ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by email at the address florian.luca@wits.ac.za with an email copy to ROBERT FRONTCZAK at Robert.Frontczak@lbbw.de as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-901 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

$$(i)\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2F_{n+1}}+L_{2F_n}}=\frac{1}{10}\qquad\text{and}\quad (ii)\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2L_{n+1}}+L_{2L_n}}=\frac{1}{15}.$$

H-902 Proposed by I. V. Fedak, Ivano-Frankivsk, Ukraine

For all positive integers n, prove that

$$2L_nL_{n+2}(\sqrt[L_n]{2}-\sqrt[L_{n+2}]{2})(\sqrt[L_n]{6}-\sqrt[L_{n+2}]{6})<L_{n+1}(\sqrt[L_n]{12}-\sqrt[L_{n+2}]{12}).$$

H-903 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that the Diophantine equations

$$3^nL_n+4^n=5^m\qquad\text{and}\qquad 3^n+4^nL_n=5^m$$

have no solutions in positive integers n and m.

H-904 Proposed by Robert Frontczak, Stuttgart, Germany

Show the following identities valid for each even integer $m\geq 2$:

$$\sum_{n=1}^{\infty}\frac{L_{2mn}-L_{2n}}{(L_{2n}+L_{2m})(L_{2mn}+L_{2m})}=\frac{m-1}{\sqrt{5}F_{2m}}$$

and

$$\sum_{n=1}^{\infty}\frac{L_{2mn}+L_{2n}+2L_{2m}}{(L_{2n}+L_{2m})(L_{2mn}+L_{2m})}=\frac{m+1}{\sqrt{5}F_{2m}}-\frac{1}{L_{2m}+2}.$$

H-905 **Proposed by Kay Wang, Henderson, NV**

Prove the following identities:

(i) $$\frac{\pi^2}{16} = \sum_{i=0}^{\infty} \left(\arctan \frac{1}{F_{4i-1}} \arctan \frac{3}{F_{4i+1}} - \arctan \frac{3}{F_{4i+1}} \arctan \frac{1}{F_{4i+3}} \right), \text{ where } F_{-1} = 1.$$

(ii) $$\frac{\pi}{4} = \sum_{i=0}^{\infty} (-1)^i \arctan \frac{3}{F_{4i+1}}.$$

SOLUTIONS

A logarithmic inequality

H-866 **Proposed by Ángel Plaza, Gran Canaria, Spain**
(Vol. 58, No. 4, November 2020)

Let a_n denote the nth number in the sequence given by $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$ with initial values $a_0 = a - 1$ and $a_1 = 1$ with some $a \geq 1$. Prove that

$$\sum_{k=1}^{n} \frac{2(a_{k+1} - a_k)}{a_{k+1} + a_k} < \ln a_{n+1} < \sum_{k=1}^{n} \frac{a_{k+1}^2 - a_k^2}{2a_{k+1}a_k}.$$

Solution by Michel Bataille, Rouen, France

We will make use of the logarithmic mean $L(a, b)$ of two distinct positve real numbers a, b defined by $L(a, b) = \frac{a-b}{\ln a - \ln b}$. It is linked to the usual means $A(a, b) = \frac{a+b}{2}$, $H(a, b) = \frac{2ab}{a+b}$ by

$$H(a, b) < \sqrt{ab} < L(a, b) < A(a, b). \tag{1}$$

(A recent reference is [**1**].)

Turning to the problem, it is readily checked that strict inequalities do not hold if $n = 1$ and $a = 1$ (if $a = 1$, then $\frac{2(a_2 - a_1)}{a_2 + a_1} = \ln a_2 = \frac{a_2^2 - a_1^2}{2a_2a_1} = 0$). In what follows, we suppose that $n \geq 2$ or $a > 1$ if $n = 1$.

