ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-720 **Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON**

Let $\lfloor \dots \rfloor$ be the largest integer function and, for a positive integer n, define $\varepsilon_n = 1$ for n even and $\varepsilon_n = 0$ for n odd. Then, with P_n the nth Pell number, prove the following identities:

(a) $$\sum_{k\ge 0} \frac{\binom{n-2k}{2k}}{25^k} = \frac{1}{5^{n/2}6}\Big[\varepsilon_n(L_{2n+2}+3L_{n+1}) + (1-\varepsilon_n)\sqrt{5}(F_{2n+2}+3F_{n+1})\Big];$$

(b) $$\sum_{k\ge 0} \frac{\binom{n-1-2k}{2k}}{16^k} = \frac{1}{2^n}[P_n + n];$$

(c) $$\sum_{k=0}^{\lfloor (n-1)/4\rfloor} \frac{\binom{n-1-2k}{2k}}{25^k(n-4k)} = \frac{1}{5^{n/2}n}\Big[\varepsilon_n(L_{2n}+L_n-2(1+(-1)^{n/2})) + (1-\varepsilon_n)\sqrt{5}(F_{2n}+F_n)\Big];$$

(d)

$$\sum_{k\ge 1} \frac{k\binom{n-1-k}{k}}{5^k}$$
$$= \frac{1}{5^{n/2}54}\Big[\varepsilon_n((45n-20)F_{2n}-15nL_{2n}) + (1-\varepsilon_n)\sqrt{5}((9n-4)L_{2n}-15nF_{2n})\Big].$$

H-721 **Khristo N. Boyadzhiev, Ohio Northern University, Ada, Ohio**

Let $H_0 = 0$ and $H_n = 1 + 1/2 + \dots + 1/n$ for $n \ge 1$ be the harmonic numbers. Show that

$$\sum_{n=0}^{\infty} F_n H_n z^n = C(z)\sum_{n=0}^{\infty} F_n z^n, \qquad \text{where} \qquad C(z) = 1 + \sum_{n=1}^{\infty}\left(\frac{F_{n-1}}{n} + \frac{F_{n+1}}{n+1}\right) z^n,$$

for $|z|$ small enough.

H-722 Proposed by Ovidiu Furdui, Campia Turzii, Romania

Let $x \in (0, 2\pi)$, $k \geq 1$ be a natural number and

$$S_k(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n(n+1)(n+2)\cdots(n+k)}.$$

Prove that $S_k(x)$ equals

$$\frac{(2\sin(x/2))^k}{k!}\left(-\cos\frac{(\pi-x)k}{2}\cdot\frac{\ln(2(1-\cos x))}{2} - \frac{\pi-x}{2}\sin\frac{(\pi-x)k}{2} + \sum_{j=1}^{k}\frac{\cos\frac{(\pi-x)(j-k)}{2}}{j(2\sin(x/2))^j}\right).$$

H-723 Proposed by Ovidiu Furdui, Campia Turzii, Romania

Let $k \geq 2$ be an integer and let m be a nonnegative integer. Prove that

$$\lim_{n\to\infty}\frac{1}{n^{k-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{i_1+i_2+\cdots+i_k+m} = \frac{k}{(k-1)!}\sum_{j=2}^{k}(-1)^{k-j}j^{k-2}\binom{k-1}{j-1}\ln j.$$

H-723 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Determine

$$\left(\sum_{k=1}^{\infty}\frac{1}{F_{4^k}^2} - \sum_{k=1}^{\infty}\frac{1}{L_{4^k}^2} + \sum_{k=1}^{\infty}\frac{1}{L_{2^k}^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{F_{2^k}^2}\right)^{-1}.$$

SOLUTIONS

A k-Fibonacci Identity

H-696 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain (Vol. 47, No. 4, November 2009/2010)

For any positive integer k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\geq 0}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$. For $n \geq 0$, and $i \geq j$ define $S_{i,j} = \sum_{r=0}^{j-1} kF_{k,i-r}F_{k,j-r}$. Prove by combinatorial arguments that

$$S_{i,j} = \begin{cases} F_{k,i}F_{k,j+1} & \text{if } j \text{ is odd,} \\ F_{k,i}F_{k,j+1} - F_{k,i-j} & \text{if } j \text{ is even.} \end{cases}$$

Solution by the proposers

It is well-known that the k-Fibonacci numbers, $F_{k,n}$, count the number of tilings of an $(n-1)$-board with k-distinguished (or colored) squares and black dominoes (see [1]). For convenience, we will use the notation $f_{k,n} = F_{k,n+1}$. For k-distinguished squares we understand that each square may be labeled (or colored) in k different ways.

