ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-854 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, Buzău, Romania

Compute

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)F_{n+1}}}-(f(x))^{\frac{L_n}{xL_{n+1}}}\right)^{\frac{L_{n-1}}{L_{n+1}}}\right),$$

where $f:\mathbb{R}^*\mapsto\mathbb{R}^*$ is a function that satisfies $\lim_{x\to\infty} f(x+1)/(xf(x))=a\in\mathbb{R}^*$.

H-855 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers given by $T_0=0$, $T_1=T_2=1$, and $T_n=T_{n-1}+T_{n-2}+T_{n-3}$ for $n\geq 3$. Define the functions

$$G_{FT}(z)=\sum_{n=0}^{\infty}F_nT_nz^n \quad\text{and}\quad G_{LT}(z)=\sum_{n=0}^{\infty}L_nT_nz^n.$$

Show that for $k\geq 1$, we have

$$G_{FT}(2^{-2k})=\frac{2^{4k}(2^{6k}-2^{2k}-1)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}$$

and

$$G_{LT}(2^{-2k})=\frac{2^{4k}(2^{6k}+2^{4k+1}+2^{2k}+3)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}.$$

H-856 Proposed by Robert Frontczak, Stuttgart, Germany

Let T_n denote the nth triangular number; i.e., $T_n=n(n+1)/2$. Show that

$$\sum_{n=0}^{\infty}T_n\cdot\frac{F_n}{2^{n+2}}=F_7 \quad\text{and}\quad \sum_{n=0}^{\infty}T_n\cdot\frac{L_n}{2^{n+2}}=L_7.$$

H-857 Proposed by T. Goy, Ivano-Frankivsk, Ukraine

Let T_n be the nth Tribonacci number given by $T_0 = T_1 = 0$, $T_2 = 1$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. For all $n \geq 2$, prove that

$$F_{n-2} = \sum_{i=1}^{n-1} (-1)^{i-1} \sum_{s_1+\cdots+s_i=n} T_{s_1-1} T_{s_2-1} \cdots T_{s_i-1}.$$

SOLUTIONS

A formula for π^2 involving Fibonacci numbers

H-821 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 56, No. 2, May 2018)

Prove that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{F_n} \tan^{-1} \frac{1}{F_{n+1}}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, Ill.

Note this inverse tangent identity among Fibonacci numbers [1]:

$$\tan^{-1}\left(\frac{1}{F_{2m}}\right) = \tan^{-1}\left(\frac{1}{F_{2m+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2m+2}}\right).$$

For brevity, denote the sum to be evaluated by S and use $t_n := \tan^{-1}(1/F_n)$, so that $S = \sum_{n\geq 1} t_n t_{n+1}$. Reindex the sum as

$$S = t_1 t_2 + \sum_{m\geq 1} (t_{2m} t_{2m+1} + t_{2m+1} t_{2m+2}) = t_1 t_2 + \sum_{m\geq 1} t_{2m+1}(t_{2m} + t_{2m+2}).$$

Using the arctangent identity above, we can replace the odd-indexed factor inside the summation with $t_{2m+1} = t_{2m} - t_{2m+2}$, so

$$S = t_1 t_2 + \sum_{m\geq 1} (t_{2m} - t_{2m+2})(t_{2m} + t_{2m+2}) = t_1 t_2 + \sum_{m\geq 1} (t_{2m}^2 - t_{2m+2}^2).$$

The above summation is telescopic in which only the $m = 1$ term survives. Therefore,

$$S = t_1 t_2 + t_2^2 = \tan^{-1}\left(\frac{1}{F_1}\right) \tan^{-1}\left(\frac{1}{F_2}\right) + \left(\tan^{-1}\left(\frac{1}{F_2}\right)\right)^2 = 2(\tan^{-1}(1))^2 = \frac{\pi^2}{8}.$$

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008, 37.

Also solved by Brian Bradie, Pridon Davlianidze, Dmitry Fleischman, Raphael Schumacher, and the proposer.

