ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-837 Proposed by Robert Frontczak, Stuttgart, Germany

The Tribonacci numbers $\{T_n\}_{n\geq 0}$ satisfy $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 3$. Prove that for any $n \geq 1$

$$\sum_{k=1}^{n} T_{2(n-k)+2}\left(\sum_{j=0}^{2(n-k)} T_j\right) = \frac{1}{2}\left(\left(\sum_{k=1}^{n} T_{2k}\right)^2 - \left(\sum_{k=1}^{n} T_{2k-1}\right)^2\right).$$

H-838 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain

Find a closed form expression for the following sum, where $r > 1$ and $n \geq r$ are integers

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{r} - \binom{r+j-1}{r} - \binom{r+j-2}{r}\right) L_{n-(n+j)}.$$

H-839 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain

For a positive integer k, the k-Fibonacci hyperbolic sine and cosine functions are defined respectively by

$$sF_kh(x) = \frac{\sigma_k^x - \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}}, \quad cF_kh(x) = \frac{\sigma_k^x + \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}},$$

where $\sigma_k = (k + \sqrt{k^2+4})/2$. If the k-Fibonacci hyperbolic tangent and cotangent are respectively $tF_kh(x) = \dfrac{sF_kh(x)}{cF_kh(x)}$ and $ctF_kh(x) = (tF_kh(x))^{-1}$, find a closed form expression for the following sum

$$\sum_{r=1}^{\infty} \frac{1}{2^r} tF_k h\left(\frac{x}{2^r}\right).$$

H-840 Proposed by Arkady Alt, San Jose, California

Prove that $(n-1)(n+1)(2nF_{n+1}-(n+6)F_n)$ is divisible by 150 for all $n \geq 1$.

H-841 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For any integer $n \geq 2$, prove that

$$\sum_{j=1}^{n} L_{a_j} < \frac{L_{n+a_n}}{L_n - 1}$$

for any integer sequence $\{a_m\}_{m\geq 1}$ with $a_1 \geq 1$ and $a_{m+1} \geq a_m + 2m + 1$ for all $m \geq 1$.

SOLUTIONS

An application of Jensen's inequality

H-805 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 55, No. 2, May 2017)

Prove that if $n \geq 2$, $p \geq 1$ are integers and $m \geq 0$, $x_k > 0$ are real numbers for $k = 1, \ldots, n$, then putting $X_n = \sum_{k=1}^{n} x_k$, we have the inequality

$$\sum_{k=1}^{n} \frac{(F_p X_n + F_{p+1} x_k)^{m+1}}{(F_{p+1}^2 X_n - F_p^2 x_k)^{2m+1}} \geq \frac{(nF_p + F_{p+1})^{m+1} n^{m+1}}{(nF_{p+1}^2 - F_p^2)^{2m+1} X_n^m}.$$

Solution by Ángel Plaza, Gran Canaria, Spain

The solution follows straightforwardly by Jensen's inequality.

First, note that the proposed inequality is homogeneous, so we may assume that $0 < x_k < 1$ for $k = 1, \ldots, n$, with $X_n = \sum_{k=1}^{n} x_k = 1$. If, in addition, we write $\alpha = F_p$ and $\beta = F_{p+1}$, the given inequality reads as

$$\sum_{k=1}^{n} \frac{(\alpha + \beta x_k)^{m+1}}{(\beta^2 - \alpha^2 x_k)^{2m+1}} \geq \frac{(n\alpha + \beta)^{m+1} n^{m+1}}{(n\beta^2 - \alpha^2)^{2m+1}}.$$

Let $f(x)$ defined by $f(x) = \dfrac{(\alpha + \beta x)^{m+1}}{(\beta^2 - \alpha^2 x)^{2m+1}}$. Then

$$f''(x) = (m+1)\frac{(\alpha + \beta x)^{m-1}}{(\beta^2 - \alpha^2 x)^{2m+3}} \cdot P,$$

where $P = \alpha^6(4m+2) + 2\alpha^5\beta(2m+1)x + \alpha^4\beta^2 m x^2 + 2\alpha^3\beta^3(2m+1) + 2\alpha^2\beta^4(m+1)x + \beta^6 m$. Since $\beta \geq \alpha$, $f''(x) > 0$ for $x \in (0,1)$ and f is convex. By Jensen's inequality, the problem follows.

