NEW IDENTITIES FOR SOME SYMMETRIC POLYNOMIALS, AND A HIGHER ORDER ANALOGUE OF THE FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. We give new identities for some symmetric polynomials. As applications of these identities, we obtain some formulas for a higher order analogue of Fibonacci and Lucas numbers.

1. INTRODUCTION

Throughout the paper, we denote the set of non-negative integers by $\mathbb{Z}_{\geq 0}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$ and $\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$. Let $z_1, \ldots, z_r$ be r independent variables and $\mathbf{z} := (z_1, \ldots, z_r)$. For each non-negative integer n, the nth elementary symmetric polynomial $e_n^{(r)}$, complete homogeneous symmetric polynomials $h_n^{(r)}$ and power symmetric polynomial $p_n^{(r)}$ are defined by

$$e_n^{(r)} = e_n^{(r)}(\mathbf{z}) := \begin{cases} \sum_{1 \leq j_1 < \cdots < j_n \leq r} z_{j_1} \cdots z_{j_n} & (1 \leq n \leq r), \\ 1 & (n = 0), \\ 0 & (n > r), \end{cases} \tag{1.1}$$

$$h_n^{(r)} = h_n^{(r)}(\mathbf{z}) := \sum_{\substack{m_1 + \cdots + m_r = n \\ m_1, \ldots, m_r \in \mathbb{Z}_{\geq 0}}} z_1^{m_1} \cdots z_r^{m_r}, \tag{1.2}$$

$$p_n^{(r)} = p_n^{(r)}(\mathbf{z}) := \sum_{j=1}^{r} z_j^n, \tag{1.3}$$

respectively. For $\mathbf{z} = (z_1, \ldots, z_r) \in {\mathbb{C}^{\times}}^r$, we put

$$(\mathbf{z} + \mathbf{z}^{-1}) := (z_1 + z_1^{-1}, \ldots, z_r + z_r^{-1}) \in \mathbb{C}^r,$$
$$(\mathbf{z}, \mathbf{z}^{-1}) := (z_1, \ldots, z_r, z_1^{-1}, \ldots, z_r^{-1}) \in {\mathbb{C}^{\times}}^{2r}.$$

Our main results are new identities for these three types of symmetric polynomials $f = e, h, p$, which are relationships between $f_n^{(r)}(\mathbf{z} + \mathbf{z}^{-1})$ and $f_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1})$. More precisely, we determine the following expansion coefficients $a_{n,k}^{(f)}$ and $b_{n,k}^{(f)}$,

$$f_n^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) = \sum_{k=0}^{n} a_{n,k}^{(f)} f_k^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}), \tag{1.4}$$

$$f_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=0}^{n} b_{n,k}^{(f)} f_k^{(r)}(\mathbf{z} + \mathbf{z}^{-1}). \tag{1.5}$$

In this article we call (1.4) (resp. (1.5)) the first kind formulas (resp. the second kind formulas).

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Theorem 1.1 (The first kind formulas)**.** *For any non-negative integer m, we have the following identities:*
(1)

$$\sum_{k=\max\left(\left\lfloor \frac{m}{2}\right\rfloor - r, 0\right)}^{\left\lfloor \frac{m}{2}\right\rfloor} c_{m-r-1,k}\, e^{(2r)}_{m-2k}(\mathbf{z}, \mathbf{z}^{-1}) = \begin{cases} e^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}) & (m = 0, 1, \ldots, r), \\ 0 & (\text{otherwise}), \end{cases} \tag{1.6}$$

where

$$c_{n,k} := \binom{n}{k} - \binom{n}{k-1}, \quad \binom{n}{k} := \begin{cases} \frac{n(n-1)\cdots(n-k+1)}{k!} & (k \neq 0), \\ 1 & (k = 0), \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer not exceeding $x \in \mathbb{R}$.
(2)

$$h^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}) = \sum_{k=0}^{\left\lfloor \frac{m}{2}\right\rfloor} c_{m+r-1,k}\, h^{(2r)}_{m-2k}(\mathbf{z}, \mathbf{z}^{-1}). \tag{1.7}$$

(3)

$$p^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}) = \frac{1}{2}\sum_{k=0}^{m} \binom{m}{k} p^{(2r)}_{|m-2k|}(\mathbf{z}, \mathbf{z}^{-1}). \tag{1.8}$$

Theorem 1.2 (The second kind formulas)**.** (1) *For $n = 0, 1, \ldots, 2r$, we have*

$$e^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=\max\left\{\left\lfloor \frac{n-r}{2}\right\rfloor, 0\right\}}^{\left\lfloor \frac{n}{2}\right\rfloor} \binom{r-n+2k}{k} e^{(r)}_{n-2k}(\mathbf{z}+\mathbf{z}^{-1}). \tag{1.9}$$

(2) *For any non-negative integer n,*

$$h^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \binom{n-k+r-1}{k} h^{(r)}_{n-2k}(\mathbf{z}+\mathbf{z}^{-1}). \tag{1.10}$$

(3) *For any positive integer n,*

$$p^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) = 2\sum_{k=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \binom{2k-n-1}{k} p^{(r)}_{n-2k}(\mathbf{z}+\mathbf{z}^{-1}) - \sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \binom{2k-n}{k} p^{(r)}_{n-2k}(\mathbf{z}+\mathbf{z}^{-1}). \tag{1.11}$$

The proofs of (1.6) and (1.7) are more difficult than (1.9) and (1.10). In particular we need a hypergeometric identity (2.20) to derive the explicit formulas of $a^{(e)}_{n,k}$ or $a^{(h)}_{n,k}$. The second kind formulas and (1.8) which are proved immediately by the binomial formula and generating functions may be well known. However we have not found appropriate references about these formulas and their interesting applications which are to propose some new formulas of Fibonacci and Lucas numbers etc. In particular, we apply Theorem 1.1 and Theorem 1.2 to some special values of $h^{(r)}_n$ and $p^{(r)}_n$;

$$F^{(r)}_{n+1} := h^{(r)}_n\left(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota}\right) = h^{(r)}_n\left(-2\cos\left(\frac{2\pi r}{2r+1}\right), \ldots, -2\cos\left(\frac{2\pi}{2r+1}\right)\right), \tag{1.12}$$

$$L^{(r)}_n := p^{(r)}_n\left(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota}\right) = p^{(r)}_n\left(-2\cos\left(\frac{2\pi r}{2r+1}\right), \ldots, -2\cos\left(\frac{2\pi}{2r+1}\right)\right), \tag{1.13}$$

where

$$\boldsymbol{\zeta}^{\pm\iota} := \left(e^{\pm\frac{2\pi\sqrt{-1}r}{2r+1}}, \dots, e^{\pm\frac{2\pi\sqrt{-1}}{2r+1}}\right).$$

Since the case of $r = 1$ gives the trivial sequences

$$F_{n+1}^{(1)} = L_n^{(1)} = 1, \quad (n \geq 0)$$

and the case of $r = 2$ is the classical Fibonacci numbers $\{F_{n+1}\}_n$ and Lucas numbers $\{L_n\}_n$

$$F_{n+1}^{(2)} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) =: F_{n+1},$$

$$L_n^{(2)} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} =: L_n,$$

the sequences $\{F_{n+1}^{(r)}\}_n$ and $\{L_n^{(r)}\}_n$ are regarded as a higher order analogues of the classical Fibonacci and Lucas numbers. From Theorem 1.1, Theorem 1.2 and some fundamental results for symmetric functions, we propose the following fundamental formulas for $F_{n+1}^{(r)}$ and $L_n^{(r)}$.