Some inequalities with Fibonacci numbers

H-822 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 2, May 2018)

Prove the following inequalities:

(a) $\dfrac{F_nF_{n+2}^2}{F_{n+3}}+\dfrac{F_{n+1}F_{n+3}^2}{F_n+F_{n+2}}+(F_n+F_{n+2})^2>2\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2}$;

(b) $F_{n+2}^2+(F_n+F_{n+2})^2+F_{n+3}^2>4\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2}$;

(c) $L_{n+2}^2+(L_n+L_{n+2})^2+L_{n+3}^2>4\sqrt{6}\sqrt{L_nL_{n+1}}L_{n+2}$;

(d) $\sqrt{2}\sqrt{1+F_n^4}+\sum_{k=1}^{n-1}\sqrt{(F_k^4+1)(F_{k+1}^4+1)}>2F_nF_{n+1}$ for $n>1$.

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, S.C.

(a) Applying $F_{n+2}=F_n+F_{n+1}$ and $F_{n+3}=F_n+2F_{n+1}$, we can rewrite the claimed inequality as:

$$\frac{F_n(F_n+F_{n+1})^2}{F_n+2F_{n+1}}+\frac{F_{n+1}(F_n+2F_{n+1})^2}{2F_n+F_{n+1}}+(2F_n+F_{n+1})^2>2\sqrt{6}\sqrt{F_nF_{n+1}}(F_n+F_{n+1}).$$

To make the calculation easier, we let $a:=F_n$, $b:=F_{n+1}$. So, the above inequality becomes

$$\frac{a(a+b)^2}{a+2b}+\frac{b(a+2b)^2}{2a+b}+(2a+b)^2>2\sqrt{6}\sqrt{ab}(a+b).$$

Multiplying by $(a+2b)(2a+b)$ and expanding all products, we get

$$5a^4+17a^3b+20a^2b^2+13ab^3+5b^4>\sqrt{6}\sqrt{ab}(2a^3+7a^2b+7ab^2+2b^3).$$

After squaring both sides, we get

$$\begin{aligned}&25a^8+170a^7b+489a^6b^2+810a^5b^3+892a^4b^4+690a^3b^5+369a^2b^6+130ab^7+25b^8\\ >\;&24a^7b+168a^6b^2+462a^5b^3+636a^4b^4+462a^3b^5+168a^2b^6+24ab^7,\end{aligned}$$

which is clearly true.

(b) Let $a:=F_n$, $b:=F_{n+1}$, $c:=F_{n+2}$, and $d:=F_{n+3}$. We want to prove that

$$c^2+(a+c)^2+d^2>4\sqrt{6}\sqrt{ab}c.$$

Since $d=b+c$, we have $c^2+(a+c)^2+d^2=a^2+b^2+3c^2+2ac+2bc$. Inserting $c=a+b$ into the products ac and bc, we have

$$a^2+b^2+3c^2+2ac+2bc=3a^2+3b^2+3c^2+4ab.$$

Applying the AM-GM inequality twice, we get

$$3a^2+3b^2+3c^2+4ab\geq 3c^2+3(36)^{1/3}(ab)\geq 6^{4/3}\sqrt{ab}c.$$

It is easy to check that $6^{4/3}>4\sqrt{6}$. Therefore, we have proved the claimed inequality.

(c) The proof in (b) is still valid if $a:=L_n$, $b:=L_{n+1}$, $c:=L_{n+2}$, and $d:=L_{n+3}$.