An induction shows that $a_{n+1} > a_n$ for $n \geq 1$. Using (1), it follows that for $k \geq 1$, we have

$$\frac{2}{a_k + a_{k+1}} = \frac{1}{A(a_k, a_{k+1})} < \frac{\ln a_{k+1} - \ln a_k}{a_{k+1} - a_k} = \frac{1}{L(a_k, a_{k+1})} < \frac{1}{H(a_k, a_{k+1})} = \frac{a_k + a_{k+1}}{2a_{k+1}a_k}.$$

Therefore, multiplying by the positive number $a_{k+1} - a_k$,

$$\frac{2(a_{k+1} - a_k)}{a_{k+1} + a_k} < \ln a_{k+1} - \ln a_k < \frac{a_{k+1}^2 - a_k^2}{2a_{k+1}a_k}$$

and, summing, we obtain

$$\sum_{k=1}^{n} \frac{2(a_{k+1} - a_k)}{a_{k+1} + a_k} < \sum_{k=1}^{n} (\ln a_{k+1} - \ln a_k) < \sum_{k=1}^{n} \frac{a_{k+1}^2 - a_k^2}{2a_{k+1}a_k}.$$

The result follows because $\sum_{k=1}^{n} (\ln a_{k+1} - \ln a_k) = \ln a_{n+1}$.

Reference

[1] G. Jameson and P. R. Mercer, *The logarithmic mean revisited*, The American Mathematical Monthly, **126.7** (2019), 641–645.

Also solved by Dmitry Fleischman, Dongsheng Li Shu, Albert Stadler, Andrés Ventas, and the proposer.

An identity with sums of ratios of Fibonacci and Lucas numbers

H-867 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 58, No. 4, November 2020)

Let a, b, c, d be even positive integers with $a+b=c+d$. Prove that

$$\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} + \sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} = \sum_{k=1}^{c} \frac{L_d}{F_k L_{k+d}} + \sum_{k=1}^{d} \frac{L_c}{L_k F_{k+c}}.$$

Solution by Andrés Ventas, Santiago de Compostela, Spain

This problem can be solved using the ideas of Rabinowitz [2] and the solution to Advanced Problem H-749 by Á. Plaza [1].

Rabinowitz showed that we must find formulas with indices $n \pm 2$. With that hint and with a even, we have

$$\frac{L_a}{F_n L_{n+a}} - \frac{L_{a-2}}{F_{n+2} L_{n+a}} = \frac{L_a F_{n+2} - L_{a-2} F_n}{F_n F_{n+2} L_{n+a}} = \frac{L_{n+a}}{F_n F_{n+2} L_{n+a}} = \frac{1}{F_n F_{n+2}}.$$

$$\frac{L_a}{L_n F_{n+a}} - \frac{L_{a-2}}{L_n F_{n+a-2}} = \frac{L_a F_{n+a-2} - L_{a-2} F_{n+a}}{L_n F_{n+a} F_{n+a-2}} = \frac{-L_n}{L_n F_{n+a} F_{n+a-2}} = \frac{-1}{F_{n+a} F_{n+a-2}}.$$

Now, we sum from 1 to N and apply Theorem 4 from Rabinowitz [2] so we get products with consecutive indices

$$\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} - \sum_{k=1}^{a} \frac{L_{b-2}}{F_{k+2} L_{k+b}} = \sum_{k=1}^{a} \frac{1}{F_k F_{k+2}} = \frac{1}{F_1 F_2} - \frac{1}{F_{a+1} F_{a+2}}.$$

$$\sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} - \sum_{k=1}^{b} \frac{L_{a-2}}{L_k F_{k+a-2}} = \sum_{k=1}^{b} \frac{-1}{F_{k+a-2} F_{k+a}} = \frac{-1}{F_{a-1} F_a} - \frac{-1}{F_{b+a-1} F_{b+a}}.$$