We use the concepts of *breakable* tiling and *unbreakable* tiling (see [1]). It is said that a tiling of an n-board is *breakable* at cell p, if the tiling can be decomposed into two tilings, one

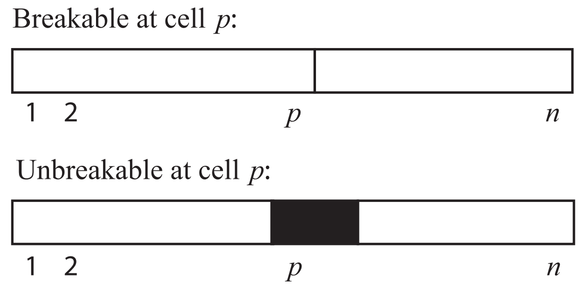

FIGURE 1. An (n)-board is either breakable or unbreakable at cell p.

covering cells 1 through p and the other covering cells $p+1$ through n. On the other hand, a tiling is said to be *unbreakable* at cell p if a domino occupies cells p and $p+1$. See Figure 1.

Consider two tilings offset as in Figure 2. The first one tiles cells 2 through 9; the second one tiles cells 1 through 6. Following again [1] we say that there is a *fault* at cell r, for $1 \le r \le 6$, if both tilings are breakable at cell r. The pair of tilings of Figure 2 has faults at cells $1, 4$, and 6.

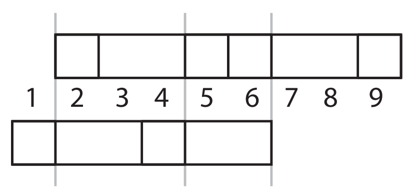

FIGURE 2. Two tilings with their faults (in gray lines).

To tackle this identity we first rewrite the identity in function of k-tiling boards using $f_{k,n} = F_{k,n+1}$ to obtain $S_{i,j} = \sum_{r=0}^{j-1} k f_{k,i-r-1} f_{k,j-r-1} = \sum_{r=1}^{j} k f_{k,i-r} f_{k,j-r}$, and then

$$S_{i,j} = \begin{cases} f_{k,i-1} f_{k,j} & \text{if } j \text{ is odd,} \\ f_{k,i-1} f_{k,j} - f_{k,i-j-1} & \text{if } j \text{ is even.} \end{cases}$$

Now we consider pairs of tilings: an $(i-1)$-tiling covering cells 2 through i and a j-tiling covering cells 1 through j, with $j \le i$.

Question: How many tilings of an $(i-1)$-board and j-board exist?

Answer 1: There are $f_{k,i-1} f_{k,j}$ such tilings.

Answer 2: Condition on the location of the first fault. See Fig. 3 left. Note that the first fault may appear at cell r, for $1 \le r \le j$, and there are $k f_{k,i-r} f_{k,j-r}$ of such tilings, where the k factor corresponds to the k possible colorings of the unique square that appears in the first r cells either in the $(i-1)$-board, or in the j-board. Summing up on r we get $S_{i,j}$ except for the case in which there are no faults in the pair of tilings; that is, if j is even (see Figure 3 right). Therefore if j is even $S_{i,j}$ is equal to $f_{k,i-1} f_{k,j} - f_{k,i-j-1}$. □