Theorem 1.3 (Explicit formulas of $F_n^{(r)}$ and $L_n^{(r)}$)**.**

$$\begin{aligned} F_n^{(r)} &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{2}\left((-1)^{\lfloor\frac{n-1-2k}{2r+1}\rfloor} - (-1)^{\lfloor\frac{n-2k-3}{2r+1}\rfloor}\right) c_{n+r-2,k} \\ &= \sum_{k=0}^{\lfloor\frac{n-1}{2r+1}\rfloor} (-1)^k c_{n+r-2,\lfloor\frac{n-1-(2r+1)k}{2}\rfloor}, \end{aligned} \tag{1.14}$$

$$\begin{aligned} L_n^{(r)} &= (-1)^{n+1}2^{n-1} + (-1)^n\frac{2r+1}{2}\sum_{k=0}^{n}\binom{n}{k}\delta_{2r+1|n-2k} \\ &= \begin{cases} -2^{2m-1} + \frac{2r+1}{2}\sum_{k=-\lfloor\frac{m}{2r+1}\rfloor}^{\lfloor\frac{m}{2r+1}\rfloor}\binom{2m}{m-(2r+1)k} & (n = 2m), \\ 4^m - \frac{2r+1}{2}\sum_{k=-\lfloor\frac{m+r+1}{2r+1}\rfloor}^{\lfloor\frac{m-r}{2r+1}\rfloor}\binom{2m+1}{m-(2r+1)k-r} & (n = 2m+1), \end{cases} \end{aligned} \tag{1.15}$$

where

$$\delta_{2r+1|n-2k} := \begin{cases} 0 & (2r+1 \nmid n-2k), \\ 1 & (2r+1 \mid n-2k). \end{cases}$$

Theorem 1.4 (Inversion of the explicit formulas)**.**

$$\begin{aligned} &\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k+r-1}{k}F_{n-2k+1}^{(r)} \\ &= \frac{1}{2}\left((-1)^{\lfloor\frac{n-1-2k}{2r+1}\rfloor} - (-1)^{\lfloor\frac{n-2k-3}{2r+1}\rfloor}\right) = \begin{cases} 1 & (n \equiv 0, 1 \pmod{4r+2}), \\ -1 & (n \equiv 2r+1, 2r+2 \pmod{4r+2}), \\ 0 & (\text{otherwise}), \end{cases} \end{aligned} \tag{1.16}$$

$$2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}L_{n-2k}^{(r)} - \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k-n}{k}L_{n-2k}^{(r)}$$

$$= (-1)^n(-1+(2r+1)\delta_{2r+1|n}) = \begin{cases} (-1)^{n-1} & (2r+1 \nmid n), \\ (-1)^n 2r & (2r+1 \mid n). \end{cases} \tag{1.17}$$

Theorem 1.5 (Initial values and recurrence formulas)**.** *Initial values of* $\{F_{n+1}^{(r)}\}_{n\in\mathbb{Z}}$ *and* $\{L_n^{(r)}\}_{n\in\mathbb{Z}}$ *are given by*

$$F_1^{(r)} = 1, \quad F_0^{(r)} = F_{-1}^{(r)} = \cdots = F_{-r+2}^{(r)} = 0, \tag{1.18}$$

$$L_n^{(r)} = \begin{cases} -2^{2m-1} + \frac{2r+1}{2}\binom{2m}{m} & (n=2m) \\ 4^m & (n=2m+1) \end{cases} \qquad (n=0,1,\ldots,r-1). \tag{1.19}$$

The sequences $\{F_{n+1}^{(r)}\}_{n\in\mathbb{Z}}$ *and* $\{L_n^{(r)}\}_{n\in\mathbb{Z}}$ *satisfy the following same recursion:*

$$a_{n+r}^{(r)} = \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^j \binom{r-1-j}{j} a_{n+r-1-2j}^{(r)} + \sum_{j=0}^{\lfloor \frac{r-2}{2} \rfloor} (-1)^j \binom{r-1-j}{j+1} a_{n+r-2-2j}^{(r)}. \tag{1.20}$$

The content of this article is as follows. In Section 2, we refer to some basic formulas for symmetric polynomials and the Gauss hypergeometric function from [2] and [4]. Section 3 is the main part of this article. In this section, we prove Theorem 1.1 and Theorem 1.2, and give their principal specializations. In Section 4, we evaluate $e_n^{(2r)}\left(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}\right)$, $h_n^{(2r)}\left(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}\right)$ and $p_n^{(2r)}\left(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}\right)$ by the definitions and generating functions. By substituting these evaluations into Theorem 1.1 and Theorem 1.2, we propose Theorem 1.3, Theorem 1.4 and Theorem 1.5. Further, we consider some specializations of our main results and derive some interesting binomial sum formulas, including new formulas for the Fibonacci and Lucas numbers (see Corollary 4.6 and Corollary 4.7). In the Appendix, we mention some congruence properties and other formulas (generating functions, determinant formulas, some relations) for $F_{n+1}^{(r)}$ and $L_n^{(r)}$ that are proven independently of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

2.1. **Symmetric polynomials.** We refer to Macdonald [4] for the details in this subsection. We fix a positive integer r, and denote the partition set of length r by

$$\lambda \in \mathcal{P}_r := \{\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}_{\geq 0}^r \mid \nu_1 \geq \ldots \geq \nu_r\}$$

and the symmetric group of order r by $\mathfrak{S}_r$. For some special partitions, we use the notations

$$(n) := (n, 0, \ldots, 0) \in \mathcal{P}_r,$$
$$(1^n) := (1, \ldots, 1, 0, \ldots, 0) \in \mathcal{P}_r, \quad (n = 1, \ldots, r).$$

The symmetric group $\mathfrak{S}_r$ acts on $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ by

$$\sigma.\mathbf{z} := (z_{\sigma(1)}, \ldots, z_{\sigma(r)}).$$

For any partition λ, we define the Schur polynomial $s_\lambda(\mathbf{z})$ and monomial symmetry polynomial $m_\lambda(\mathbf{z})$ by

$$s_\lambda(\mathbf{z}) := \frac{\det\left(z_i^{\lambda_j+r-j}\right)_{i,j=1,\ldots r}}{\det\left(z_i^{r-j}\right)_{i,j=1,\ldots r}} = \frac{\det\left(z_i^{\lambda_j+r-j}\right)_{i,j=1,\ldots r}}{\prod_{1\leq i<j\leq r}(z_i - z_j)}, \tag{2.1}$$

$$m_\lambda(\mathbf{z}) := \sum_{\nu \in \mathfrak{S}_r.\lambda} z_1^{\nu_1} \cdots z_r^{\nu_r}, \tag{2.2}$$

where det is the usual determinant and

$$\mathfrak{S}_r.\lambda := \{\sigma.\lambda := (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(r)}) \mid \sigma \in \mathfrak{S}_r\}.$$

We remark that the Schur polynomial extends to the Schur function $s_\lambda(\mathbf{z})$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ by (2.1).

It is well known that

$$e_n^{(r)}(\mathbf{z}) = s_{(1^n)}(\mathbf{z}) = m_{(1^n)}(\mathbf{z}), \tag{2.3}$$

$$p_n^{(r)}(\mathbf{z}) = m_{(n)}(\mathbf{z}), \tag{2.4}$$

$$h_n^{(r)}(\mathbf{z}) = s_{(n)}(\mathbf{z}), \tag{2.5}$$

which is Schur $s_{(1^n)}$ or monomial $m_{(1^n)}$ where one column is $e_n^{(r)}$, and monomial $m_{(n)}$ where one row is $p_n^{(r)}$, and Schur $s_{(n)}$ where one row is $h_n^{(r)}$ [4, Chapter I Section 3 (3.9)]. From (2.5), we extend the complete homogeneous polynomials to $h_n^{(r)}(\mathbf{z})$ $(n \in \mathbb{Z})$: namely

$$h_{-n}^{(r)}(\mathbf{z}) := s_{(-n)}(\mathbf{z}) \quad (n \geq 0). \tag{2.6}$$

By this extension (2.6) and the definition of the Schur function, for any r we have

$$h_{-n}^{(r)}(\mathbf{z}) = 0 \quad (n = 1, 2, \ldots, r-1). \tag{2.7}$$

We list some required formulas for symmetric polynomials, given in [4].

