ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-809 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\left(1-\frac{\alpha}{L_2}\right)\left(1-\frac{\beta}{L_{2^2}}\right)\left(1-\frac{\alpha}{L_{2^3}}\right)\left(1-\frac{\beta}{L_{2^4}}\right)\cdots=\frac{7\sqrt{5}-5}{22}.$$

H-810 Proposed by Ángel Plaza, Gran Canaria, Spain.

Prove that

$$\sum_{n=3}^{\infty}\frac{1}{L_n^4-25}=\frac{5}{63}-\frac{1}{6\sqrt{5}}.$$

H-811 Proposed by Ángel Plaza, Gran Canaria, Spain.

For any positive integer k let $\{F_{k,n}\}_{n\geq 0}$ be defined by $F_{k,n+2}=kF_{k,n+1}+F_{k,n}$ for $n\geq 0$ with $F_{k,0}=0,\ F_{k,1}=1$. Prove that

$$\sum_{n=0}^{\infty}\frac{1}{1+F_{k,2n+1}}=\frac{\sqrt{k^2+4}}{2k}.$$

H-812 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{i+j=F_{n+1}}\binom{F_{n+1}}{i}\binom{F_n}{i}\binom{L_n}{j}=\sum_{i+j=F_{n+1}}\binom{F_n}{i}\binom{L_n}{i}\binom{2j}{F_{n+1}}.$$

SOLUTIONS

A Series Related to the Sum of the Reciprocals of the Fibonacci Numbers

H-775 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 3, August 2015)

Let c be any real number $c \neq 2,\ -L_{2^n}$ for $n \geq 0$. Let

$$\gamma_c = \sqrt{5}\prod_{n=1}^{\infty}\left(1+\frac{c}{L_{2^n}}\right)^{-1}.$$

Prove that

$$\sum_{k=1}^{\infty}\frac{1}{(L_2+c)(L_4+c)\cdots(L_{2^k}+c)} = \frac{\gamma_c+c-3}{c^2-c-2}.$$

Solution by the proposer.

Let $P_n = (L_2+c)(L_{2^2}+c)\cdots(L_{2^n}+c)$. For $n \geq 1$, we show that

$$(c^2-c-2)\sum_{k=1}^{n}\frac{1}{P_k} = \frac{L_{2^{n+1}}-c}{P_n}+c-3. \tag{1}$$

The proof of (1) is by mathematical induction on n. For $n=1$, we have

$$LHS = \frac{c^2-c-2}{P_1} = \frac{c^2-c-2}{3+c} = \frac{7-c}{3+c}+c-3 = \frac{L_4-c}{P_1}+c-3 = RHS.$$

We assume that (1) holds for n. For $n+1$, we have

$$\begin{aligned}(c^2-c-2)\sum_{k=1}^{n+1}\frac{1}{P_k} &= (c^2-c-2)\left(\frac{1}{P_{n+1}}+\sum_{k=1}^{n}\frac{1}{P_k}\right)\\ &= \frac{c^2-c-2}{P_{n+1}}+\frac{L_{2^{n+1}}-c}{P_n}+c-3\\ &= \frac{c^2-c-2+(L_{2^{n+1}}-c)(L_{2^{n+1}}+c)}{P_{n+1}}+c-3\\ &= \frac{L_{2^{n+1}}^2-2-c}{P_{n+1}}+c-3\\ &= \frac{L_{2^{n+2}}-c}{P_{n+1}}+c-3,\end{aligned}$$

since $L_m^2-2(-1)^m = L_{2m}$. Thus, (1) holds for $n+1$. Therefore (1) is proved. We have

$$P_n = \prod_{k=1}^{n}(L_{2^k}+c) = \prod_{k=1}^{n}L_{2^k}\prod_{k=1}^{n}\left(1+\frac{c}{L_{2^k}}\right).$$

Hence, using $F_mL_m = F_{2m}$, we have

$$F_2L_2L_4L_8\cdots L_{2^n} = F_4L_4L_8\cdots L_{2^n} = \cdots = F_{2^n}L_{2^n} = F_{2^{n+1}}.$$

Thus,

$$P_n = F_{2^{n+1}}\prod_{k=1}^{n}\left(1+\frac{c}{L_{2^k}}\right). \tag{2}$$

Therefore, by (1) and (2), we have

$$\sum_{k=1}^{n}\frac{1}{P_k}=\frac{1}{c^2-c-2}\left[\frac{L_{2^{n+1}}-c}{F_{2^{n+1}}}\prod_{k=1}^{n}\left(1+\frac{c}{L_{2^k}}\right)^{-1}+c-3\right]\to\frac{\gamma_c+c-3}{c^2-c-2}$$

as $n\to\infty$ since $L_m/F_m\to\sqrt{5}$ as $m\to\infty$.