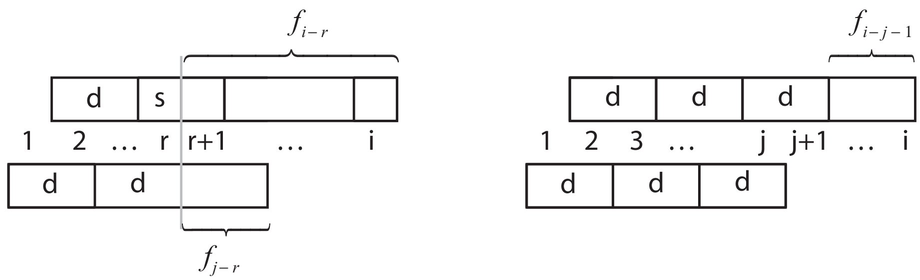

FIGURE 3. Two tilings with their faults (in gray lines), and two tilings with no faults if j is even.

References

[1] A. T. Benjamin and J. J.Quinn, *Fibonacci and Lucas identities through colored tilings*, Utilit. Math., **56** (1999), 137–142.

Also solved by Paul S. Bruckman.

Cubonomials

H-697 Proposed by N. Gauthier, Kingston, ON (Vol. 48, No. 1, February 2011)

Define $K_0 = 1$ and, for a positive integer n, let K_n represent the sum of the cubes of the first n positive integers. Then define

$$\begin{bmatrix} n \\ k \end{bmatrix}_K = \frac{K_n K_{n-1} \cdots K_{n-k+1}}{K_k K_{k-1} \cdots K_1 K_0}, \qquad \text{for} \quad 0 \le k \le n.$$

a) Show that $\begin{bmatrix} n \\ n-k \end{bmatrix}_K = \begin{bmatrix} n \\ k \end{bmatrix}_K$.

b) Show that $\begin{bmatrix} n \\ k \end{bmatrix}_K = m^2$, where $m = m(n,k)$ is a positive integer.

c) Find a closed form expression for $S_n = \sum_{k \ge 0} m(n,k)$.

Solution by Ángel Plaza and Sergio Falcón

a) $\begin{bmatrix} n \\ n-k \end{bmatrix}_K = \begin{bmatrix} n \\ k \end{bmatrix}_K \Leftrightarrow \dfrac{K_n K_{n-1} \cdots K_{n-k+1}}{K_k K_{k-1} \cdots K_1 K_0} = \dfrac{K_n K_{n-1} \cdots K_{k+1}}{K_{n-k} K_{n-k-1} \cdots K_1 K_0}$, which it is true since the cross product is the same: $K_n K_{n-1} \cdots K_1 K_0$. For convenience we can denote it by $(K_n)!$. □

b) We use that $K_n = \frac{1}{4} n^2 (n+1)^2$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_K = \frac{K_n K_{n-1} \cdots K_{n-k+1}}{K_k K_{k-1} \cdots K_1} = \left(\frac{\prod_{n-k+1}^{n} i(i+1)}{\prod_{i=1}^{k} i(i+1)} \right)^2$$

$$= \left(\frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \cdot \frac{(n+1)n(n-1)\cdots(n-k+2)}{(k+1)k(k-1)\cdots 2} \right)^2$$

$$= \left(\frac{1}{k+1} \binom{n}{k} \binom{n+1}{k} \right)^2 = \left(\binom{n}{k}\binom{n+1}{k+1} - \binom{n+1}{k}\binom{n}{k+1} \right)^2.$$

So $m = m(n,k) = \binom{n}{k}\binom{n+1}{k+1} - \binom{n+1}{k}\binom{n}{k+1}$. □

c) We use the Vandermonde convolution:

$$S_n = \sum_{k\geq 0} m(n,k) = \sum_{k\geq 0} \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$$
$$= \frac{1}{n+1}\binom{2n+2}{n}.$$

□

Also solved by Paul S. Bruckman and the proposer.

Sums of Reciprocals of Squares of Fibonacci Numbers

H-698 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48, No. 1, February 2011)

i) Prove that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k^2}\right)^{-1} = F_{n-1}F_n - \frac{(-1)^n}{3} + O\left(\frac{1}{F_n^2}\right).$$

ii) Is it true that for all nonnegative integers m we have the estimate

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_kF_{k+m}}\right)^{-1} = \sum_{k=1}^{n-1} F_kF_{k+m} + \frac{1}{3}F_{m-2(-1)^n} + O\left(\frac{1}{F_n^2}\right),$$

where the constant implied by the above O might depend on m?

