PROBLEMS

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1. Burghard Herrmann: Winners and Transitions

Let α be irrational. For a positive real number ξ and $i \in \mathbb{Z}$ we define

$$\delta(\xi, i) = \begin{cases} \|(\llbracket i\alpha \rrbracket, i/\xi)\| & \text{if } i \text{ is positive} \\ \|(1 - \llbracket -i\alpha \rrbracket, i/\xi)\| & \text{otherwise,} \end{cases}$$

where $\llbracket x \rrbracket$ is the fractional part of x and $\|(x, y)\| = \sqrt{x^2 + y^2}$.

Definition: W_ξ denotes the minimal absolute value of an integer that *wins* by means of minimizing $\delta(\xi, i)$, i.e., $W_\xi = \min\{|w| \mid w \in \mathbb{Z}, \delta(\xi, w) = \min_{i\in\mathbb{Z}} \delta(\xi, i)\}$.

Fact: $\xi \mapsto W_\xi$ is increasing. Thus, there are increasing sequences of non-negative integers $(W_n)_{n\geq 1}$ and *transitions* $(\xi_n)_{n\geq 0}$ such that

$$W_n = W_\xi \Leftrightarrow \xi \in (\xi_{n-1}, \xi_n].$$

Problem: For $n \geq 1$ show that W_n is a denominator of a principal convergent of the continued fraction expansion of α.

Example: Let $\alpha = \sqrt{3}$. The denominators of the continued fraction convergents to $\sqrt{3}$ are

$0, 1, 1, 3, 4, 11, 15, 41, 56, 153, 209, 571, 780, 2131, 2911,$
$7953, 10864, 29681, 40545, 110771, 151316, 413403, 564719, 1542841, \ldots,$

whereas the absolute winners are

$0, 1, 4, 15, 56, 209, 780, 2911,$
$7953, 10864, 29681, 40545, 110771, 151316, 413403, 564719, 1542841, \ldots$

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2. Clark Kimberling: Cardinality of Difference Sequences

Let r be an irrational number with fractional part between $1/3$ and $2/3$. Let C_n be the number of distinct nth differences of the sequence $(\lfloor kr \rfloor)$.

Problem 2.1. *Prove or disprove that*

$$C_n = (2, 3, 3, 5, 4, 7, 5, 9, 6, 11, 7, 13, 8, 15, 9, \ldots),$$

which is simply a riffle of $(2, 3, 4, 5, 6, \ldots)$ *and* $(3, 5, 7, 9, 11, \ldots)$.

We thank the participants of the 19th International Fibonacci Conference for comments on these proposed problems.

For example if $r = (1+\sqrt{5})/2$, then $(\lfloor kr \rfloor)$ is the lower Wythoff sequence, given by $(1,3,4,6,8,9,11,12,14,\ldots)$, and the first three difference sequences are

$$\begin{aligned}
\Delta^1 &= (2,1,2,2,1,2,1,2,2,1,2,2,1,2,1,2,2,1,2,1,\ldots);\\
\Delta^2 &= (-1,1,0,-1,1,-1,1,0,-1,1,0,-1,1,-1,\ldots);\\
\Delta^3 &= (2,-1,-1,2,-2,2,-1,-1,2,-1,-1,2,-2,\ldots),
\end{aligned}$$

so that $C_1 = 2, C_2 = 3, C_3 = 3$.

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3. Kai Wang: Integer Sequences For Elementary Symmetric Polynomials Of Powers Of Roots

Problem 3.1. *Let $f(x)$ be an integral polynomial of degree k and $\{x_1,\ldots,x_k\}$ be its roots. For each $1 \le q \le k$, in terms of elementary symmetrical polynomials, let*

$$Q_{q,n} = \sum_{j_1,\ldots,j_q} x_{j_1}^n \cdots x_{j_q}^n \tag{3.1}$$

where the indices $\{1 \le j_i \le k \mid i = 1,\ldots,q\}$ satisfy $\{j_1 < j_2 < \cdots < j_q\}$. Then for each $1 \le q \le k$, $\{Q_{q,n} | n = 0,1,\ldots\}$ is a recursive integer sequence of order $m = \binom{k}{q}$. Let $g(x)$ be the characteristic polynomial for $Q_{q,*}$ and $\{y_1,\ldots,y_m\}$ be the roots of $g(x)$.
1. How does one compute $g(x)$?
2. Is following equation true?