Note: If $c=0$, we then have

$$\sum_{k=1}^{\infty}\frac{1}{L_2L_4\cdots L_{2^k}}=\frac{\gamma_0-3}{-2}\qquad\text{i.e.,}\qquad\sum_{k=1}^{\infty}\frac{1}{F_{2^{k+1}}}=\frac{3-\sqrt{5}}{2}.$$

From the above identity, we obtain the well-known identity

$$\sum_{k=0}^{\infty}\frac{1}{F_{2^k}}=\frac{7-\sqrt{5}}{2}.$$

Also solved by Dmitry Fleischman.

A Series of Inverse Tangents of Reciprocals of Lucas Numbers

H-776 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 3, August 2015)

Determine

$$\text{(i)}\quad\sum_{n=0}^{\infty}(-1)^n\tan^{-1}\frac{1}{L_{3^n}}\qquad\text{and}\qquad\text{(ii)}\quad\sum_{n=1}^{\infty}\tan^{-1}\frac{1}{F_{2n}}\tan^{-1}\frac{1}{L_{2n}}.$$

Solution by the proposer.

(i) For $n\ge 0$, we have

$$\begin{aligned}\tan^{-1}\frac{1}{\alpha^{3^n}}+\tan^{-1}\frac{1}{\alpha^{3^{n+1}}}&=\tan^{-1}\left(\frac{\frac{1}{\alpha^{3^n}}+\frac{1}{\alpha^{3^{n+1}}}}{1-\frac{1}{\alpha^{3^n}}\cdot\frac{1}{\alpha^{3^{n+1}}}}\right)\\&=\tan^{-1}\left(\frac{\alpha^{3^{n+1}}+\alpha^{3^n}}{\alpha^{4\cdot 3^n}-1}\right)\\&=\tan^{-1}\left(\frac{\alpha^{2\cdot 3^n}\left(\alpha^{3^n}+\alpha^{-3^n}\right)}{\alpha^{2\cdot 3^n}\left(\alpha^{3^n}+\alpha^{-3^n}\right)\left(\alpha^{3^n}-\alpha^{-3^n}\right)}\right)\\&=\tan^{-1}\left(\frac{1}{\alpha^{3^n}+\beta^{3^n}}\right)=\tan^{-1}\frac{1}{L_{3^n}}.\end{aligned}$$

Using the above identity, we have

$$\begin{aligned}\sum_{n=0}^{m}(-1)^n\tan^{-1}\frac{1}{L_{3^n}}&=\sum_{n=0}^{m}\left[(-1)^n\tan^{-1}\frac{1}{\alpha^{3^n}}-(-1)^{n+1}\tan^{-1}\frac{1}{\alpha^{3^{n+1}}}\right]\\&=\tan^{-1}\frac{1}{\alpha}-(-1)^{m+1}\tan^{-1}\frac{1}{\alpha^{3^{m+1}}}\to\tan^{-1}\frac{1}{\alpha}\end{aligned}$$

as $m\to\infty$.

(ii) We have

$$\tan^{-1}\frac{1}{\alpha^{2n-1}}+\tan^{-1}\frac{1}{\alpha^{2n+1}}=\tan^{-1}\left(\frac{\frac{1}{\alpha^{2n-1}}+\frac{1}{\alpha^{2n+1}}}{1-\frac{1}{\alpha^{2n-1}}\cdot\frac{1}{\alpha^{2n+1}}}\right)$$
$$=\tan^{-1}\left(\frac{\alpha+\alpha^{-1}}{\alpha^{2n}-\alpha^{-2n}}\right)$$
$$=\tan^{-1}\frac{\sqrt{5}}{\alpha^{2n}-\beta^{2n}}=\tan^{-1}\frac{1}{F_{2n}},$$

and similarly

$$\tan^{-1}\frac{1}{\alpha^{2n-1}}-\tan^{-1}\frac{1}{\alpha^{2n+1}}=\tan^{-1}\left(\frac{\frac{1}{\alpha^{2n-1}}-\frac{1}{\alpha^{2n+1}}}{1+\frac{1}{\alpha^{2n-1}}\cdot\frac{1}{\alpha^{2n+1}}}\right)$$
$$=\tan^{-1}\left(\frac{\alpha-\alpha^{-1}}{\alpha^{2n}+\alpha^{-2n}}\right)$$
$$=\tan^{-1}\frac{1}{\alpha^{2n}+\beta^{2n}}=\tan^{-1}\frac{1}{L_{2n}}.$$