Solution by Paul Bruckman

We will prove only part (b) since part (a) is the special case of part (b) with $m = 0$. Let

$$A_{m,n} = \sum_{k=n}^{\infty} \frac{1}{F_kF_{k+m}}.$$

Then

$$A_{m,n} = \sum_{k=n}^{\infty} \frac{5}{(\alpha^k - \beta^k)(\alpha^{k+m} - \beta^{k+m})} = \frac{5}{\alpha^m}\sum_{k=n}^{\infty} \frac{\beta^{2k}}{(1-c^k)(1-c^{k+m})},$$

where $c = \beta/\alpha = -\beta^2$. Then

$$A_{m,n} = \frac{5}{\alpha^m}\sum_{k=n}^{\infty} \beta^{2k}\left\{\frac{1}{1-c^k} - \frac{c^m}{1-c^{k+m}}\right\}\frac{1}{1-c^m}$$
$$= \frac{5}{\alpha^m}\sum_{k=n}^{\infty} \frac{\beta^{2k}}{1-c^m}\left\{1 + c^k - c^m(1+c^{k+m}) + O_m(c^{2k})\right\}.$$

Thus,

$$A_{m,n} = \frac{5}{\alpha^m(1-c^m)} \sum_{k=n}^{\infty} \left\{ \beta^{2k}(1-c^m) + (\beta^2 c)^k (1-c^{2m}) + O_m((\beta^2 c^2)^k) \right\}$$
$$= \frac{5}{\alpha^m} \sum_{k=n}^{\infty} \left\{ \beta^{2k} + (\beta^2 c)^k (1+c^m) + O_m((\beta^2 c^2)^k \right\}.$$

Note that $\beta^2 c = -\beta^4,\ \beta^2 c^2 = \beta^6$. Then

$$A_{m,n} = \frac{5}{\alpha^m} \left\{ \frac{\beta^{2n}}{1-\beta^2} + (1+c^m)\frac{(-\beta^4)^n}{1+\beta^4} + O_m(\beta^{6n}) \right\}$$
$$= \frac{5}{\alpha^m} \left\{ \frac{\beta^{2n}}{-\beta} + (1+c^m)\frac{(-\beta^4)^n}{3\beta^2} + O_m(\beta^{6n}) \right\}$$
$$= \frac{5}{\alpha^m} \left\{ \frac{1}{\alpha^{2n-1}} + (1+c^m)(-1)^n \frac{1}{3\alpha^{4n-2}} + O_m(\beta^{6n}) \right\}$$
$$= \frac{5}{\alpha^{m+2n-1}} \left\{ 1 + (1+c^m)(-1)^n \frac{1}{3\alpha^{2n-1}} + O_m(\beta^{4n}) \right\}.$$

We are interested in computing a comparable expression for $1/A_{m,n}$, which we compute as

$$\frac{1}{A_{m,n}} = \frac{\alpha^{m+2n-1}}{5} \left\{ 1 - (1+c^m)(-1)^n \frac{1}{3\alpha^{2n-1}} + O_m(\beta^{4n}) \right\}.$$

Expanding the above expression out we find

$$\frac{1}{A_{m,n}} = \frac{\alpha^{m+2n-1}}{5} - \frac{(-1)^n}{15} L_m + O_m\left(\frac{1}{F_n^2}\right). \tag{1}$$

We seek to equate the above expression asymptotically with $B_{m,n}$, where

$$B_{m,n} = \sum_{k=1}^{n-1} F_k F_{k+m} + \frac{1}{3} F_{m-2(-1)^n} + O_m\left(\frac{1}{F_n^2}\right).$$