$$Q_{q,n} = \sum_{i=1}^{m} y_i^n. \tag{3.2}$$

Example 3.2.
1. [OEIS A074193] $q = 2$:

$$f(x) = x^4 - x^3 - x^2 - x - 1,$$

$$g(x) = x^6 + x^5 + 2x^4 + 2x^3 - 2x^2 + x - 1.$$

2. [OEIS A073937] $q = 3$:

$$f(x) = x^4 - x^3 - x^2 - x - 1,$$

$$g(x) = x^4 - x^3 + x^2 - x - 1.$$

3. [OEIS A123127] $q = 2$:

$$f(x) = x^5 - x^4 - x^3 - x^2 - x - 1.$$

$$g(x) = x^{10} + x^9 + 2x^8 + 3x^7 + 3x^6 - 6x^5 + x^4 - x^3 - x + 1.$$

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4. Carlo Sanna: Practical Numbers

A practical number is a positive integer n such that all the positive integers $m \leq n$ can be written as a sum of distinct divisors of n. The term "practical" was coined by Srinivasan.

Problem 4.1 (Open question Q1 from [Sa])**.** *Let $\mathcal{A}$ be the set of natural numbers k such that the kth Fibonacci number is a practical number. Is it true that $\mathcal{A}$ has natural density equal to zero? That is,* $\lim_{x\to\infty} \frac{\#(\mathcal{A} \cap [1,x])}{x} = 0$.

Note. The problem was posed in a slightly more general setting of Lucas sequences satisfying some mild hypotheses, and it was proved that $\#(\mathcal{A} \cap [1,x]) \gg x/\log x$ for $x \geq 2$.

References

[Sa] C. Sanna, Practical numbers in Lucas sequences, *Quaest. Math.* **42** (2019), no. 7, 977–983.

[Sr] A. K. Srinivasan, Practical numbers, *Current Sci.* **17** (1948), 179–180.

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5. Steven J. Miller: Zeckendorf Games

The Zeckendorf Game was introduced in [BEFM1, BEFM2]. We quote from [BEFM2] to describe the game.

We introduce some notation. By $\{1^n\}$ or $\{{F_1}^n\}$ we mean n copies of 1, the first Fibonacci number. If we have 3 copies of F_1, 2 copies of F_2, and 7 copies of F_4, we write either $\{{F_1}^3 \wedge {F_2}^2 \wedge {F_4}^7\}$ or $\{1^3 \wedge 2^2 \wedge 5^7\}$.

Definition 5.1 (The Two Player Zeckendorf Game)**.** *At the beginning of the game, there is an unordered list of n 1's. Let $F_1 = 1, F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{{F_1}^n\}$. On each turn, a player can do one of the following moves.*

1. *If the list contains two consecutive Fibonacci numbers, F_{i-1}, F_i, then a player can change these to F_{i+1}. We denote this move $\{F_{i-1} \wedge F_i \to F_{i+1}\}$.*
2. *If the list has two of the same Fibonacci number, F_i, F_i, then*
   - (a) *if $i = 1$, a player can change F_1, F_1 to F_2, denoted by $\{F_1 \wedge F_1 \to F_2\}$,*
   - (b) *if $i = 2$, a player can change F_2, F_2 to F_1, F_3, denoted by $\{F_2 \wedge F_2 \to F_1 \wedge F_3\}$, and*
   - (c) *if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}, denoted by $\{F_i \wedge F_i \to F_{i-2} \wedge F_{i+1}\}$.*

The players alternate moving. The game ends when one player moves to create the Zeckendorf decomposition.

The moves of the game are derived from the Fibonacci recurrence, either combining terms to make the next in the sequence or splitting terms with multiple copies. A proof that this game is well defined and ends at the Zeckendorf decomposition can be found in [BEFM2].

Several papers in this volume examine this game and its generalizations, in particular proving tight upper and lower bounds on how long games take and generalizing to other sequences and more players. However, none of these address a particularly interesting issue.

Problem 5.2. *A non-constructive proof was given that for every $n > 2$, Player Two has a winning strategy. Find a constructive proof; in other words, what is the winning strategy which is known to exist?*

References

[BEFM1] P. Baird-Smith, A. Epstein, K. Flynt and S. J. Miller, The Generalized Zeckendorf Game, *Proceedings of the 18th Conference, Fibonacci Quart.* **57** (2019), no. 5, 1–15. https://arxiv.org/pdf/1809.04883.

[BEFM2] P. Baird-Smith, A. Epstein, K. Flynt and S. J. Miller, *The Zeckendorf Game*, Combinatorial and Additive Number Theory III, CANT, New York, USA, 2017 and 2018, Springer Proceedings in Mathematics & Statistics **297** (2020), 25–38. https://arxiv.org/pdf/1809.04881.

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