Also solved by Dmitry Fleischman, Dmitriy Shtefan and Irina Dobrovolska (jointly), and the proposers.

An identity involving Tribonacci numbers

H-806 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 55, No. 2, May 2017)

The two sequences $\{T_n\}_{n\in\mathbb{Z}}$ and $\{S_n\}_{n\in\mathbb{Z}}$ satisfy

$$\begin{array}{rclcl} T_{n+3} & = & T_{n+2}+T_{n+1}+T_n & \text{with} & T_0=0,\ T_1=T_2=1, \\ S_{n+3} & = & S_{n+2}+S_{n+1}+S_n & \text{with} & S_0=3,\ S_1=1,\ S_2=3 \end{array}$$

for all integers n. For $n\geq 0$, prove that

$$\sum_{k=0}^{n} T_{(-2)^k}S_{(-2)^k} = T_{2(-2)^n}.$$

Solution by the proposer

In [1], Howard showed that

$$T_{n+2a} = S_aT_{n+a} - S_{-a}T_n + T_{n-a}.$$

Putting $n=(-2)^k$ and $a=-(-2)^k$ in the above identity, we have

$$T_{-(-2)^k} = -S_{(-2)^k}T_{(-2)^k} + T_{2(-2)^k}.$$

That is,

$$T_{(-2)^k}S_{(-2)^k} = T_{2(-2)^k} - T_{2(-2)^{k-1}}.$$

Using this identity, we have

$$\sum_{k=0}^{n} T_{(-2)^k}S_{(-2)^k} = \sum_{k=0}^{n}(T_{2(-2)^k} - T_{2(-2)^{k-1}}) = T_{2(-2)^n} - T_{-1} = T_{2(-2)^n}.$$

[1] F. T. Howard, *A Tribonacci identity*, The Fibonacci Quarterly, **39.4** (2001), 352–357.

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, and Raphael Schumacher.

Identities with sums of Euler and number of squarefree divisors functions

H-807 Proposed by Mehtaab Sawhney, Commack, NY (Vol. 55, No. 2, May 2017)

Prove for positive integers n that

$$\sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor\sum_{j=1}^{i}\mu(\gcd(i,j)) = \sum_{k=1}^{n}\phi(k),$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\mu(\gcd(i,j))\left\lfloor\sqrt{\frac{n}{ij}}\right\rfloor = \sum_{k=1}^{n}2^{\omega(k)}.$$

Solution by the proposer

Let S be the set of integral points (x, y) with $1 \leq y \leq x \leq n$, and $\gcd(x, y) = 1$. The key to the proof of the first identity is to demonstrate

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor \sum_{j=1}^{i} \mu(\gcd(i, j)) = |S| = \sum_{k=1}^{n} \phi(k).$$

Notice that for $y = i$ there are $\phi(i)$ possible y-coordinates such that $\gcd(x, y) = 1$. However, it is also possible to consider any point (x, y) with $1 \leq y \leq x \leq n$. Suppose that (x, y) lies on the ray $k\langle i, j\rangle$ with $1 \leq j \leq i \leq n$. Then, notice that there are $\lfloor \frac{n}{i} \rfloor$ points that lie in $1 \leq x \leq y \leq n$ on the ray $k\langle i, j\rangle$ with $k \in \mathbb{Z}^+$. However, using

$$\sum_{d|\gcd(x,y)} \mu(d) = 0$$

if $\gcd(x, y) \neq 1$ and that $\mu(1) = 1$, it follows that the left side also counts the number of points such that $1 \leq y \leq x \leq n$, and $\gcd(x, y) = 1$. The result follows accordingly.
Let T be the set of integral points (x, y) with $1 \leq y \leq n$, $1 \leq x \leq n$, $1 \leq xy \leq n$, and $\gcd(x, y) = 1$. The key to the proof of the second identity is to demonstrate

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mu(\gcd(i, j)) \left\lfloor \sqrt{\frac{n}{ij}} \right\rfloor = |T| = \sum_{k=1}^{n} 2^{\omega(k)}.$$