Now

$$\sum_{k=1}^{n-1} F_k F_{k+m} = \frac{1}{5} \sum_{k=0}^{n-1} (\alpha^k - \beta^k)(\alpha^{k+m} - \beta^{k+m})$$
$$= \frac{1}{5} \sum_{k=0}^{n-1} (\alpha^{2k+m} + \beta^{2k+m} - (-1)^k(\alpha^m + \beta^m))$$
$$= \frac{\alpha^m}{5}\left(\frac{\alpha^{2n}-1}{\alpha^2-1}\right) + \frac{\beta^m}{5}\left(\frac{\beta^{2n}-1}{\beta^2-1}\right) - \frac{L_m}{10}\{1-(-1)^n\}.$$

Thus,

$$\sum_{k=1}^{n-1} F_k F_{k+m} = \frac{L_{m+2n-1} - L_{m-1}}{5} - \frac{L_m}{10}\{1-(-1)^n\} = \frac{\alpha^{m+2n-1}}{5} - \frac{1}{5} L_{m-(-1)^n} + O_m\left(\frac{1}{F_n^2}\right).$$

To the above expression we add $F_{m-2(-1)^n}/3$. We find

$$\begin{aligned}\frac{1}{3}F_{m-2(-1)^n} - \frac{1}{5}L_{m-(-1)^n} &= \frac{1}{15}\left\{5F_{m-2(-1)^n} - 3L_{m-(-1)^n}\right\}\\ &= \frac{1}{15}\left\{L_{m+1-2(-1)^n} + L_{m-1-2(-1)^n} - 3L_{m-(-1)^n}\right\}\\ &= -\frac{(-1)^n}{15}L_m,\end{aligned}$$

after some simplification. That is,

$$\sum_{k=1}^{n-1} F_kF_{k+m} + \frac{1}{3}F_{m-2(-1)^n} = \frac{\alpha^{m+2n-1}}{5} - \frac{(-1)^n}{15}L_m + O_m\left(\frac{1}{F_n^2}\right). \qquad (2)$$

Comparing (1) with (2), gives the desired result.

Part a) also solved by the proposer.

A Sequence Involving nth Roots of the Γ Function

H-699 Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China (Vol. 48, No. 1, February 2011) Let $k \geq 0$ be a natural number and let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\begin{aligned}x_n = &\sqrt[n]{\Gamma\left(-2k+\frac{1}{2}\right)\Gamma\left(-2k+\frac{1}{3}\right)\cdots\Gamma\left(-2k+\frac{1}{n}\right)}\\ &-\sqrt[n]{(-1)^{n-1}\Gamma\left(-(2k+1)+\frac{1}{2}\right)\Gamma\left(-(2k+1)+\frac{1}{3}\right)\cdots\Gamma\left(-(2k+1)+\frac{1}{n}\right)},\end{aligned}$$

where Γ denotes the classical Gamma function. Find $\lim_{n\to\infty} x_n/n$.

Solution by the proposers

The limit equals

$$\frac{2k}{e(2k+1)!}.$$

We have, since $\Gamma(1-z) = -z\Gamma(-z)$, that for a positive integer a one has

$$\Gamma\left(-a+\frac{1}{i}\right) = (-1)^a\Gamma\left(\frac{1}{i}\right)\prod_{j=1}^{a}\frac{1}{j-1/i}.$$

It follows that

$$\Gamma\left(-2k+\frac{1}{i}\right) = \Gamma\left(\frac{1}{i}\right)\prod_{j=1}^{2k}\frac{1}{(j-\frac{1}{i})} \quad \text{and} \quad \Gamma\left(-(2k+1)+\frac{1}{i}\right) = -\Gamma\left(\frac{1}{i}\right)\prod_{j=1}^{2k+1}\frac{1}{(j-1/i)},$$

which implies that

$$\ln\Gamma\left(-2k+\frac{1}{i}\right) = \ln\Gamma\left(\frac{1}{i}\right) - \sum_{j=1}^{2k}\ln\left(j-\frac{1}{i}\right).$$

Also,

$$\ln\Gamma\left(\frac{1}{i}\right) = -\frac{\gamma}{i} + \ln i + \sum_{m=1}^{\infty}\left(\frac{1}{im} - \ln\left(1+\frac{1}{im}\right)\right).$$