And now, we use telescoping with L_b and L_{b-2}, $b/2$ times, and with L_a and L_{a-2}, $a/2$ times, erasing all intermediate sums. In the first sum, F_k increases because we have F_{n+2} in the denominator of the telescoping. In the second sum, F_k decreases because we have F_{n-2} in the denominator of the telescoping.

$$\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} - \sum_{k=1}^{a} \frac{L_0}{F_{k+b} L_{k+b}} = \sum_{k=0}^{b/2-1} \left(\frac{1}{F_{2k+1} F_{2k+2}} - \frac{1}{F_{2k+a+1} F_{2k+a+2}} \right).$$

$$\sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} - \sum_{k=1}^{b} \frac{L_0}{L_k F_k} = \sum_{k=0}^{a/2-1} \left(\frac{-1}{F_{a-1-2k} F_{a-2k}} - \frac{-1}{F_{b+a-1-2k} F_{b+a-2k}} \right)$$

$$\text{(Reversing the order of the elements)} = \sum_{k=0}^{a/2-1} \left(\frac{-1}{F_{2k+1} F_{2k+2}} - \frac{-1}{F_{2k+1+b} F_{2k+2+b}} \right).$$

As a final step, we sum both formulas to get the left sums of the problem, then the elements of the right side cancel.

$$\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} + \sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} - \sum_{k=1}^{a} \frac{L_0}{F_{k+b} L_{k+b}} - \sum_{k=1}^{b} \frac{L_0}{L_k F_k} = 0.$$

$$\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} + \sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} = \sum_{k=1}^{a+b} \frac{2}{F_k L_k} = \sum_{k=1}^{a+b} \frac{2}{F_{2k}}$$

that is the same result for c and d where $a+b=c+d$.

References

[1] A. Plaza, *Advanced Problems and Solutions*, H-749, The Fibonacci Quarterly, **53.4** (2015), 375.

[2] S. Rabinowitz, *Algorithmic summation of reciprocals of products of Fibonacci numbers*, The Fibonacci Quarterly, **37.2** (1999), 122–127.

Also solved by Dmitry Fleischman, Albert Stadler, and the proposer.

Odd perfect numbers and values of the Riemann ζ-function

H-868 Proposed by Juan Lopez Gonzalez, Madrid, Spain (Vol. 59, No. 1, February 2021)

Prove that if N is an odd perfect number, then it satisfies

$$\frac{\sigma_0(N)\ln 2}{2} = N\ln 2 - \sum_{\substack{d|N \\ d>1}} \sum_{k=1}^{(d-1)/2} \sum_{\ell\geq 1} \frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell}} \zeta(2\ell),$$

where $\sigma(N)$ is the sum of divisors of N and for $k>1$, $\zeta(k)$ is the Riemann zeta function.

Solution by Albert Stadler, Herrliberg, Switzerland

We use the product representation of the sine function

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$$

to deduce

$$\begin{aligned}
\sum_{\ell\geq 1} \frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell}} \zeta(2\ell) &= \sum_{n=1}^{\infty}\sum_{\ell\geq 1} \frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell} n^{2\ell}} \\
&= -\sum_{n=1}^{\infty} \ln\left(1 - \frac{4k^2}{d^2n^2}\right) + \sum_{n=1}^{\infty} \ln\left(1 - \frac{k^2}{d^2n^2}\right) \\
&= -\ln\left(\frac{\sin\left(\frac{2\pi k}{d}\right)}{\frac{2\pi k}{d}}\right) + \ln\left(\frac{\sin\left(\frac{\pi k}{d}\right)}{\frac{\pi k}{d}}\right) \\
&= -\ln\left(\frac{\sin\left(\frac{2\pi k}{d}\right)}{2\sin\left(\frac{\pi k}{d}\right)}\right) \\
&= -\ln\left(\cos\left(\frac{\pi k}{d}\right)\right).
\end{aligned}$$

