rewrite $q_{n,i} \le t < q_{n,i} + q_{n-1}$. □

3.3. **Key findings.**

The following lemmas are useful for the proof of our main theorem. The first key result links residual lengths and denominators.

Lemma 3.5. *Let α be irrational and $[b_0, b_1, b_2, \ldots]$ its continued fraction representation. For positive integers n and $i \le b_n$ we have*

$$\mu_{n,i} = \langle (-1)^n q_{n,i}\alpha \rangle.$$

Moreover, the equation $\mu_n = \langle (-1)^n q_n \alpha \rangle$ holds for all $n \ge 0$.

Proof. Let $n \ge 1$ and $1 \le i \le b_n$. By Lemma 3.3, for $t = q_{n,i}$ we get $Q(t) = Q_{n,i}$, i.e.,

$$(l_t, r_t, \lambda_t, \varrho_t) = \begin{cases} (q_{n-1}, q_{n,i}, \mu_{n-1}, \mu_{n,i}) & \text{if } n \text{ is odd,} \\ (q_{n,i}, q_{n-1}, \mu_{n,i}, \mu_{n-1}) & \text{else.} \end{cases}$$

As $\lambda_t = \langle l_t \alpha \rangle$ and $\varrho_t = 1 - \langle r_t \alpha \rangle$, it follows either

$\mu_{n-1} = \langle q_{n-1}\alpha \rangle$ and $\mu_{n,i} = 1 - \langle q_{n,i}\alpha \rangle$ if n is odd, or

$\mu_{n,i} = \langle q_{n,i}\alpha \rangle$ and $\mu_{n-1} = 1 - \langle q_{n-1}\alpha \rangle$ if n is even.

Thus, $\mu_{n-1} = \langle (-1)^{n-1} q_{n-1}\alpha \rangle$ and $\mu_{n,i} = \langle (-1)^n q_{n,i}\alpha \rangle$. By these two things, the "moreover"-part and the general equation are proved. □

Reversion of the "moreover"-part yields that if $n \ge 0$, then

$$\langle q_n \alpha \rangle = \begin{cases} \mu_n & \text{if } n \text{ is even,} \\ 1 - \mu_n & \text{else.} \end{cases}$$

In the case of the golden ratio $\Phi = [1, 1, 1, \ldots]$, we rewrite:

$$\langle F_{n+1} \Phi \rangle = \begin{cases} \Phi^{-(n+1)} & \text{if } n \text{ is even,} \\ 1 - \Phi^{-(n+1)} & \text{else,} \end{cases}$$

using $q_n = F_{n+1}$ and $\mu_n = \Phi^{-(n+1)}$ where $(F_n)_{n \ge 0}$ is the sequence of Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$ Substituting $n + 1$ by n provides a solution to [5, exercise 31]:

$$\langle F_n \Phi \rangle = \begin{cases} \Phi^{-n} & \text{if } n \text{ is odd,} \\ 1 - \Phi^{-n} & \text{else.} \end{cases}$$

Notably, the latter is generalized by our reversion of the "moreover"-part. We may also observe that this exhibits a parallel between the proof of Theorem 4.2 and the proof of the special case for $\alpha = \Phi$ in [3].

The second key result concerns the minimal marginal length.

Lemma 3.6. *Let α be irrational and t a positive integer. If $n \geq 0$ satisfies $q_n \leq t < q_{n+1}$ then* $\min(\lambda_t, \varrho_t) = \mu_n$.

Proof. Let $t \geq 1$ and $n \geq 0$ such that $q_n \leq t < q_{n+1}$. Of course, $q_n \leq t < q_{n+1,1}$ or $q_{n+1,i} \leq t < q_{n+1,i+1}$ for some $1 \leq i < b_{n+1}$. By Lemma 3.4, in the first case, $Q(t) = Q_{n,b_n}$ and so $\{\lambda_t, \varrho_t\} = \{\mu_{n-1}, \mu_n\}$ and, in the second case, $Q(t) = Q_{n+1,i}$ and so $\{\lambda_t, \varrho_t\} = \{\mu_n, \mu_{n+1,i}\}$. As $\mu_n < \mu_{n-1}$ and $\mu_n < \mu_{n+1,i}$ for $1 \leq i < b_{n+1}$, we conclude $\min(\lambda_t, \varrho_t) = \mu_n$. □

4. Distances between lattice points

As is well known, the appearance of the Fibonacci numbers in criss-crossing spiral patterns of phyllotaxis (e.g. capitulum of a sunflower) is due to the golden divergence angle. The so-called cylindrical model (e.g. for a pineapple) requires two parameters: a real number α that determines the divergence angle $2\pi\alpha$ and a positive real number ξ that determines the density or vertical compression. Following Coxeter [2], we consider the unrolled cylinder as a strip over the unit interval with lattice points $(\langle i\alpha\rangle, i/\xi)$.