Using the above identities, we have

$$\sum_{n=1}^{m}\tan^{-1}\frac{1}{F_{2n}}\tan^{-1}\frac{1}{L_{2n}}=\sum_{n=1}^{m}\left[\left(\tan^{-1}\frac{1}{\alpha^{2n-1}}\right)^2-\left(\tan^{-1}\frac{1}{\alpha^{2n+1}}\right)^2\right]$$
$$=\left(\tan^{-1}\frac{1}{\alpha}\right)^2-\left(\tan^{-1}\frac{1}{\alpha^{2m+1}}\right)^2\to\left(\tan^{-1}\frac{1}{\alpha}\right)^2$$

as $m\to\infty$.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, and David Terr.

Sums of Products of Binomial Coefficients

H-777 Proposed by Kiyoshi Kawazu, Izumi Kubo, and Toshio Nakata, Japan. (Vol. 53, No. 4, November 2015)

For any nonnegative integers $n,\ m,\ l$ prove that

$$\sum_{k=0}^{n}\binom{n}{k}^2\sum_{i\ge 0}\binom{2k}{i}\binom{2n-2k}{m-i}(-1)^{m-i}=\begin{cases}\binom{2l}{l}\binom{2n-2l}{n-l} & \text{if}\quad m=2l;\\ 0 & \text{if}\quad m=2l+1.\end{cases}$$

Solution by the proposers.

For any nonnegative integer n and formal power series $f(x)=\sum_{k=0}^{\infty}a_kx^k$, let $[x^n]f(x)$ denote a_n. Let $a(n,m)$ be the left-hand side of the identity to be proved. Then we have

$$a(n,m)=[z^m]\sum_{k=0}^{n}\binom{n}{k}^2(1+z)^{2k}(1-z)^{2n-2k},$$

since we obtain

$$(1+z)^{2k}(1-z)^{2n-2k}=\sum_{m,i}\binom{2k}{i}\binom{2n-2k}{m-i}(-1)^{m-i}z^m\qquad\text{for}\qquad n\ge k\ge 0$$

using the binomial theorem. We have

$$a(n,m) = [z^m t^n]\{(1+z)^2t^2 + 2(1+z^2)t + (1-z)^2\}^n,$$

since

$$[t^n]\{(1+z)^2t + (1-z)^2\}^n(1+t)^n = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}(1+z)^{2k}(1-z)^{2n-2k}.$$

Using the trinomial theorem,

$$a(n,m) = [z^m t^n] \sum_{k_0+k_1+k_2=n} \binom{n}{k_0,k_1,k_2}(1-z)^{2k_0}(2+2z^2)^{k_1}(1+z)^{2k_2}t^{k_1+2k_2},$$

where the sum runs over all nonnegative integers k_0, k_1, k_2 satisfying $k_0 + k_1 + k_2 = n$. Emphasizing the coefficient of t^n, we have

$$a(n,m) = [z^m]\sum_k \binom{n}{k, n-2k, k} 2^{n-2k}(1+z^2)^{n-k}(1-z^2)^{2k}.$$