Next, we use (see for example [1]), that

$$\prod_{k=1}^{n} \cos\left(\frac{\pi k}{2n+1}\right) = \frac{1}{2^n},$$

to deduce

$$-\sum_{k=1}^{\frac{d-1}{2}} \ln\left(\cos\left(\frac{\pi k}{d}\right)\right) = -\ln\left(\prod_{k=1}^{\frac{d-1}{2}} \cos\left(\frac{\pi k}{d}\right)\right) = \frac{(d-1)}{2}\ln 2.$$

Finally,

$$\begin{aligned}\frac{\sigma_0(N)\ln 2}{2} - N\ln 2 \quad &+ \quad \sum_{\substack{d\mid N\\ d>1}} \sum_{k=1}^{\frac{d-1}{2}} \sum_{\ell\geq 1} \frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell}}\zeta(2\ell) = \frac{\sigma_0(N)\ln 2}{2} - N\ln 2 + \sum_{\substack{d\mid N\\ d>1}} \frac{d-1}{2}\ln 2 \\ &= \quad \frac{\ln 2}{2}(\sigma_0(N) - 2N + (\sigma_1(N)-1) - (\sigma_0(N)-1)) = 0,\end{aligned}$$

by the definition of a perfect number.

Reference

[1] https://en.wikipedia.org/wiki/List of trigonometric identities.

Also solved by the proposer.

An identity with fifth powers of the Fibonacci numbers

H-869 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 59, No. 1, February 2021)

For positive integer n, prove that

$$\sum_{k=1}^{n}(-1)^k L_k F_k^5 = \frac{(-1)^n(F_n^5F_{n+3} - F_n^2)}{2}.$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

We have, by using the Binet formulas for F_k and L_k, the following expansions

$$\begin{aligned}(-1)^k L_k F_k^5 &= \frac{1}{25\sqrt{5}}\left((-1)^k\alpha^{6k} - (-1)^k\beta^{6k} - 4\alpha^{4k} + 4\beta^{4k} + 5(-1)^k\alpha^{2k} - 5(-1)^k\beta^{2k}\right),\\ (-1)^k F_k^5 F_{k+3} &= \frac{1}{125}\big((-1)^k\alpha^{6k+3} + (-1)^k\beta^{6k+3} - 5\alpha^{4k+3} - 5\beta^{4k+3} + 10(-1)^k\alpha^{2k+3}\\ &\quad + 10(-1)^k\beta^{2k+3} - 10\alpha^3 - 10\beta^3 + 5\beta^3(-1)^k\alpha^{2k}\\ &\quad + 5\alpha^3(-1)^k\beta^{2k} - \beta^3\alpha^{4k} - \alpha^3\beta^{4k}\big),\\ (-1)^k F_k^2 &= \frac{1}{5}\left((-1)^k\alpha^{2k} + (-1)^k\beta^{2k} - 2\right),\end{aligned}$$

which imply that

$$\sum_{k=0}^{\infty}(-1)^k L_k F_k^5 x^k = \frac{p_1(x)}{(x^2-7x+1)(x^2+3x+1)(x^2+18x+1)},$$

$$\sum_{k=0}^{\infty}(-1)^k F_k^5 F_{k+3} x^k = \frac{p_2(x)}{(1-x)(x^2-7x+1)(x^2+3x+1)(x^2+18x+1)},$$

$$\sum_{k=0}^{\infty}(-1)^k F_k^2 x^k = \frac{p_3(x)}{(1-x)(x^2+3x+1)},$$

where $p_1(x)$, $p_2(x)$, and $p_3(x)$ are the unique polynomials of degrees ≤ 5, ≤ 6, and ≤ 2, such that they generate the first 6, 7, and 3 terms from their corresponding generating functions, because then they generate all of its terms.