Lemma 2.1. (1) *Generating functions*

$$\prod_{j=1}^{r}(1 + z_j y) = \sum_{n=0}^{r} e_n^{(r)}(\mathbf{z}) y^n, \tag{2.8}$$

$$\prod_{j=1}^{r} \frac{1}{1 - z_j y} = \sum_{n=0}^{\infty} h_n^{(r)}(\mathbf{z}) y^n, \tag{2.9}$$

$$\sum_{j=1}^{r} \frac{1}{1 - z_j y} = \sum_{n=0}^{\infty} p_n^{(r)}(\mathbf{z}) y^n. \tag{2.10}$$

(2) *q-binomial formulas*

$$\prod_{j=0}^{n-1}(1 + q^j y) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} y^k, \tag{2.11}$$

$$\prod_{j=0}^{n-1} \frac{1}{1 - q^j y} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q y^k, \tag{2.12}$$

where $\binom{n}{k}_q$ *denotes the q-binomial coefficient*

$$\binom{n}{k}_q := \begin{cases} \frac{(1-q^n)\cdots(1-q^{n-k+1})}{(1-q)\cdots(1-q^k)} & (k \neq 0) \\ 1 & (k = 0) \end{cases}.$$

(3) *Wronski relation and Newton's formula*

$$\sum_{j=0}^{\min(n,r)} (-1)^{j-1} e_j^{(r)}(\mathbf{z}) h_{n-j}^{(r)}(\mathbf{z}) = 0, \tag{2.13}$$

$$\sum_{j=0}^{\min(n,r)} (-1)^{n-j} p_{n-j+1}^{(r)}(\mathbf{z}) e_j^{(r)}(\mathbf{z}) = (n+1) e_{n+1}^{(r)}(\mathbf{z}). \tag{2.14}$$

Actually, (2.8) is [4], p. 19 (2.2), and (2.9) is [4], p. 21 (2.5) exactly. For (2.11) and (2.12), see [4], p. 26, Examples 3. Similarly, (2.13) and (2.14) are [4], p. 21 (2.6′) and p. 23 (2.11′), respectively.

2.2. **The Gauss hypergeometric function.** Let a, b, c, z be complex numbers such that c is not non-negative integers, and $(a)_m$ be the rising factorial defined by

$$(a)_m := \begin{cases} a(a+1)\cdots(a+m-1) & (m \neq 0), \\ 1 & (m = 0). \end{cases}$$

We recall the Gauss hypergeometric function

$${}_2F_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m!(c)_m} z^m \quad (|z| < 1),$$

and for any complex numbers α and x we put

$$\psi(\alpha; x) := \sum_{k=0}^{\infty} \frac{(\alpha)_{2k}}{k!(\alpha+1)_k} x^k \quad (|4x| < 1). \tag{2.15}$$

Since

$$\psi(\alpha; x) = {}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha+1 \end{matrix}; 4x\right),$$

the function $\psi(\alpha; x)$ is analytically continued to $4x \in \mathbb{C} \setminus \{0, 1\}$ by analytic continuation of ${}_2F_1(z)$.

Lemma 2.2. (1) *Another expression*

$$\psi(\alpha; x) = \sum_{k=0}^{\infty} c_{\alpha+2k-1,k} x^k. \tag{2.16}$$

(2) *Closed form*

$$\psi(\alpha; x) = \left(\frac{1 - \sqrt{1-4x}}{2x}\right)^{\alpha}. \tag{2.17}$$

(3) *Index law*

$$\psi(\alpha; x)\psi(\beta; x) = \psi(\alpha+\beta; x). \tag{2.18}$$

(4) *Quadratic formula*

$$x(x\psi(1; x^2))^2 - (x\psi(1; x^2)) + x = 0. \tag{2.19}$$

Proof. (1) By the definition of $\psi(\alpha;x)$ and $c_{a,k}$, we have

$$\begin{aligned}\psi(\alpha;x) &= \sum_{k=0}^{\infty}\frac{(\alpha)_{2k}}{k!(\alpha+1)_k}x^k\\ &= 1+\sum_{k=1}^{\infty}(\alpha+k-k)\frac{(\alpha+k+1)_{k-1}}{k!}x^k\\ &= 1+\sum_{k=1}^{\infty}\left(\frac{(\alpha+k)_k}{k!}-\frac{(\alpha+k+1)_{k-1}}{(k-1)!}\right)x^k\\ &= 1+\sum_{k=1}^{\infty}\left(\binom{\alpha+2k-1}{k}-\binom{\alpha+2k-1}{k-1}\right)x^k\\ &= \sum_{k=0}^{\infty}c_{\alpha+2k-1,k}x^k.\end{aligned}$$

(2) We use the hypergeometric transformation [2, (3.1.10)]

$${}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2}\\ \alpha-\beta+1\end{matrix};x\right)=\left(2\frac{1-\sqrt{1-x}}{x}\right)^{\alpha}{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha-\beta+1\end{matrix};\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right).$$

Thus,

$$\psi(\alpha;x)={}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2}\\ \alpha+1\end{matrix};4x\right)=\left(\frac{1-\sqrt{1-4x}}{2x}\right)^{\alpha}.$$

The formulas (2.18) and (2.19) follow from (2.17) immediately. □

The following Lemma is a corollary of Lemma 2.2 and is the key step in the proof of Theorem 1.1.

Lemma 2.3. *If*

$$y:=\frac{1-\sqrt{1-4x^2}}{2x},$$

then

$$x=\frac{y}{1+y^2}$$

and for any non-negative integer N,

$$y^N=x^N\psi\left(N;x^2\right)=x^N\sum_{k=0}^{\infty}c_{N+2k-1,k}x^{2k}. \tag{2.20}$$

Proof. By the quadratic formula,

$$y=x\frac{1-\sqrt{1-4x^2}}{2x^2}=x\psi\left(1;x^2\right).$$

Thus, we have

$$y^N=x^N\psi\left(1;x^2\right)^N=x^N\psi\left(N;x^2\right)=x^N\sum_{k=0}^{\infty}c_{N+2k-1,k}x^{2k}.$$

Here the second and third equalities follow from (2.18) and (2.16) respectively. □

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	1									
3	1	2									
4	1	3	2								
5	1	4	5								
6	1	5	9	5							
7	1	6	14	14							
8	1	7	20	28	14						
9	1	8	27	48	42						
10	1	9	35	75	90	42					
11	1	10	44	110	165	132					
12	1	11	54	154	275	297	132				
13	1	12	65	208	429	572	429				
14	1	13	77	273	637	1001	1001	429			
15	1	14	90	350	910	1638	2002	1430			
16	1	15	104	440	1260	2548	3640	3432	1430		
17	1	16	119	544	1700	3808	6188	7072	4862		
18	1	17	135	663	2244	5508	9996	13260	11934	4862	
19	1	18	152	798	2907	7752	15504	23256	25194	16796	
20	1	19	170	950	3705	10659	23256	38760	48450	41990	16796
21	1	20	189	1120	4655	14364	33915	62016	87210	90440	58786

TABLE 1. $c_{n,k}$

Remark 2.4. We mention some properties of $c_{n,k}$. For non-negative integers n and k, we make a table of $c_{n,k} > 0$. This table is determined exactly by the initial conditions

$$\begin{aligned}
c_{n,0} &= 1 \quad (n \geq 0), \\
c_{n,k} &= 0 \quad \left(n > 0, \left\lfloor \frac{n}{2} \right\rfloor < k\right), \\
c_{2n,n+1} &= \frac{1}{n+2}\binom{2n+2}{n+1} \quad (n \geq 0), \\
c_{2n,n} &= \frac{1}{n+1}\binom{2n}{n} \quad (n \geq 0) \quad \text{Catalan numbers}
\end{aligned}$$

and the recursion formula

$$c_{n,k} = c_{n-1,k-1} + c_{n-1,k}.$$

The sequence $c_{n,k}$ is a kind of Clebsch-Gordan coefficients for the Lie group SU(2) or the Lie algebra sl_2. In fact, from the above initial conditions and recursion of $c_{n,k}$, we have

$$\left(\frac{\sin(2\theta)}{\sin\theta}\right)^n = (2\cos\theta)^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{n,k} \frac{\sin((n-2k+1)\theta)}{\sin\theta}. \tag{2.21}$$

Since $\frac{\sin((m+1)\theta)}{\sin\theta}$ is the character of the irreducible representation of SU(2) with the highest weight m [6, Section 6.9], the formula (2.21) means the irreducible decomposition of a tensor product representation which is a variation of the classical Clebsch-Gordan rule for SU(2).