We have,

$$\begin{aligned}\frac{x_n}{n} &= e^{-\ln n}\left(e^{\frac{1}{n}\sum_{i=2}^{n}\ln\Gamma\left(-2k+\frac{1}{i}\right)} - e^{\frac{1}{n}\sum_{i=2}^{n}\ln\left(-\Gamma\left(-(2k+1)+\frac{1}{i}\right)\right)}\right)\\ &= e^{-\ln n}\left(e^{\frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\sum_{j=1}^{2k}\ln\left(j-\frac{1}{i}\right)\right)} - e^{\frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\sum_{j=1}^{2k+1}\ln\left(j-\frac{1}{i}\right)\right)}\right) \qquad (3)\\ &= \left(1-e^{-\frac{1}{n}\sum_{i=2}^{n}\ln(2k+1-1/i)}\right)e^{\frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\sum_{j=1}^{2k}\ln\left(j-\frac{1}{i}\right)\right)-\ln n}.\end{aligned}$$

We have,

$$\begin{aligned}\frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\sum_{j=1}^{2k}\ln\left(j-\frac{1}{i}\right)\right) &= \frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\ln(2k)!-\sum_{j=1}^{2k}\ln\left(1-\frac{1}{ij}\right)\right)\\ &= -\ln(2k)!\frac{n-1}{n}+\frac{1}{n}\sum_{i=2}^{n}\ln\Gamma\left(\frac{1}{i}\right)\\ &\quad -\frac{1}{n}\sum_{i=2}^{n}\sum_{j=1}^{2k}\ln\left(1-\frac{1}{ij}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=2}^{n}\ln\Gamma\left(\frac{1}{i}\right) &= \sum_{i=2}^{n}\left(-\frac{\gamma}{i}+\ln i+\sum_{m=1}^{\infty}\left(\frac{1}{im}-\ln\left(1+\frac{1}{im}\right)\right)\right)\\ &= -\gamma(H_n-1)+\ln(n!)+\sum_{i=2}^{n}\left(\sum_{m=1}^{\infty}\left(\frac{1}{im}-\ln\left(1+\frac{1}{im}\right)\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{n}\sum_{i=2}^{n}\left(\ln\Gamma\left(\frac{1}{i}\right)-\sum_{j=1}^{2k}\ln\left(j-\frac{1}{i}\right)\right)-\ln n &= \frac{-\gamma(H_n-1)}{n}+\frac{\ln n!-n\ln n}{n}\\ -\ln(2k)!\frac{n-1}{n}+\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{m=1}^{\infty}\left(\frac{1}{im}-\ln\left(1+\frac{1}{im}\right)\right)\right) &- \frac{1}{n}\sum_{i=2}^{n}\sum_{j=1}^{2k}\ln\left(1-\frac{1}{ij}\right).\end{aligned}$$

Now we calculate the following three limits by using the **Cesaro Stolz** Lemma.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\ln(2k+1-1/i) = \lim_{n\to\infty}\ln(2k+1-1/(n+1)) = \ln(2k+1).$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{m=1}^{\infty}\left(\frac{1}{im}-\ln\left(1+\frac{1}{im}\right)\right)\right)=\lim_{n\to\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(n+1)m}-\ln\left(1+\frac{1}{(n+1)m}\right)\right)=0.$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{j=1}^{2k}\ln\left(1-\frac{1}{ij}\right)\right)=\lim_{n\to\infty}\sum_{j=1}^{2k}\ln\left(1-\frac{1}{j(n+1)}\right)=\sum_{j=1}^{2k}\lim_{n\to\infty}\ln\left(1-\frac{1}{j(n+1)}\right)=0.$$

$$\lim_{n\to\infty}\frac{-\gamma(H_n-1)}{n}=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{\ln n!-n\ln n}{n}=-1.$$

It follows based on (3) and the preceding limits that

$$\lim_{n\to\infty}\frac{x_n}{n}=(1-e^{-\ln(2k+1)})e^{-1-\ln(2k)!}=\frac{2k}{e(2k+1)!},$$

and the problem is solved.

Also solved by Paul Bruckman.

Errata: The first problem labeled **H-717** in Volume **49** no. 2, May 2012 should read **H-716**.