Since all terms among the sum are polynomials in z^2, we have that $a(n,m) = 0$ if $m = 2l+1$ for some integer $l \geq 0$. So, suppose that $m = 2l$. Letting $y = z^2$, we have

$$\begin{aligned}
a(n,2l) &= [y^l]\sum_k \binom{n}{k, n-2k, k} 2^{n-2k}(1+y)^{n-2k}(1-y)^{2k} \\
&= [y^l]\sum_k \binom{n}{2k}\binom{2k}{k} 2^{n-2k}(1+y)^{n-2k}(1-y)^{2k} \\
&= [x^n y^l]\sum_k \sum_n \binom{n}{2k}\{2x(1+y)\}^n \binom{2k}{k} 2^{-2k}(1+y)^{-2k}(1-y)^{2k} \\
&= [x^n y^l]\sum_k \frac{\{2x(1+y)\}^{2k}}{(1-2x(1+y))^{2k+1}} \binom{2k}{k} 2^{-2k}(1+y)^{-2k}(1-y)^{2k} \\
&= [x^n y^l]\frac{1}{1-2x(1+y)} \sum_k \binom{2k}{k}\left(\frac{x(1-y)}{1-2x(1+y)}\right)^{2k} \\
&= [x^n y^l]\frac{1}{1-2x(1+y)} \left(1 - 4\left(\frac{x(1-y)}{1-2x(1+y)}\right)^2\right)^{-1/2} \\
&= [x^n y^l]\{(1-4x)(1-4xy)\}^{-1/2}.
\end{aligned}$$

The second equality and the fourth equality hold by

$$\binom{n}{k, n-2k, k} = \binom{n}{2k}\binom{2k}{k} \qquad \text{and} \qquad \sum_n \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}},$$

respectively. Since

$$(1-4x)^{-1/2} = \sum_{k=0}^{\infty}\binom{-1/2}{k}(-4x)^k = \sum_{k=0}^{\infty}\binom{2k}{k}x^k,$$

we have

$$(1-4x)^{-1/2}(1-4xy)^{-1/2} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{2k}{k}\binom{2l}{l}x^{k+l}y^l.$$

Hence,

$$a(n, 2l) = [x^n y^l]\{(1-4x)(1-4xy)\}^{-1/2} = \binom{2l}{l}\binom{2n-2l}{n-l}.$$

Also solved by Dmitry Fleischman.

<u>**A Series with Reciprocals of Products of Fibonacci Numbers**</u>

<u>**H-778**</u> **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**
(Vol. 53, No. 4, November 2015)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(-\sqrt{5})^n F_2 F_4 F_8 \cdots F_{2^n}} = \frac{\sqrt{5}-3}{2}.$$

Solution by the proposer.

Let $F'_n = \sqrt{5}F_n$. For $n \geq 2$, we have

$$\begin{aligned}
\frac{(-1)^n}{\alpha^{2^n} F'_2 F'_4 \cdots F'_{2^{n-1}}} - \frac{(-1)^{n+1}}{\alpha^{2^{n+1}} F'_2 F'_4 \cdots F'_{2^n}} &= \frac{(-1)^n(\alpha^{2^n} F'_{2^n} + 1)}{\alpha^{2^{n+1}} F'_2 F'_4 \cdots F'_{2^n}} \\
&= \frac{(-1)^n(\alpha^{2^n}(\alpha^{2^n} - \beta^{2^n}) + 1}{\alpha^{2^{n+1}} F'_2 F'_4 \cdots F'_{2^n}} \\
&= \frac{(-1)^n \alpha^{2^{n+1}}}{\alpha^{2^{n+1}} F'_2 F'_4 \cdots F'_{2^n}} \\
&= \frac{(-1)^n}{F'_2 F'_4 \cdots F'_{2^n}}.
\end{aligned}$$

Using the above identity, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{(-\sqrt{5})^n F_2 F_4 F_8 \cdots F_{2^n}} &= \sum_{n=1}^{m} \frac{(-1)^n}{F'_2 F'_4 F'_8 \cdots F'_{2^n}} \\
&= \frac{-1}{F'_2} + \sum_{n=2}^{m} \left(\frac{(-1)^n}{\alpha^{2^n} F'_2 F'_4 \cdots F'_{2^{n-1}}} - \frac{(-1)^{n+1}}{\alpha^{2^{n+1}} F'_2 F'_4 \cdots F'_{2^n}} \right) \\
&= \frac{-1}{F'_2} + \frac{1}{\alpha^4 F'_2} - \frac{(-1)^{m+1}}{\alpha^{2^{m+1}} F'_2 F'_4 \cdots F'_{2^m}} \to \frac{\sqrt{5}-3}{2}
\end{aligned}$$

as $m \to \infty$.

Also solved by Kenneth B. Davenport and Dmitry Fleischman.

Late acknowledgement. Both Kenneth B. Davenport and J. M. Jarvie (solution submitted via Kenneth B. Davenport) solved H-767.