Further $c_{n,k}$ is also a typical example of Kostka numbers $K_{\lambda\mu}$ (see [5, Section 2.11 and Section 4.9]). We remark Young's rule

$$s_{(\mu_1)}(\mathbf{z})\cdots s_{(\mu_n)}(\mathbf{z}) = \sum_{\lambda} K_{\lambda\mu} s_\lambda(\mathbf{z}) \quad (\mu_1 \geq \cdots \geq \mu_n \geq 1), \tag{2.22}$$

and

$$s_{(\lambda_1,\lambda_2)}\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right) = \frac{e^{(\lambda_1-\lambda_2+1)\sqrt{-1}\theta} - e^{-(\lambda_1-\lambda_2+1)\sqrt{-1}\theta}}{e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta}} = \frac{\sin((\lambda_1-\lambda_2+1)\theta)}{\sin\theta}.$$

By putting $r=2$, $\mu_1 = \cdots = \mu_n = 1, z_1 = e^{\sqrt{-1}\theta}, z_2 = e^{-\sqrt{-1}\theta}$ in (2.22), we have

$$\begin{aligned}\left(\frac{\sin(2\theta)}{\sin\theta}\right)^n &= s_{(1,0)}\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right)^n \\ &= \sum_\lambda K_{\lambda(1^n)} s_\lambda\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right) \\ &= \sum_\lambda K_{\lambda(1^n)} \frac{\sin((\lambda_1-\lambda_2+1)\theta)}{\sin\theta}. \end{aligned} \tag{2.23}$$

Finally, by comparing (2.21) and (2.23), we have

$$K_{(n-k,k),(1^n)} = c_{n,k}.$$

3. Proofs of Theorem 1.1 and Theorem 1.2

From (2.8), (2.9) and simple calculation

$$(1 \pm z_j y)(1 \pm z_j^{-1} y) = 1 \pm (z_j + z_j^{-1})y + y^2 = (1+y^2)\left(1 \pm (z_j + z_j^{-1})\frac{y}{1+y^2}\right),$$

we obtain the following key lemma.

Lemma 3.1. *If*

$$|z_j y|, |z_j^{-1} y|, \left|(z_j + z_j^{-1})\frac{y}{1+y^2}\right| < 1 \quad (j = 1, \ldots, r),$$

then

$$\begin{aligned}\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n &= \prod_{j=1}^{r} (1 + z_j y)(1 + z_j^{-1} y) \\ &= (1+y^2)^r \sum_{m=0}^{r} e_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) \left(\frac{y}{1+y^2}\right)^m, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n &= \prod_{j=1}^{r} \frac{1}{(1 - z_j y)(1 - z_j^{-1} y)} \\ &= (1+y^2)^{-r} \sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) \left(\frac{y}{1+y^2}\right)^m. \end{aligned} \tag{3.2}$$

By Lemma 3.1 and the definition of the power symmetric polynomials, we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 (1) By (3.1),

$$\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n-r} = x^{-r}\sum_{m=0}^{r} e_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})x^m, \tag{3.3}$$

where

$$x = \frac{y}{1+y^2}.$$

Hence, from (2.20) we have

$$\begin{aligned}\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n-r} &= \sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})x^{n-r}\sum_{k=0}^{\infty} c_{n-r+2k-1,k}x^{2k}\\ &= x^{-r}\sum_{m=0}^{\infty}\sum_{k=\max\left(\left\lfloor\frac{m}{2}\right\rfloor-r,0\right)}^{\left\lfloor\frac{m}{2}\right\rfloor} e_{m-2k}^{(2r)}(\mathbf{z},\mathbf{z}^{-1})c_{m-r-1,k}x^m. \end{aligned}\tag{3.4}$$

By comparing coefficients of (3.3) and (3.4), we obtain the conclusion.
(2) Similarly, from (3.2),

$$\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n+r} = x^{r}\sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})x^m \tag{3.5}$$

and by (2.20) we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n+r} &= \sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})x^{n+r}\sum_{k=0}^{\infty} c_{n+r+2k-1,k}x^{2k}\\ &= x^{r}\sum_{m=0}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} h_{m-2k}^{(2r)}(\mathbf{z},\mathbf{z}^{-1})c_{m+r-1,k}x^m. \end{aligned}\tag{3.6}$$

The formula (1.7) follows from (3.5) and (3.6).
(3) We prove this formula without generating function and other lemmas. In fact, by applying the usual binomial formula we have

$$\begin{aligned}p_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1}) &= \sum_{i=1}^{r}(z_i+z_i^{-1})^m\\ &= \sum_{i=1}^{r}\sum_{k=0}^{m}\binom{m}{k}z_i^{2k-m}\\ &= \frac{1}{2}\sum_{i=1}^{r}\sum_{k=0}^{m}\left(\binom{m}{k}z_i^{2k-m}+\binom{m}{m-k}z_i^{m-2k}\right)\\ &= \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\sum_{i=1}^{r}(z_i^{2k-m}+z_i^{m-2k})\\ &= \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}p_{|m-2k|}^{(2r)}(\mathbf{z},\mathbf{z}^{-1}).\end{aligned}$$

□

Proof of Theorem 1.2 (1) From (3.1) and the binomial theorem, we have

$$\begin{aligned}\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^n &= \sum_{m=0}^{r} e_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^m(1+y^2)^{r-m}\\ &= \sum_{m=0}^{r}\sum_{k=0}^{r-m}\binom{r-m}{k} e_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^{m+2k}\\ &= \sum_{n=0}^{2r}\sum_{k=\max\{\lfloor\frac{n-r}{2}\rfloor,0\}}^{\lfloor\frac{n}{2}\rfloor}\binom{r-n+2k}{k} e_{n-2k}^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^n.\end{aligned}$$

(2) Similarly, we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^n &= \sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^m(1+y^2)^{-m-r}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(m+r)_k}{k!} h_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^{m+2k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(n-2k+r)_k}{k!} h_{n-2k}^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k+r-1}{k} h_{n-2k}^{(r)}(\mathbf{z}+\mathbf{z}^{-1})y^n.\end{aligned}$$

(3) It is enough to show that the case of $r = 1$ is

$$z^n + z^{-n} = 2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}(z+z^{-1})^{n-2k} - \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k-n}{k}(z+z^{-1})^{n-2k}.$$

From (3.1) and simple calculation,

$$\begin{aligned}\sum_{n=0}^{\infty}(z^n+z^{-n})y^n &= \frac{1}{1-zy} + \frac{1}{1-z^{-1}y}\\ &= \frac{2-(z+z^{-1})y}{1-(z+z^{-1})y+y^2}\\ &= \frac{1}{1+y^2}\frac{2-(z+z^{-1})y}{1-(z+z^{-1})\frac{y}{1+y^2}}.\end{aligned}$$

If $|zy| < 1$, $|z^{-1}y| < 1$ and $|y| < 1$, then

$$\begin{aligned}\sum_{n=0}^{\infty}(z^n+z^{-n})y^n &= \sum_{m=0}^{\infty}\left(\frac{2}{y}-(z+z^{-1})\right)(z+z^{-1})^m\left(\frac{y}{1+y^2}\right)^{m+1}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2}{y}-(z+z^{-1})\right)(z+z^{-1})^m y^{m+1}\frac{(m+1)_k}{k!}(-1)^k y^{2k}\end{aligned}$$

$$= \sum_{n=0}^{\infty} 2 \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{2k-n-1}{k} (z+z^{-1})^{n-2k} y^n$$
$$- \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{2k-n}{k} (z+z^{-1})^{n-2k+1} y^{n+1}.$$

□

Finally we consider the principal specializations of Theorem 1.1 and Theorem 1.2, which means substituting

$$\boldsymbol{q}^{\pm\iota} := (q^{\pm r}, \ldots, q^{\pm 1})$$

for $\mathbf{z}$. In this special case, we evaluate $e_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota})$, $h_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota})$ and $p_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota})$ explicitly.