Definition 4.1. *For real numbers α and ξ we define $\delta(\xi, 0) = 1$ and $\delta(\xi, i) = \sqrt{\langle i\alpha\rangle^2 + (i/\xi)^2}$ for non-zero integers i.*

Let t be a positive integer. Then $\delta(\xi, t)$ is the distance between the point $(\langle t\alpha\rangle, t/\xi)$ and the left-hand end-point of the unit interval $(0, 0)$. Whereas, $\delta(\xi, -t)$ is the distance between the same point $(\langle t\alpha\rangle, t/\xi)$ and the right-hand end-point of the unit interval $(1, 0)$, since $\langle -t\alpha\rangle = 1 - \langle t\alpha\rangle$. Moreover, $\delta(\xi, 0) = 1$ is the distance between $(0, 0)$ and $(1, 0)$.

One more helpful fact is that $\delta(\xi, i) \geq |i|/\xi$, and this one ensures that there is a smallest distance by means of $\delta(\xi, w) = \min_{i\in\mathbb{Z}} \delta(\xi, i)$ for some integer w. Likewise, a simple algorithm ranks $\delta(\xi, \mathbb{Z})$. The first and second rank characterize the criss-crossing pattern of the lattice. Usually, their signs can be ignored. As an illustration, for $\alpha \in (0, 0.5)$ and $\xi \in (2.71, 1618.2)$ the transitions of absolute first and second rank are shown in Figure 1. For $\alpha \in (0.5, 1)$ the figure is mirror-inverted. The blue curves show transitions of the absolute first rank and expand the classical "Van Iterson Diagram", cf. [8, Fig. 5].

Following [2, FIG. 3. Klein's geometrization of the continued fraction for $\sqrt{2}$], Rothen and Koch provide a geometric argument for [7, rule 3.5.2] which corresponds to our main result:

Theorem 4.2. *Let α be irrational, ξ a positive real number, and w a non-zero integer such that $\delta(\xi, w) = \min_{i\in\mathbb{Z}} \delta(\xi, i)$. Then $|w|$ is a denominator of a principal convergent, i.e., $|w| = q_n$ for some $n \geq 0$.*

Proof. The assumptions of the theorem being given, we put $k = |w|$. We suppose $k \neq q_n$ for all $n \geq 0$ and deduce a contradiction. As k is positive and $q_0 = 1 \leq q_1 < q_2 < q_3 < \ldots$, we find an unique $m \geq 0$ such that $q_m < k < q_{m+1}$. Of course, $\langle k\alpha\rangle \geq \lambda_k$ and $\langle -k\alpha\rangle \geq \varrho_k$. Hence, $\langle w\alpha\rangle \geq \min(\lambda_k, \varrho_k)$. By Lemma 3.6, $\min(\lambda_k, \varrho_k) = \mu_m$. Therefore, $\langle w\alpha\rangle \geq \mu_m$. Furthermore, by $w^2 = k^2 > q_m^2$, we derive $\langle w\alpha\rangle^2 + (w/\xi)^2 > \mu_m^2 + (q_m/\xi)^2$, so $\delta(\xi, w) > \sqrt{\mu_m^2 + (q_m/\xi)^2}$. By Lemma 3.5, $\mu_m = \langle(-1)^m q_m\alpha\rangle$. Thus, $\sqrt{\mu_m^2 + (q_m/\xi)^2} = \delta(\xi, (-1)^m q_m)$ and we conclude $\delta(\xi, w) > \delta(\xi, (-1)^m q_m)$ which contradicts our assumption that $\delta(\xi, w) = \min_{i\in\mathbb{Z}} \delta(\xi, i)$. □

Theorem 4.2 provides a necessary but not a sufficient condition, as we find divergences (e.g. Example 4.3) where single principal denominators are *skipped*, i.e., for all $\xi > 0$ neither $\delta(\xi, q_n)$ nor $\delta(\xi, -q_n)$ is the smallest distance.