(d) Since $F_1=1$, we may rewrite the claimed inequality as a cyclic form:

$$\sqrt{(F_1^4+1)(F_2^4+1)}+\cdots+\sqrt{(F_{n-1}^4+1)(F_n^4+1)}+\sqrt{(F_n^4+1)(F_1^4+1)}>2F_nF_{n+1}.$$

Because the square mean (SM) is greater than or equal to the arithmetic mean (AM), we have

$$\begin{aligned}&\sqrt{(F_1^4+1)(F_2^4+1)}+\cdots+\sqrt{(F_{n-1}^4+1)(F_n^4+1)}+\sqrt{(F_n^4+1)(F_1^4+1)}\\ \geq\quad&\frac{(F_1^2+1)(F_2^2+1)}{2}+\cdots+\frac{(F_{n-1}^2+1)(F_n^2+1)}{2}+\frac{(F_n^2+1)(F_1^2+1)}{2}.\end{aligned}$$

We therefore only need to prove that

$$(F_1^2+1)(F_2^2+1)+\cdots+(F_{n-1}^2+1)(F_n^2+1)+(F_n^2+1)(F_1^2+1)>4F_nF_{n+1}.$$

Since $F_nF_{n+1}=\sum_{i=1}^n F_i^2$, the above inequality is equivalent to

$$(F_1^2F_2^2+1)+\cdots+(F_{n-1}^2F_n^2+1)+(F_n^2F_1^2+1)>2\sum_{i=1}^n F_i^2.$$

However, the above inequality can be transformed into

$$\begin{aligned}&(F_1^2F_2^2-F_1^2-F_2^2+1)+\cdots+(F_{n-1}^2F_n^2-F_{n-1}^2-F_n^2-1)+(F_n^2F_1^2-F_n^2-F_1^2+1)\\ =\quad&(F_1^2-1(F_2^2-1)+\cdots+(F_{n-1}^2-1)(F_n^2-1)+(F_n^2-1)(F_1^2-1)>0.\end{aligned}$$

The proof is then complete. Note that the equality does not occur in either of the above inequalities or in the SM-AM inequality for $n\geq 3$.

Also solved by Kenneth B. Davenport, Dmitry Fleichman, and the proposers.

Some summation formulas with general recurrences

H-823 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 56, No. 2, May 2018)

Given an integer $r\geq 2$, define the sequence $\{G_n\}_{n\geq -r+1}$ by

$$G_n=G_{n-1}+G_{n-2}+\cdots+G_{n-r}\quad\text{for}\quad n\geq 1$$

with arbitrary G_0, G_{-1}, G_{-2}, …, G_{-r+1}. For an integer $n\geq 1$, prove that

(i) $$\sum_{k=1}^n G_kG_{k+r}=\sum_{k=1}^r\frac{k(r-k-1)+r+1}{2(r-1)}\sum_{i=1}^k(G_{n+i-k}G_{n+i}-G_{i-k}G_i);$$

(ii) $$\sum_{k=1}^n G_kG_{k+r+1}=\sum_{k=1}^r\frac{k(r-k-1)+2r}{2(r-1)}\sum_{i=1}^k(G_{n+i-k}G_{n+i}-G_{i-k}G_i).$$

Solution by the proposer

Let $S_m:=\sum_{k=1}^n G_kG_{k+m}$ and $A_k:=\sum_{i=1}^k(G_{n+i-k}G_{n+i}-G_{i-k}G_i)$. We use the identity

$$S_0=\sum_{k=1}^r\frac{k(r-k-1)+2}{2(r-1)}A_k\qquad\text{(see [1]).}$$

For $m\geq 0$, since

$$2G_m=G_m+G_{m-1}+\cdots+G_{m-r+1}+G_{m-r}=G_{m+1}+G_{m-r},$$

we have

$$2S_m=S_{m+1}+S_{m-r}.\tag{1}$$

For $1 \leq k \leq r$, we have

$$S_k - S_{-k} = \sum_{i=1}^{n}(G_iG_{i+k} - G_{i-k}G_i) = \sum_{i=1}^{k}(G_{n+i-k}G_{n+i} - G_{i-k}G_i) = A_k. \qquad (2)$$