Notice that for $xy = i \leq n$ and $1 \leq y \leq n$, $1 \leq x \leq n$ there are $2^{\omega(i)}$ points (assign each prime factor independently) such that $\gcd(x, y) = 1$. However, it is also possible to consider any point (x, y) with $1 \leq y \leq n$, $1 \leq x \leq n$. Consider the points that lie on the ray $k\langle i, j\rangle$ with $1 \leq i \leq n$ and $1 \leq j \leq n$ and $k \in \mathbb{R}^+$. The intersection of this ray and the curve $xy = n$ is at distance $\sqrt{\dfrac{n(i^2 + j^2)}{ij}}$ from the origin. Therefore, there are $\left\lfloor \sqrt{\dfrac{n}{ij}} \right\rfloor$ positive integral points along this ray. However, using

$$\sum_{d|\gcd(x,y)} \mu(d) = 0$$

if $\gcd(x, y) \neq 1$, it follows that the left side sum only accounts for the points (x, y) with $1 \leq y \leq n$, $1 \leq x \leq n$, $xy \leq n$, and $\gcd(x, y) = 1$. The second identity follows accordingly.

Also solved by Jean-Marie De Koninck, Dmitry Fleischman, and Raphael Schumacher.

An identity with binomial coefficients

H-808 Proposed by Mehtaab Sawhney, Commack, NY (Vol. 55, No. 2, May 2017)

Prove that

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{j,j,n-2j}=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n}{i}\binom{2n-1-3i}{n-1}.$$

Solution by Dmitriy Shtefan and Irina Dobrovolska, Zaporizhzhya, Ukraine

Let us consider the following expansion

$$(1+x+x^2)^n=\sum_{l=0}^{2n}a_lx^l. \tag{1}$$

According to the multinomial theorem, we have

$$(1+x+x^2)^n=\sum_{i+j+k=n}\binom{n}{i,j,k}1^ix^jx^{2k}. \tag{2}$$

Now, we focus on a term from (1) with $l=n$, and calculate a_n. It can be expressed easily through the multinomials coefficients from (2). It is clear that only terms with $j+2k=n$ (or, equivalently, $k=i$) contribute to a_n, and we find

$$a_n=\sum_{j=0}^{[n/2]}\binom{n}{j,j,n-2j}. \tag{3}$$

On the other hand, applying the Maclaurin expansion to both sides of the expression

$$1+x+x^2=\frac{x^3-1}{x-1}, \tag{4}$$

we obtain

$$\begin{aligned}\left(\frac{x^3-1}{x-1}\right)^n&=(1-x^3)^n(1-x)^{-n}\\&=\sum_{i=0}^{n}(-1)^ix^{3i}\binom{n}{i}\sum_{j=0}^{\infty}(-1)^jx^j\binom{-n}{j}.\end{aligned} \tag{5}$$

Note, that for calculating the coefficient a_n it is necessary that $3i+j=n$. Also, taking into account that $\binom{n}{k}=0$ if $n>0, k>0, k>n$, we obtain

$$\begin{aligned} a_n &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(-1)^{n-3i}\binom{-n}{n-3i} \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(-1)^{2(n-3i)}\frac{(2n-3i-1)!}{(n-3i)!(n-1)!} \\ &= \sum_{i=0}^{[n/2]}(-1)^i\binom{n}{i}\binom{2n-1-3i}{n-1}. \end{aligned} \tag{6}$$

Therefore, using the identities (3) and (6), we obtain that

$$\sum_{j=0}^{[n/2]}\binom{n}{j,j,n-2j} = \sum_{i=0}^{[n/2]}(-1)^i\binom{n}{i}\binom{2n-1-3i}{n-1},$$

which is what we wanted to prove.

Also solved by the proposer.

Evaluating an infinite product

H-809 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 55, No. 3, August 2017)

Prove that

$$\left(1-\frac{\alpha}{L_2}\right)\left(1-\frac{\beta}{L_{2^2}}\right)\left(1-\frac{\alpha}{L_{2^3}}\right)\left(1-\frac{\beta}{L_{2^4}}\right)\cdots = \frac{7\sqrt{5}-5}{22}.$$

Solution by David Terr, Oceanside, California

Define the sequence $(p_n)_{n\geq 0}$ as follows:

$$\begin{aligned} p_{2m} &= \frac{7\sqrt{5}+5}{10}\prod_{k=1}^{m}\left(1-\frac{\alpha}{L_{2^{k-1}}}\right)\left(1-\frac{\beta}{L_{2^k}}\right); \\ p_{2m+1} &= \left(1-\frac{\alpha}{L_{2^{m+1}}}\right)p_{2m}. \end{aligned}$$