Therefore, we have the generating function identities

$$f_1(x) := \sum_{k=0}^{\infty}(-1)^k L_k F_k^5 x^k = -\frac{x\left(x^4+11x^3-4x^2+11x+1\right)}{(x^2-7x+1)(x^2+3x+1)(x^2+18x+1)},$$

$$f_2(x) := \sum_{k=0}^{\infty}(-1)^k F_k^5 F_{k+3} x^k = -\frac{x\left(x^5+14x^4-91x^3-121x^2+34x+3\right)}{(1-x)(x^2-7x+1)(x^2+3x+1)(x^2+18x+1)},$$

$$f_3(x) := \sum_{k=0}^{\infty}(-1)^k F_k^2 x^k = -\frac{x(x+1)}{(1-x)(x^2+3x+1)}.$$

The claimed identity follows from $(-1)^0 L_0 F_0^5 = 0$ and the equation

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k L_k F_k^5\right)x^n = \frac{1}{1-x}f_1(x) = \frac{1}{2}f_2(x) - \frac{1}{2}f_3(x)$$
$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n\left(F_n^5 F_{n+3} - F_n^2\right)}{2}\right)x^n,$$

which holds because we have that

$$\begin{aligned} &x^4+11x^3-4x^2+11x+1 \\ &= \frac{x^5+14x^4-91x^3-121x^2+34x+3-(x^5+12x^4-113x^3-113x^2+12x+1)}{2} \\ &= \frac{x^5+14x^4-91x^3-121x^2+34x+3-(x+1)(x^2-7x+1)(x^2+18x+1)}{2}. \end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai, Ángel Plaza, Albert Stadler, Liyang Zhang, and the proposer.

Formulas with Fibonacci numbers whose indices are Fibonacci numbers

H-870 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 59, No. 1, February 2021)

For any positive integer n, find closed form expressions for the sums

$$(i) \quad \sum_{k=1}^{n}(L_{F_k}L_{F_{k+1}})(F_{F_k}F_{F_{k+1}})^3 \qquad \text{and} \qquad (ii) \quad \sum_{k=1}^{n}(F_{F_k}F_{F_{k+1}})(L_{F_k}L_{F_{k+1}})^3.$$

Solution by Brian Bradie, Newport News, VA

By Catalan's identity

$$F_{2F_k}F_{2F_{k+1}} = F_{F_{k+2}}^2 - F_{F_{k-1}}^2.$$

Combining the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ with $F_k \pmod 2$ has period 3 so F_{k-1} and F_{k+2} have the same parity, it follows that

$$F_{F_{k+2}}^2 - F_{F_{k-1}}^2 = \frac{1}{5}\left(L_{F_{k+2}}^2 - L_{F_{k-1}}^2\right).$$

(i) Because

$$\begin{aligned}
(L_{F_k}L_{F_{k+1}})(F_{F_k}F_{F_{k+1}})^3 &= (F_{F_k}F_{F_{k+1}})^2(F_{F_k}L_{F_k}F_{F_{k+1}}L_{F_{k+1}}) \\
&= (F_{F_k}F_{F_{k+1}})^2(F_{2F_k}F_{2F_{k+1}}) \\
&= (F_{F_k}F_{F_{k+1}})^2(F_{F_{k+2}}^2 - F_{F_{k-1}}^2),
\end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{n}(L_{F_k}L_{F_{k+1}})(F_{F_k}F_{F_{k+1}})^3 &= \sum_{k=1}^{n}\left((F_{F_k}F_{F_{k+1}}F_{F_{k+2}})^2 - (F_{F_{k-1}}F_{F_k}F_{F_{k+1}})^2\right) \\
&= (F_{F_n}F_{F_{n+1}}F_{F_{n+2}})^2 - (F_{F_0}F_{F_1}F_{F_2})^2 \\
&= (F_{F_n}F_{F_{n+1}}F_{F_{n+2}})^2.
\end{aligned}$$