Proposition 3.2. *For any non-negative integer n, we have the following identities:*
(1)

$$\sum_{k=0}^{\min(n,2r+1)} (-1)^{n-k} \binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}} = \begin{cases} e_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) & (n = 0, 1, \ldots, 2r), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.7}$$

(2)

$$h_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) = q^{-nr} \left(\binom{2r+n}{n}_q - \binom{2r+n-1}{n-1}_q q^r \right). \tag{3.8}$$

(3)

$$p_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) = -1 + q^{-rn} \frac{1-q^{(2r+1)n}}{1-q^n}. \tag{3.9}$$

Proof. (1) From the generating function of the elementary symmetric polynomials (2.8),

$$\sum_{n=0}^{2r} e_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) y^n = \prod_{j=1}^{r} (1+q^j y)(1+q^{-j} y) = \frac{1}{1+y} \prod_{j=0}^{2r} (1+q^j q^{-r} y).$$

By the q-binomial formula (2.11), we have

$$\sum_{n=0}^{2r} e_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) y^n = \sum_{i=0}^{\infty} (-1)^i y^i \sum_{k=0}^{2r+1} \binom{2r+1}{k}_q q^{\frac{k(k-1)}{2}} q^{-kr} y^k$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,2r+1)} (-1)^{n-k} \binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}} y^n.$$

(2) Similarly, we have

$$\sum_{n=0}^{\infty} h_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) y^n = \prod_{j=1}^{r} \frac{1}{(1-q^j y)(1-q^{-j} y)}$$
$$= (1-y) \prod_{j=0}^{2r} \frac{1}{1-q^j q^{-r} y}$$

$$= (1-y)\sum_{k=0}^{\infty}\binom{2r+k}{k}_q q^{-kr}y^k$$
$$= \sum_{n=0}^{\infty}\left(\binom{2r+n}{n}_q - \binom{2r+n-1}{n-1}_q q^r\right) q^{-nr}y^n.$$

(3) By the definition of power sum and geometric series, we have

$$p_n^{(2r)}(\boldsymbol{q}^{+\iota}, \boldsymbol{q}^{-\iota}) = -1 + \sum_{k=-r}^{r} q^{kn} = -1 + \frac{q^{-rn} - q^{(r+1)n}}{1-q^n}.$$

We remark that this formula holds in the limit $q \to 1$. □

Corollary 3.3. (1) *For any non-negative integer* m*,*

$$\sum_{k=\max\{\lfloor \frac{m-r}{2}\rfloor, 0\}}^{\lfloor \frac{m}{2}\rfloor} \sum_{l=0}^{\min(m-2k,2r+1)} (-1)^{m-l}\binom{2r+1}{l}_q q^{\frac{l(l-2r-1)}{2}} c_{m-r-1,k}$$
$$= \begin{cases} e_m^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}) & (m = 0, 1, \ldots, r), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.10}$$

For $n = 0, 1, \ldots, 2r$*,*

$$\sum_{k=0}^{n}(-1)^k\binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}} = \sum_{k=\max\{\lfloor \frac{n-r}{2}\rfloor, 0\}}^{\lfloor \frac{n}{2}\rfloor} \binom{r-n+2k}{k} e_{n-2k}^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}). \tag{3.11}$$

(2) *For any non-negative integers* n *and* m*, we have*

$$h_m^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota})$$
$$= \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^m q^{-(m-2k)r}\left(\binom{2r+m-2k}{m-2k}_q - \binom{2r+m-2k-1}{m-2k-1}_q q^r\right) c_{m+r-1,k}, \tag{3.12}$$

$$q^{-nr}\left(\binom{2r+n}{n}_q - \binom{2r+n-1}{n-1}_q q^r\right) = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k+r-1}{k} h_{n-2k}^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}). \tag{3.13}$$

(3) *For any non-negative integer* m*,*

$$p_m^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}) = -2^{m-1} + \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\frac{1-q^{(2r+1)|m-2k|}}{1-q^{|m-2k|}} q^{-|m-2k|r}. \tag{3.14}$$

For any positive integer n*,*

$$-1 + \frac{q^{-rn} - q^{(r+1)n}}{1-q^n}$$
$$= 2\sum_{k=0}^{\lfloor \frac{n+1}{2}\rfloor}\binom{2k-n-1}{k} p_{n-2k}^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}) - \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{2k-n}{k} p_{n-2k}^{(r)}(\boldsymbol{q}^{+\iota} + \boldsymbol{q}^{-\iota}). \tag{3.15}$$

4. Applications to $F_n^{(r)}$ and $L_n^{(r)}$

In this section we investigate more specializations of Theorem 1.1 and Theorem 1.2, and prove Theorem 1.3, Theorem 1.4 and Theorem 1.5. To apply Theorem 1.1 and Theorem 1.2 to $F_n^{(r)}$ and $L_n^{(r)}$, we evaluate $e_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota})$, $h_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota})$, $p_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota})$ and $e_m^{(r)}(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota})$.

Proposition 4.1. (1) *For $n = 0, 1, \ldots, 2r$, we have*

$$e_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) = 1. \tag{4.1}$$

(2) *For any non-negative integer n,*

$$\begin{aligned} h_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) &= \frac{1}{2}\left((-1)^{\lfloor \frac{n}{2r+1} \rfloor} - (-1)^{\lfloor \frac{n-2}{2r+1} \rfloor}\right) \\ &= \begin{cases} 1 & (n \equiv 0, 1 \pmod{4r+2}), \\ -1 & (n \equiv 2r+1, 2r+2 \pmod{4r+2}), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned} \tag{4.2}$$

(3) *For any non-negative integer n,*

$$p_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) = (-1)^n(-1 + (2r+1)\delta_{2r+1|n}) = \begin{cases} (-1)^{n-1} & (2r+1 \nmid n), \\ (-1)^n 2r & (2r+1 \mid n). \end{cases} \tag{4.3}$$

where

$$\delta_{2r+1|n} := \begin{cases} 0 & (2r+1 \nmid n), \\ 1 & (2r+1 \mid n). \end{cases}$$

Proof. (1) From (2.8), we have

$$\begin{aligned} \sum_{n=0}^{2r} e_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) y^n &= \prod_{j=1}^{r} \left(1 - e^{2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)\left(1 - e^{-2\pi\sqrt{-1}\frac{j}{2r+1}} y\right) \\ &= \frac{1 - y^{2r+1}}{1 - y} \\ &= \sum_{n=0}^{2r} y^n. \end{aligned}$$

(2) From (2.9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} h_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) y^n &= \prod_{j=1}^{r} \frac{1}{\left(1 + e^{2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)\left(1 + e^{-2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)} \\ &= \frac{1 + y}{1 + y^{2r+1}} \\ &= \sum_{k=0}^{\infty} (-1)^k (y^{(2r+1)k} + y^{(2r+1)k+1}) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}\left((-1)^{\lfloor \frac{n}{2r+1} \rfloor} - (-1)^{\lfloor \frac{n-2}{2r+1} \rfloor}\right) y^n. \end{aligned}$$

(3) By the definition of $p_n^{(2r)}$,

$$p_n^{(2r)}(-\boldsymbol{\zeta}^{+\iota}, -\boldsymbol{\zeta}^{-\iota}) = (-1)^n \left(-1 + \sum_{k=-r}^{r} \zeta_{2r+1}^{kn}\right) = (-1)^n(-1 + (2r+1)\delta_{2r+1|n}).$$

□

Proof of Theorem 1.3 and Theorem 1.4 From Theorem 1.1 (2) and (3) and Proposition 4.1 (2) and (3), we derive explicit formulas of $F_n^{(r)}$ and $L_n^{(r)}$. Similarly, Theorem 1.4 follows from Theorem 1.1 (2) and (3) and Proposition 4.1 (2) and (3). □

By specialization of Theorem 1.3 (2) and (3), we obtain the initial values of $F_{n+1}^{(r)}$ and $L_n^{(r)}$.

Corollary 4.2. (1) *If $m \leq r$, then we have*

$$F_{2m-1}^{(r+1)} = F_{2m}^{(r)} = \binom{2m+r-2}{m-1} - \binom{2m+r-2}{m-2}. \tag{4.4}$$

(2) *If $m < 2r+1$, then we have*

$$L_{2m}^{(r)} = -2^{2m-1} + \frac{2r+1}{2}\binom{2m}{m}. \tag{4.5}$$

If $m < r$, then we have

$$L_{2m+1}^{(r)} = 4^m. \tag{4.6}$$

To prove Theorem 1.5, we need to evaluate $e_m^{(r)}(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota})$, which can be computed from (1.6) and (4.1).