(i) We have

$$\begin{aligned} S_r &= S_0 + \sum_{k=1}^{r}(S_k - S_{k-1}) = S_0 + \frac{1}{2}\sum_{k=1}^{r}(S_k - S_{k-1-r}) \qquad \text{(by (1))} \\ &= S_0 + \frac{1}{2}\sum_{k=1}^{r}(S_k - S_{-k}) = \sum_{k=1}^{r}\frac{k(r-k-1)+2}{2(r-1)}A_k + \frac{1}{2}\sum_{k=1}^{r}A_k \qquad \text{(by (2))} \\ &= \sum_{k=1}^{r}\frac{k(r-k-1)+r+1}{2(r-1)}A_k. \end{aligned}$$

(ii) We have

$$\begin{aligned} S_{r+1} &= 2S_r - S_0 \qquad \text{(by (1))} \\ &= 2\sum_{k=1}^{r}\frac{k(r-k-1)+r+1}{2(r-1)}A_k - \sum_{k=1}^{r}\frac{k(r-k-1)+2}{2(r-1)}A_k \\ &= \sum_{k=1}^{r}\frac{k(r-k-1)+2r}{2(r-1)}A_k. \end{aligned}$$

[1] Hideyuki Ohtsuka, *Sums of squares of members of r-generalized Fibonacci like sequences (solution to Advanced Problem H-759)*, The Fibonacci Quarterly, **54.3** (2016), 281–282.

Also solved by Dmitry Fleischman.

Identities with generalized Fibonomial coefficients

H-824 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 56, No. 2, May 2018)

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;r}$ by

$$\binom{n}{k}_{F;r} = \frac{F_{rn}F_{r(n-1)}F_{r(n-2)}\cdots F_{r(n-k+1)}}{F_{rk}F_{r(k-1)}F_{r(k-2)}\cdots F_r} \quad \text{for} \quad 0 < k \leq n,$$

with $\binom{n}{0}_{F;r} = 1$ and $\binom{n}{k}_{F;r} = 0$ (otherwise). For positive integers n, r, and s, find closed form expressions for the sums

(i) $\displaystyle\sum_{i+j=2s-1}(-1)^{(r+1)i}\binom{n-1}{i}_{F;r}\binom{n+1}{j}_{F;r}$;

(ii) $\displaystyle\sum_{i+j=2s}(-1)^{i}\binom{n-1}{i}_{F;r}\binom{n+1}{j}_{F;r}$.

Solution by the proposer

Let m, n, r, and s be positive integers. We use the identity

$$\sum_{k=0}^{n}(-1)^{\frac{rk(k+1)}{2}+mk}\binom{n}{k}_{F;r}z^k=\prod_{k=1}^{n}\left(1+(-1)^m\alpha^{r(n-k+1)}\beta^{rk}z\right)\qquad\text{(see [1])}.$$

We have

$$\begin{aligned}
&\sum_{l=0}^{2n}\left(\sum_{i+j=l}(1-)^{\frac{r}{2}(i^2+i+j^2+j)+i+j+rj}\binom{n+1}{i}_{F;r}\binom{n-1}{j}_{F;r}\right)z^l\\
=&\left(\sum_{i=0}^{n+1}(-1)^{\frac{ri(i+1)}{2}+i}\binom{n+1}{i}_{F;r}z^i\right)\left(\sum_{j=0}^{n-1}(-1)^{\frac{rj(j+1)}{2}+rj}\binom{n-1}{j}_{F;r}z^j\right)\\
=&\prod_{k=1}^{n+1}\left(1-\alpha^{r(n-k+2)}\beta^{rk}z\right)\prod_{k=1}^{n-1}\left(1+(-1)^r\alpha^{r(n-k)}\beta^{rk}z\right)\\
=&\prod_{k=0}^{n}\left(1-\alpha^{r(n-k+1)}\beta^{r(k+1)}z\right)\prod_{k=1}^{n-1}\left(1+(-1)^r\alpha^{r(n-k)}\beta^{rk}z\right)\\
=&\prod_{k=0}^{n}\left(1-(-1)^r\alpha^{r(n-k)}\beta^{rk}z\right)\prod_{k=1}^{n-1}\left(1+(-1)^r\alpha^{r(n-k)}\beta^{rk}z\right)\\
=&\left(1-(-1)^r\alpha^{rn}z\right)\left(1-(-1)^r\beta^{rn}z\right)\prod_{k=1}^{n-1}\left(1-\alpha^{2r(n-k)}\beta^{2rk}z^2\right)\\
=&\left(1-(-1)^rL_{rn}z+(-1)^{rn}z^2\right)\sum_{k=0}^{n-1}\binom{n-1}{k}_{F;2r}z^{2k}\qquad(3)
\end{aligned}$$