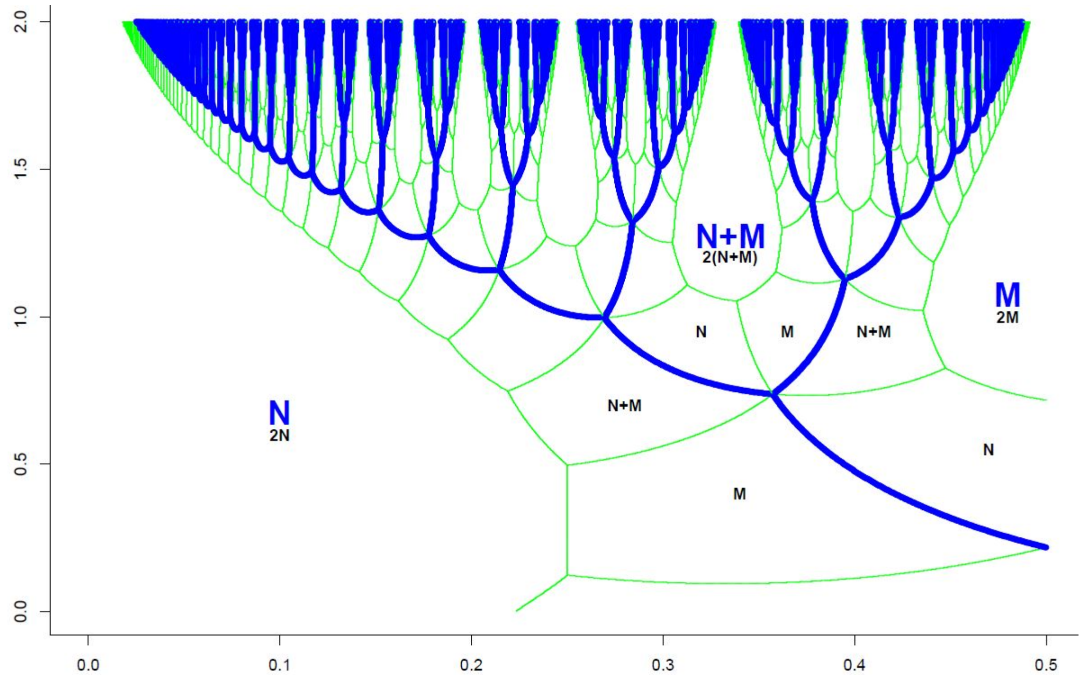

FIGURE 1. Classification of lattices with divergence α on x-axis and density ξ as $\ln(\ln(\xi))$ on y-axis where $e^{(e^y)}$ corresponds to $\xi \approx 5.2$ for $y = 0.5$, $\xi \approx 15.2$ for $y = 1$, $\xi \approx 88.4$ for $y = 1.5$, and $\xi \approx 1618.2$ for $y = 2$. Blue curves show the transitions of absolute first rank, e.g. from large blue **N=1** to large blue **M=2**. Bifurcation opens a blue-rimmed area for large blue **N+M=3**. Green curves decompose blue-rimmed areas by the transitions of absolute second rank, e.g. from small black M=2 to small black N+M=3. The inner green "drops" are areas where the second rank is twice the first rank - that is, they consist of lattices with stripe patterns similar to the vertical stripes of dense lattices with rational divergence.

Example 4.3. *As is well known, $\sqrt{3}$ is represented by the continued fraction $[1, 1, 2, 1, 2, \ldots]$ which is periodic. We consider $\alpha = 2 - \sqrt{3} \approx 0.268$ which is represented by $[0, 3, 1, 2, 1, 2, \ldots]$. The principal denominators are $1, 3, 4, 11, 15, 41, \ldots$ The transitions of $\min_{i \in \mathbb{Z}} \delta(\xi, i)$ have been calculated for increasing ξ by the algorithm underlying Figure 1. There are transitions from 1 to 4, from 4 to 15, and from 15 to 56. So they tell us that the principal denominators $3, 11$ and 41 are skipped.*

If q_n is not skipped, then a more sophisticated method for finding a skipped principal denominators compares the transitions $\xi_{n,m}$ between q_n and q_m for $m > n$. The transition between q_n and q_m solves the equation $\delta(\xi, (-1)^n q_n) = \delta(\xi, (-1)^m q_m)$ which, essentially by Lemma 3.5, leads to $\xi_{n,m} = \sqrt{\frac{q_m^2 - q_n^2}{\mu_n^2 - \mu_m^2}}$. The latter term looks like a "reciprocity law". If $\xi_{n,n+2} < \xi_{n,n+1}$ then q_{n+1} is skipped. This is a compelling ansatz for further investigations, in particular, for finding new integer sequences. Invitingly, it has been applied for finding 106 and 33102 as skipped principal denominators of $\pi = [3, 7, 15, 1, 292, 1, \ldots]$.

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