Proposition 4.3. *For $m = 0, 1, \ldots, r$, we have*

$$e_m^{(r)}(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota}) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-r-1,k} = \binom{m-r-1}{\lfloor \frac{m}{2} \rfloor} = (-1)^{\lfloor \frac{m}{2} \rfloor}\binom{r - \lfloor \frac{m+1}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}. \tag{4.7}$$

Remark 4.4. (1) *For (4.7), we give another proof without using (1.6). Let $x := z + z^{-1}$. First, we remark*

$$\sum_{k=-r}^{r} z^k = \prod_{j=1}^{r}\left(x - 2\cos\left(\frac{2\pi j}{2r+1}\right)\right) = \sum_{m=0}^{r} e_m^{(r)}(-\boldsymbol{\zeta}^{+\iota} - \boldsymbol{\zeta}^{-\iota})x^{r-m}.$$

On the other hand, if $|u| < |z| < |u|^{-1}$, then

$$\begin{aligned}
\sum_{r=0}^{\infty} u^r \sum_{k=-r}^{r} z^k &= \sum_{r=0}^{\infty} u^r \left(\frac{z^{-r}}{1-z} - \frac{z^{r+1}}{1-z}\right) \\
&= \frac{1}{1-z}\left(\frac{1}{1-z^{-1}u} - \frac{z}{1-zu}\right) \\
&= \frac{1+u}{(1-z^{-1}u)(1-zu)} \\
&= \frac{1+u}{1+u^2}\frac{1}{1-x\frac{u}{1+u^2}} \\
&= (1+u)\sum_{N=0}^{\infty} x^N u^N (1+u^2)^{-N-1}
\end{aligned}$$

$$= (1+u)\sum_{N=0}^{\infty} x^N \sum_{k=0}^{\infty} \binom{-N-1}{k} u^{N+2k}$$

$$= \sum_{r=0}^{\infty} u^r \left(\sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^m \binom{r-m}{m} x^{r-2m} + \sum_{m=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^m \binom{r-1-m}{m} x^{r-1-2m} \right).$$

Hence we obtain the conclusion (4.7).
(2) *The formula (4.7) is obtained by substituting (4.1) into (1.6). Similarly, by substituting (4.1) for (1.9), for $n = 0, 1, \ldots, 2r$ we obtain*

$$\sum_{k=\max\{\lfloor \frac{n-r}{2} \rfloor, 0\}}^{\lfloor \frac{n}{2} \rfloor} \binom{r-n+2k}{k} \binom{n-2k-r-1}{\lfloor \frac{n}{2} \rfloor - k} = 1. \tag{4.8}$$

Proof of Theorem 1.5 From the Wronski relations, Newton's formulas and (4.7), for any non-negative integer n we have

$$\sum_{j=0}^{\min(n,r)} (-1)^{\lfloor \frac{j-1}{2} \rfloor} \binom{r - \lfloor \frac{j+1}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} F_{n-j+1}^{(r)} = 0,$$

$$\sum_{j=0}^{\min(n,r)} (-1)^{n-\lfloor \frac{j+1}{2} \rfloor} \binom{r - \lfloor \frac{j+1}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} L_{n-j+1}^{(r)}$$

$$= \begin{cases} (-1)^{\lfloor \frac{n+1}{2} \rfloor} (n+1) \binom{r-\lfloor \frac{n}{2} \rfloor - 1}{\lfloor \frac{n+1}{2} \rfloor} & (n = 0, 1, \ldots, r-1), \\ 0 & (n \geq r). \end{cases}$$

Then $\{F_{n+1}^{(r)}\}_{n\geq 0}$ and $\{L_n^{(r)}\}_{n\geq 0}$ satisfy the recursion (1.20).

The initial values of $\{F_{n+1}^{(r)}\}_{n\geq 0}$ are determined by the vanishing property (2.7)

$$F_0^{(r)} = \cdots = F_{-(r-2)}^{(r)} = 0$$

and $F_1^{(r)} = h_0^{(r)} = 1$. The initial values of $\{L_n^{(r)}\}_{n\geq 0}$ follows from Corollary 4.2 (2). □

Example 4.5. $r = 1$*:*

$$F_1^{(1)} = 1,\ L_0^{(1)} = 1,\ a_{n+1}^{(1)} = a_n^{(1)} = 1.$$

$r = 2$ (Fibonacci numbers and Lucas numbers):

$$F_{-1}^{(2)} = 0,\ F_0^{(2)} = 1,\ L_0^{(2)} = 2,\ L_1^{(1)} = 1,\ a_{n+2}^{(2)} = a_{n+1}^{(2)} + a_n^{(2)}.$$

$$F_{n+1}^{(2)} : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

$$L_n^{(2)} : 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots$$

$r = 3$ (OEIS A006053 and OEIS A096975):

$$F_{-1}^{(3)} = F_0^{(3)} = 0,\ F_1^{(3)} = 1,\ L_0^{(3)} = 3,\ L_1^{(3)} = 1,\ L_2^{(3)} = 5,$$

$$a_{n+3}^{(3)} = a_{n+2}^{(3)} + 2a_{n+1}^{(3)} - a_n^{(3)}.$$

$$F_{n+1}^{(3)} : 1, 1, 3, 4, 9, 14, 28, 47, 89, 155, 286, 507, 924, 1652, 2993, \ldots$$

$$L_n^{(3)} : 3, 1, 5, 4, 13, 16, 38, 57, 117, 193, 370, 639, 1186, 2094, \dots$$

$r = 4$ (OEIS A188021 and OEIS A094649):

$$F_{-2}^{(4)} = F_{-1}^{(4)} = F_0^{(4)} = 0,\ F_1^{(4)} = 1,$$
$$L_0^{(4)} = 4,\ L_1^{(4)} = 1,\ L_2^{(4)} = 7,\ L_3^{(4)} = 4,$$
$$a_{n+4}^{(4)} = a_{n+3}^{(4)} + 3a_{n+2}^{(4)} - 2a_{n+1}^{(4)} - a_n^{(4)}.$$
$$F_{n+1}^{(4)} : 1, 1, 4, 5, 14, 20, 48, 75, 165, 274, 571, 988, 1988, 3536, 6953, \dots$$
$$L_n^{(4)} : 4, 1, 7, 4, 19, 16, 58, 64, 187, 247, 622, 925, 2110, 3394, 7252, \dots$$

$r = 5$ (OEIS A231181 and OEIS A189234):

$$F_{-3}^{(5)} = F_{-2}^{(5)} = F_{-1}^{(5)} = F_0^{(5)} = 0,\ F_1^{(5)} = 1,$$
$$L_0^{(5)} = 5,\ L_1^{(5)} = 1,\ L_2^{(5)} = 9,\ L_3^{(5)} = 4,\ L_4^{(5)} = 25,$$
$$a_{n+5}^{(5)} = a_{n+4}^{(5)} + 4a_{n+3}^{(5)} - 3a_{n+2}^{(5)} - 3a_{n+1}^{(5)} + a_n^{(5)}.$$
$$F_{n+1}^{(5)} : 1, 1, 5, 6, 20, 27, 75, 110, 275, 429, 1001, 1637, 3639, 6172, \dots$$
$$L_n^{(5)} : 5, 1, 9, 4, 25, 16, 78, 64, 257, 256, 874, 1013, 3034, 3953, \dots$$

$r = 6$:

$$F_{-4}^{(6)} = F_{-3}^{(6)} = F_{-2}^{(6)} = F_{-1}^{(6)} = F_0^{(6)} = 0,\ F_1^{(6)} = 1,$$
$$L_0^{(6)} = 6,\ L_1^{(6)} = 1,\ L_2^{(6)} = 11,\ L_3^{(6)} = 4,\ L_4^{(6)} = 31,\ L_5^{(6)} = 16,$$
$$a_{n+6}^{(6)} = a_{n+5}^{(6)} + 5a_{n+4}^{(6)} - 4a_{n+3}^{(6)} - 6a_{n+2}^{(6)} + 3a_{n+1}^{(6)} + a_n^{(6)}.$$
$$F_{n+1}^{(6)} : 1, 1, 6, 7, 27, 35, 110, 154, 429, 637, 1638, 2548, 6188, 9995, \dots$$
$$L_n^{(6)} : 6, 1, 11, 4, 31, 16, 98, 64, 327, 256, 1126, 1024, 3958, 4083, \dots$$

Finally, we mention some interesting examples of our results, including seemingly new formulas for the Fibonacci and Lucas numbers.