(by $\alpha\beta=-1$ and $L_{rn}=\alpha^{rn}+\beta^{rn}$).

(i) In (3), by comparing the coefficient of z^{2s-1}, we have

$$\sum_{i+j=2s-1}(-1)^{\frac{r}{2}(i^2+i+j^2+j)+i+j+rj}\binom{n+1}{i}_{F;r}\binom{n-1}{j}_{F;r}=(-1)^{r+s}L_{rn}\binom{n-1}{s-1}_{F;2r}.$$

By $j=2s-1-i$, we have

$$(-1)^{\frac{r}{2}(i^2+i+j^2+j)+i+rj+r+s}=(-1)^{ri^2+i+rs+s-2rsi+2rs^2}=(-1)^{(r+1)i+(r+1)s}.$$

Therefore, we obtain

$$\sum_{i+j=2s-1}(-1)^{(r+1)i}\binom{n+1}{i}_{F;r}\binom{n-1}{j}_{F;r}=(-1)^{(r+1)s}L_{rn}\binom{n-1}{s-1}_{F;2r}.$$

(ii) In (3), by comparing the coefficient of z^{2s}, we have

$$\begin{aligned}
&\sum_{i+j=2s}(-1)^{\frac{r}{2}(i^2+i+j^2+j)+i+rj}\binom{n+1}{i}_{F;r}\binom{n-1}{j}_{F;r}\\
=&(-1)^s\binom{n-1}{s}_{F;2r}-(-1)^{rn+s}\binom{n-1}{s-1}_{F;2r}.
\end{aligned}$$

By $j = 2s - i$, we have

$$(-1)^{\frac{r}{2}(i^2+i+j^2+j)+i+rj} = (-1)^{i+ri^2-ri+3rs-2rsi+2rs^2} = (-1)^{i+rs}.$$

Therefore, we obtain

$$\sum_{i+j=2s} (-1)^i \binom{n-1}{i}_{F;r} \binom{n+1}{j}_{F;r} = (-1)^{(r+1)s} \left(\binom{n-1}{s}_{F;2r} - (-1)^{rn} \binom{n-1}{s-1}_{F;2r} \right).$$

[1] Hideyuki Ohtsuka, *An identity with Fibonomial coefficients (solution to Advanced Problem H-746)*, The Fibonacci Quarterly, **53.3** (2015), 283–285.

Also solved by Dmitry Fleischman.

Errata: There are some typos in the Advanced Problem Section of Volume **58** Number 1, February 2020, Pages 89–95 as follows:

(i) Page 89, Line -1: The exponent "$1-(n-k)-1$" in "$2^{1-(n-k)-1}$" should be "$1-(n-k)$". Also, the two occurrences of "B_{rn}" from lines -2 and -5 (right) at this page should be "F_{rn}".
(ii) Page 90, Lines 4-5: In the left sides of (i) and (ii), the denominators should be under a square root "$\sqrt{\dots}$".
(iii) Page 92, Line 10: The numerator "$F_{n+4} - F_{n+1}$" should be "$F_{n+1} - F_n$".
(iv) Page 94, Line 7: The expression "$F - n^b$" should be "F_n^b".

The Editor apologizes for these typos.