Since

$$\left(\frac{7\sqrt{5}+5}{10}\right)\left(\frac{7\sqrt{5}-5}{22}\right)=1,$$

we see that the desired limit is equivalent to

$$\lim_{n\to\infty} p_n = 1. \tag{7}$$

We will prove (7) and therefore the desired limit by first proving the following formula for p_n:

$$p_n = \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{n+1}} + 1 + (-1)^n \sqrt{5}}{F_{2^{n+1}}} \right). \tag{8}$$

To see that (7) follows from (8) note that (8) implies that

$$\lim_{n\to\infty} p_n = \frac{1}{\sqrt{5}} \lim_{n\to\infty} \frac{L_{2^{n+1}}}{F_{2^{n+1}}} = 1.$$

Thus, it suffices to prove (8). We prove this formula by induction. Checking the case $n = 0$ is straightforward. For the induction step, we consider two cases, n odd and n even. First, consider the case in which n is odd, that is $n = 2m - 1$ for some $m \geq 1$. Here, we have

$$\begin{aligned}
p_{n+1} &= p_{2m} = \left(1 - \frac{\beta}{L_{2^{2m}}}\right) p_{2m-1} = \frac{1}{2\sqrt{5}} \left(1 - \frac{\beta}{L_{2^{2m}}}\right) \left(\frac{2L_{2^{2m}} + 1 - \sqrt{5}}{F_{2^{2m}}} \right) \\
&= \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{2m}} - 1 + \sqrt{5}}{2L_{2^{2m}}} \right) \left(\frac{2L_{2^{2m}} + 1 - \sqrt{5}}{F_{2^{2m}}} \right) = \frac{1}{4\sqrt{5}} \left(\frac{4L^2_{2^{2m}} - (1 - \sqrt{5})^2}{F_{2^{2m+1}}} \right) \\
&= \frac{1}{4\sqrt{5}} \left(\frac{4L_{2^{2m+1}} + 8 - 6 + 2\sqrt{5}}{F_{2^{2m+1}}} \right) = \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{2m+1}} + 1 + \sqrt{5}}{F_{2^{2m+1}}} \right) \\
&= \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{n+2}} + 1 + (-1)^{n+1}\sqrt{5}}{F_{2^{n+2}}} \right),
\end{aligned}$$

which verifies (8) for $n + 1$. Finally, we consider the case when n is even, which is $n = 2m$ for some integer $m \geq 1$. Here, we have

$$\begin{aligned}
p_{n+1} &= p_{2m+1} = \left(1 - \frac{\alpha}{L_{2^{2m+1}}}\right) p_{2m} = \frac{1}{2\sqrt{5}} \left(1 - \frac{\alpha}{L_{2^{2m+1}}}\right) \left(\frac{2L_{2^{2m+1}} + 1 + \sqrt{5}}{F_{2^{2m+1}}} \right) \\
&= \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{2m+1}} - 1 - \sqrt{5}}{2L_{2^{2m+1}}} \right) \left(\frac{2L_{2^{2m+1}} + 1 + \sqrt{5}}{F_{2^{2m+1}}} \right) \\
&= \frac{1}{4\sqrt{5}} \left(\frac{4L^2_{2^{2m+1}} - (1 + \sqrt{5})^2}{F_{2^{2m+2}}} \right) = \frac{1}{4\sqrt{5}} \left(\frac{4L_{2^{2m+2}} + 8 - 6 - 2\sqrt{5}}{F_{2^{2m+2}}} \right) \\
&= \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{2m+2}} + 1 - \sqrt{5}}{F_{2^{2m+2}}} \right) = \frac{1}{2\sqrt{5}} \left(\frac{2L_{2^{n+2}} + 1 + (-1)^{n+1}\sqrt{5}}{F_{2^{n+2}}} \right),
\end{aligned}$$

again verifying (8) for $n + 1$. This completes the proof of (8), therefore of the desired limit.

Also solved by Raphael Schumacher and the proposer.

Errata: At Advanced Problem **H-829 (Vol. 56, No. 4, November 2018)** the recurrence for $\{F_{k,n}\}_{n\geq 0}$ should be "$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$" instead of "$F_{k,n+1} = F_{k,n} + F_{k,n-1}$". The editor apologizes for the inconvenience.