Corollary 4.6. *For any non-negative integers m and n, we have*

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2}\left((-1)^{\lfloor \frac{m-2k}{3} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{3} \rfloor}\right) c_{m,k} = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} (-1)^k c_{n-1, \lfloor \frac{n-1-3k}{2} \rfloor} = 1, \tag{4.9}$$

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2}\left((-1)^{\lfloor \frac{m-2k}{5} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{5} \rfloor}\right) c_{m+1,k} = \sum_{k=0}^{\lfloor \frac{n-1}{5} \rfloor} (-1)^k c_{n, \lfloor \frac{n-1-5k}{2} \rfloor} = F_{m+1} \tag{4.10}$$

and

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} 1 & (n \equiv 0, 1 \pmod 6), \\ -1 & (n \equiv 3, 4 \pmod 6), \\ 0 & (\text{otherwise}), \end{cases} \tag{4.11}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k+1}{k} F_{n-2k+1} = \begin{cases} 1 & (n \equiv 0, 1 \pmod{10}), \\ -1 & (n \equiv 5, 6 \pmod{10}), \\ 0 & (\text{otherwise}). \end{cases} \tag{4.12}$$

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	3	5	8	13	21	34	55	89	144
3	1	1	3	4	9	14	28	47	89	155	286	507
4	1	1	4	5	14	20	48	75	165	274	571	988
5	1	1	5	6	20	27	75	110	275	429	1001	1637
6	1	1	6	7	27	35	110	154	429	637	1638	2548
7	1	1	7	8	35	44	154	208	637	910	2548	3808
8	1	1	8	9	44	54	208	273	910	1260	3808	5508
9	1	1	9	10	54	65	273	350	1260	1700	5508	7752
10	1	1	10	11	65	77	350	440	1700	2244	7752	10659
11	1	1	11	12	77	90	440	544	2244	2907	10659	14364
12	1	1	12	13	90	104	544	663	2907	3705	14364	19019
13	1	1	13	14	104	119	663	798	3705	4655	19019	24794
14	1	1	14	15	119	135	798	950	4655	5775	24794	31878
15	1	1	15	16	135	152	950	1120	5775	7084	31878	40480
16	1	1	16	17	152	170	1120	1309	7084	8602	40480	50830

TABLE 2. $F_n^{(r)}$

$r \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	3	4	7	11	18	29	47	76	123	199	322	521
3	3	1	5	4	13	16	38	57	117	193	370	639	1186	2094
4	4	1	7	4	19	16	58	64	187	247	622	925	2110	3394
5	5	1	9	4	25	16	78	64	257	256	874	1013	3034	3953
6	6	1	11	4	31	16	98	64	327	256	1126	1024	3958	4083
7	7	1	13	4	37	16	118	64	397	256	1378	1024	4882	4096
8	8	1	15	4	43	16	138	64	467	256	1630	1024	5806	4096
9	9	1	17	4	49	16	158	64	537	256	1882	1024	6730	4096
10	10	1	19	4	55	16	178	64	607	256	2134	1024	7654	4096
11	11	1	21	4	61	16	198	64	677	256	2386	1024	8578	4096
12	12	1	23	4	67	16	218	64	747	256	2638	1024	9502	4096
13	13	1	25	4	73	16	238	64	817	256	2890	1024	10426	4096
14	14	1	27	4	79	16	258	64	887	256	3142	1024	11350	4096
15	15	1	29	4	85	16	278	64	957	256	3394	1024	11274	4096
16	16	1	31	4	91	16	298	64	1027	256	3646	1024	13198	4096

TABLE 3. $L_n^{(r)}$

The formula (4.10) was given by Andrews [1].

Corollary 4.7. *For any non-negative integer m, we have*

$$\frac{3}{2}\sum_{k=-\lfloor \frac{m}{3} \rfloor}^{\lfloor \frac{m}{3} \rfloor} \binom{2m}{m-3k} = 2^{2m-1}+1, \tag{4.13}$$

$$\frac{3}{2}\sum_{k=-\lfloor\frac{m+2}{3}\rfloor}^{\lfloor\frac{m-1}{3}\rfloor}\binom{2m+1}{m-3k-1}=3\sum_{k=0}^{\lfloor\frac{m-1}{3}\rfloor}\binom{2m+1}{m-3k-1}=4^m-1, \tag{4.14}$$

$$L_{2m}=-2^{2m-1}+\frac{5}{2}\sum_{k=-\lfloor\frac{m}{5}\rfloor}^{\lfloor\frac{m}{5}\rfloor}\binom{2m}{m-5k}, \tag{4.15}$$

$$L_{2m+1}=4^m-\frac{5}{2}\sum_{k=-\lfloor\frac{m+3}{5}\rfloor}^{\lfloor\frac{m-2}{5}\rfloor}\binom{2m+1}{m-5k-2}. \tag{4.16}$$

For any positive integer n, we have

$$2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}-\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k-n}{k}=\begin{cases}(-1)^{n-1} & (3\nmid n),\\ (-1)^n 2 & (3\mid n),\end{cases} \tag{4.17}$$

$$2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}L_{n-2k}-\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k-n}{k}L_{n-2k}=\begin{cases}(-1)^{n-1} & (5\nmid n),\\ (-1)^n 4 & (5\mid n).\end{cases} \tag{4.18}$$

Appendix A. Some congruence relations for $F_n^{(r)}$ and $L_n^{(r)}$

All of the results so far have been obtained as specializations of Theorem 1.1 and Theorem 1.2, but in this section, we mention some properties for $F_n^{(r)}$ and $L_n^{(r)}$ that can be obtained independently of Theorem 1.1 and Theorem 1.2.

Theorem A.1. *Let $p := 2r+1$ be a prime number. If q is an odd prime number such that $q \equiv \pm 1 \pmod p$, then*

$$F_{n+q-1}^{(r)} \equiv F_n^{(r)} \pmod q, \tag{A.1}$$

$$L_{n+q-1}^{(r)} \equiv L_n^{(r)} \pmod q. \tag{A.2}$$

In particular, for any non-negative integer k we have

$$F_{k(q-1)} \equiv 0 \pmod q, \tag{A.3}$$

$$F_{k(q-1)+1} \equiv F_{k(q-1)+2} \equiv 1 \pmod q, \tag{A.4}$$

$$L_{k(q-1)} \equiv \frac{p-1}{2} \pmod q, \tag{A.5}$$

$$L_{k(q-1)+1} \equiv 1 \pmod q, \tag{A.6}$$

$$L_{k(q-1)+2} \equiv p-2 \pmod q. \tag{A.7}$$

Proof. Let $\mathbb{F}_q$ be a finite field of order q and ζ_p be a primitive p-th root of unity. Since the both sides of (A.1) and (A.2) are integers, it is enough to show that the equalities hold in $\mathbb{F}_q[\zeta_p]$. For any integer i a simple calculation shows that

$$(\zeta_p^i+\zeta_p^{-i})^q=\zeta_p^{qi}+\zeta_p^{-qi}=\zeta_p^i+\zeta_p^{-i} \quad (\text{in } \mathbb{F}_q[\zeta_p]).$$

Hence by the definition of L_n^r we obtain

$$L_{n+q-1}^{(r)}=\sum_{j=1}^{r}(-\zeta_p^j-\zeta_p^{-j})^{n-1}(-\zeta_p^j-\zeta_p^{-j})^q$$

$$= \sum_{j=1}^{r} (-\zeta_p^{j} - \zeta_p^{-j})^{n-1}(-\zeta_p^{j} - \zeta_p^{-j})$$
$$= L_n \quad (\text{in } \mathbb{F}_q[\zeta_p]).$$

To prove (A.1), we need the discriminant of $-\zeta_p - \zeta_p^{-1}$ [3, Theorem 3.8]

$$\det\left((-\zeta_p^{r-i} - \zeta_p^{-(r-i)})^{r-j}\right)^2_{i,j=1,\dots r} = p^{\frac{p-3}{2}}.$$

From this evaluation, we have

$$\det\left((-\zeta_p^{r-i} - \zeta_p^{-(r-i)})^{r-j}\right)_{i,j=1,\dots r} = p^{\frac{p-3}{4}} = p^{\frac{r-1}{2}}. \tag{A.8}$$

We point out even if r is even then $p^{\frac{r-1}{2}} \in \mathbb{F}_p$ by the first supplement to quadratic reciprocity. Thus we have

$$p^{\frac{p-3}{4}} F^{(r)}_{n+q}$$
$$= \det\begin{pmatrix} (-\zeta_p^{r-1} - \zeta_p^{-(r-1)})^{n+q-1+r-1} & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)})^{n+q-1+r-1} & \cdots & (-\zeta_p - \zeta_p^{-1})^{n+q-1+r-1} \\ (-\zeta_p^{r-1} - \zeta_p^{-(r-1)})^{r-2} & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)})^{r-2} & \cdots & (-\zeta_p - \zeta_p^{-1})^{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ (-\zeta_p^{r-1} - \zeta_p^{-(r-1)}) & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)}) & \cdots & (-\zeta_p - \zeta_p^{-1}) \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$
$$= \det\begin{pmatrix} (-\zeta_p^{r-1} - \zeta_p^{-(r-1)})^{n+r-1} & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)})^{n+r-1} & \cdots & (-\zeta_p - \zeta_p^{-1})^{n+r-1} \\ (-\zeta_p^{r-1} - \zeta_p^{-(r-1)})^{r-2} & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)})^{r-2} & \cdots & (-\zeta_p - \zeta_p^{-1})^{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ (-\zeta_p^{r-1} - \zeta_p^{-(r-1)}) & (-\zeta_p^{r-2} - \zeta_p^{-(r-2)}) & \cdots & (-\zeta_p - \zeta_p^{-1}) \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$
$$= p^{\frac{p-3}{4}} F^{(r)}_{n+1} \quad (\text{in } \mathbb{F}_q[\zeta_p]).$$

Here the first equality follows from (2.5). Finally, since p does not divide q, we obtain (A.1).