(ii) Because

$$\begin{aligned}
(F_{F_k}F_{F_{k+1}})(L_{F_k}L_{F_{k+1}})^3 &= (L_{F_k}L_{F_{k+1}})^2(F_{F_k}L_{F_k}F_{F_{k+1}}L_{F_{k+1}}) \\
&= (L_{F_k}L_{F_{k+1}})^2(F_{2F_k}F_{2F_{k+1}}) \\
&= (L_{F_k}L_{F_{k+1}})^2(F_{F_{k+2}}^2 - F_{F_{k-1}}^2) \\
&= \frac{1}{5}(L_{F_k}L_{F_{k+1}})^2(L_{F_{k+2}}^2 - L_{F_{k-1}}^2),
\end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{n}(F_{F_k}F_{F_{k+1}})(L_{F_k}L_{F_{k+1}})^3 &= \frac{1}{5}\sum_{k=1}^{n}\left((L_{F_k}L_{F_{k+1}}L_{F_{k+2}})^2 - (L_{F_{k-1}}L_{F_k}L_{F_{k+1}})^2\right) \\
&= \frac{1}{5}\left((L_{F_n}L_{F_{n+1}}L_{F_{n+2}})^2 - (L_{F_0}L_{F_1}L_{F_2})^2\right) \\
&= \frac{1}{5}\left((L_{F_n}L_{F_{n+1}}L_{F_{n+2}})^2 - 4\right).
\end{aligned}$$

Also solved by Robert Frontczak, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.

The sum of a series involving balancing numbers

H-871 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 1, February 2021)

Let $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be the balancing and Lucas-balancing numbers, respectively, i.e., $B_{n+1} = 6B_n - B_{n-1}$ and $C_{n+1} = 6C_n - C_{n-1}$ for all $n \geq 1$ and $B_0 = 0,\ B_1 = 1,\ C_0 = 1,\ C_1 = 3$. Show that

$$\sum_{n=1}^{\infty} \frac{B_n}{n(n+1)6^n} = 6\ln 6 - \frac{17}{\sqrt{8}}\ln(3+\sqrt{8}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{C_n}{n(n+1)6^n} = 1 - 17\ln 6 + 6\sqrt{8}\ln(3+\sqrt{8}).$$

Solution by Ángel Plaza, Gran Canaria, Spain

We will use the generating functions of these sequences $\sum_{n=0}^{\infty} B_n x^n = \frac{x}{1-6x+x^2} = b(x)$, and $\sum_{n=0}^{\infty} C_n x^n = \frac{1-3x}{1-6x+x^2} = c(x)$. Since $\sum_{n=1}^{\infty} \frac{B_n}{n(n+1)6^n} = \sum_{n=1}^{\infty}\left(\frac{B_n}{n6^n} - \frac{B_n}{(n+1)6^n}\right)$, and $\sum_{n=1}^{\infty} \frac{C_n}{n(n+1)6^n} = \sum_{n=1}^{\infty}\left(\frac{C_n}{n6^n} - \frac{C_n}{(n+1)6^n}\right)$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{B_n}{n(n+1)6^n} &= \int_0^1 \left(\frac{b(x/6)}{x} - b(x/6)\right) \mathrm{d}x \\
&= \int_0^1 \frac{1-x}{6-6x+x^2/6} \mathrm{d}x \\
&= 6\ln 6 - \frac{17}{\sqrt{8}}\ln(3+\sqrt{8}).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{C_n}{n(n+1)6^n} &= \int_0^1 \left(\frac{c(x/6)-1}{x} - (c(x/6)-1)\right) \mathrm{d}x \\
&= \int_0^1 \frac{1/2 - 19x/36 + x^2/36}{1-x+x^2/36} \mathrm{d}x \\
&= 1 - 17\ln 6 + 6\sqrt{8}\ln(3+\sqrt{8}).
\end{aligned}$$

□

Also solved by Brian Bradie, Dmitry Fleischman, Haydn Gwyn, Raphael Schumacher, Seán M. Stewart, David Terr, Andrés Ventas, and the proposer.