The formulas (A.3) - (A.7) follow from (A.1), (A.2) and Corollary 4.2 immediately. □

Appendix B. Other formulas for $F_n^{(r)}$ and $L_n^{(r)}$ from symmetric polynomials

Since the sequences $\{F_n^{(r)}\}_n$ and $\{L_n^{(r)}\}_n$ are special values of $h_n^{(r)}(\mathbf{z})$ and $p_n^{(r)}(\mathbf{z})$ respectively, various formulas for $F_n^{(r)}$ and $L_n^{(r)}$ are derived immediately from specializations of some formulas for symmetric polynomials [4]. In this section, we list some typical formulas obtained from symmetric polynomials.

Generating functions.

$$\sum_{n\geq 0} F^{(r)}_{n+1} u^n = \frac{1}{\sum_{m=0}^{\lfloor \frac{r}{2}\rfloor} (-1)^m \binom{r-m}{m} u^{2m} - \sum_{m=0}^{\lfloor \frac{r-1}{2}\rfloor} (-1)^m \binom{r-1-m}{m} u^{2m+1}}, \tag{B.1}$$

$$\sum_{n\geq 0} L^{(r)}_{n+1} u^n = \frac{\sum_{m=0}^{\lfloor \frac{r-1}{2}\rfloor} (-1)^m (2m+1) \binom{r-1-m}{m} u^{2m} - \sum_{m=0}^{\lfloor \frac{r}{2}\rfloor} (-1)^m 2m \binom{r-m}{m} u^{2m-1}}{\sum_{m=0}^{\lfloor \frac{r}{2}\rfloor} (-1)^m \binom{r-m}{m} u^{2m} - \sum_{m=0}^{\lfloor \frac{r-1}{2}\rfloor} (-1)^m \binom{r-1-m}{m} u^{2m+1}}. \tag{B.2}$$

Generating functions (B.1) and (B.2) are obtained by substituting (4.7) into [4, (2.5) and (2.10)].

Determinant formulas. For convenience, put

$$\alpha_{r,j} := -2\cos\left(\frac{2\pi(r+1-j)}{2r+1}\right), \quad (j=1,\dots,r)$$

and

$$C_n^{(r)} := e_n^{(r)}(-\boldsymbol{\zeta}^{+\iota}-\boldsymbol{\zeta}^{-\iota}) = \begin{cases} \binom{n-r-1}{\lfloor \frac{n}{2}\rfloor} = (-1)^{\lfloor \frac{n}{2}\rfloor}\binom{r-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor} & (n=0,1,\dots,r) \\ 0 & (n\neq 0,1,\dots,r) \end{cases}.$$

From (2.5), (A.8) and the determinant formulas on [4, p28], we obtain the following determinant formulas for $F_n^{(r)}$ and $L_n^{(r)}$.

$$F_{n+1}^{(r)} = \frac{1}{(2r+1)^{\frac{r-1}{2}}}\det\begin{pmatrix} \alpha_{r,1}^{n+r-1} & \alpha_{r,2}^{n+r-1} & \cdots & \alpha_{r,r-1}^{n+r-1} & \alpha_{r,r}^{n+r-1} \\ \alpha_{r,1}^{r-2} & \alpha_{r,2}^{r-2} & \cdots & \alpha_{r,r-1}^{r-2} & \alpha_{r,r}^{r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{r,1} & \alpha_{r,2} & \cdots & \alpha_{r,r-1} & \alpha_{r,r} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \tag{B.3}$$

$$= \det\left(C_{1-i+j}^{(r)}\right)_{1\le i,j\le n} \tag{B.4}$$

$$= \frac{1}{n!}\det\begin{pmatrix} L_1^{(r)} & -1 & 0 & \cdots & 0 & 0 \\ L_2^{(r)} & L_1^{(r)} & -2 & \cdots & 0 & 0 \\ L_3^{(r)} & L_2^{(r)} & L_1^{(r)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ L_{n-1}^{(r)} & L_{n-2}^{(r)} & L_{n-3}^{(r)} & \cdots & L_1^{(r)} & -n+1 \\ L_n^{(r)} & L_{n-1}^{(r)} & L_{n-2}^{(r)} & \cdots & L_2^{(r)} & L_1^{(r)} \end{pmatrix}, \tag{B.5}$$

$$L_n^{(r)} = \det\begin{pmatrix} C_1^{(r)} & 1 & 0 & \cdots & 0 & 0 \\ 2C_2^{(r)} & C_1^{(r)} & 1 & \cdots & 0 & 0 \\ 3C_3^{(r)} & C_2^{(r)} & C_1^{(r)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ (n-1)C_{n-1}^{(r)} & C_{n-2}^{(r)} & C_{n-3}^{(r)} & \cdots & C_1^{(r)} & 1 \\ nC_n^{(r)} & C_{n-1}^{(r)} & C_{n-2}^{(r)} & \cdots & C_2^{(r)} & C_1^{(r)} \end{pmatrix} \tag{B.6}$$

$$= (-1)^{n-1}\det\begin{pmatrix} F_2^{(r)} & F_1^{(r)} & 0 & \cdots & 0 & 0 \\ 2F_3^{(r)} & F_2^{(r)} & F_1^{(r)} & \cdots & 0 & 0 \\ 3F_4^{(r)} & F_3^{(r)} & F_2^{(r)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ (n-1)F_n^{(r)} & F_{n-1}^{(r)} & F_{n-2}^{(r)} & \cdots & F_2^{(r)} & F_1^{(r)} \\ nF_{n+1}^{(r)} & F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_3^{(r)} & F_2^{(r)} \end{pmatrix}, \tag{B.7}$$

$$C_n^{(r)} = \det\left(F_{2-i+j}^{(r)}\right)_{1\le i,j\le n} \tag{B.8}$$

$$= \frac{1}{n!} \det \begin{pmatrix} L_1^{(r)} & 1 & 0 & \cdots & 0 & 0 \\ L_2^{(r)} & L_1^{(r)} & 2 & \cdots & 0 & 0 \\ L_3^{(r)} & L_2^{(r)} & L_1^{(r)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ L_{n-1}^{(r)} & L_{n-2}^{(r)} & L_{n-3}^{(r)} & \cdots & L_1^{(r)} & n-1 \\ L_n^{(r)} & L_{n-1}^{(r)} & L_{n-2}^{(r)} & \cdots & L_2^{(r)} & L_1^{(r)} \end{pmatrix}. \tag{B.9}$$

Some relations. For any partition λ let z_λ denote the product

$$z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. Then we have

$$F_{n+1}^{(r)} = \frac{1}{n} \sum_{i=1}^{n} L_i^{(r)} F_{n+1-i}^{(r)} \tag{B.10}$$

$$= \sum_{|\lambda|=n} \frac{L_{\lambda_1}^{(r)} \cdots L_{\lambda_r}^{(r)}}{z_\lambda}, \tag{B.11}$$

$$C_n^{(r)} = \sum_{|\lambda|=n} (-1)^{n-r} \frac{L_{\lambda_1}^{(r)} \cdots L_{\lambda_r}^{(r)}}{z_\lambda} \tag{B.12}$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$ runs over partitions and $|\lambda|$ denotes the sum of the parts

$$|\lambda| := \lambda_1 + \cdots + \lambda_r.$$

These formulas follow from [4, (2.11) and (2.14′)].

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