ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-895 Proposed by Andrei K. Svinin, Irkutsk, Russia

Consider the Genocchi numbers $G_{2n} = (-1)^{n-1}2(4^n - 1)B_{2n}$ for $n \geq 1$, where B_{2n} is the Bernoulli number.

(1) Prove that $\displaystyle\sum_{j=0}^{\lfloor (n-1)/3 \rfloor} \frac{1}{2j+1}\binom{n-j-1}{2j}\left(\frac{4}{27}\right)^j = \frac{4^n - 1}{3^{n-1}(2n+1)}$ and deduce that

$$G_{2n} = \sum_{j=0}^{\lfloor (n-1)/3 \rfloor} G_{2n}^{(j)}, \text{ where } G_{2n}^{(j)} = (-1)^{n-1}\frac{2^{2j+1}}{3^{3j-n+1}}\frac{2n+1}{2j+1}\binom{n-j-1}{2j}B_{2n}.$$

(2) Show that $G_{2p}^{(j)} \in \mathbb{N}$ for all $j = 0, 1, \ldots, \lfloor (p-1)/3 \rfloor$ if and only if p is prime.

(3) Prove that the g.c.d. of the set of numbers $\{G_{2p}^{(j)} : j = 0, \ldots, \lfloor (p-1)/3 \rfloor\}$ with a fixed prime $p \geq 5$ is the numerator of the Bernoulli number B_{2p}.

H-896 Proposed by Mihály Bencze, Braşov, Romania

Prove that

(1) $\displaystyle n\sum_{k=1}^{n} F_k^3 + (F_{n+2} - 1)^3 \leq (n+1)F_nF_{n+1}(F_{n+2} - 1)$ holds for all $n \geq 1$;

(2) $\displaystyle n\sum_{k=1}^{n} L_k^3 + (L_{n+2} - 1)^3 \leq (n+1)(L_nL_{n+1} - 2)(L_{n+2} - 1)$ holds for all $n \geq 1$.

H-897 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}} = \sum_{n=0}^{\infty} \frac{1}{L_{2F_{2n}}L_{2F_{2n+3}}}$;

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{L_{F_n}^2 L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}} = \sum_{n=0}^{\infty} \frac{1}{L_{F_n}^2 L_{F_{n+3}}^2} + \frac{1}{4}$.

H-898 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

Compute

$$\lim_{n\to\infty}(\sqrt[n]{n!})^2\left(\frac{\sqrt[n]{n!L_n}}{n^2}-\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\right).$$

H-899 Proposed by Robert Frontczak, Stuttgart, Germany

Show that

$$\sum_{n=1}^{\infty}\sinh^{-1}\left(\frac{1}{5F_nF_{n+1}}(L_{n+1}\sqrt{2L_{2n}}-L_n\sqrt{2L_{2n+2}})\right)=\frac{1}{2}\ln\left(\frac{(3+2\sqrt{2})(7-2\sqrt{6})}{5}\right).$$

H-900 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $\mathbf{i}=\sqrt{-1}$. For any odd integer $m\geq 1$, prove that

$$\sum_{n=0}^{\infty}\frac{1}{L_{m(2n+1)}+L_{2m}\mathbf{i}}=\frac{2}{5F_mF_{2m}}-\frac{\mathbf{i}}{\sqrt{5}F_{2m}}.$$

SOLUTIONS

An identity with Bernoulli and Lucas numbers

H-860 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 3, August 2020)

Let $(B_n)_{n\geq 0}$ be the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{n\geq 0}B_n\frac{z^n}{n!}\quad(|z|<2\pi).$$

Show that for all $n\geq 0$, we have

$$\sum_{\substack{k=0\\ k\equiv n \pmod 2}}^{n}\binom{n}{k}(2^kL_k-2)5^{(n-k)/2}\frac{B_{n-k+2}}{n-k+2}=\frac{2^{n+2}L_{n+2}-2}{5(n+1)(n+2)}-1.$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

Because $B_{2m+1}=0$ for all $m\in\mathbb{N}=\{1,2,3,4,\ldots\}$, we have that

$$\sum_{k=0}^{n}\binom{n}{k}\left(2^kL_k-2\right)5^{(n-k)/2}\frac{B_{n-k+2}}{n-k+2}=\sum_{\substack{k=0\\ k\equiv n \pmod 2}}^{n}\binom{n}{k}\left(2^kL_k-2\right)5^{(n-k)/2}\frac{B_{n-k+2}}{n-k+2}.$$

We have the generating function identities

$$f_1(x) := \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1},$$

$$f_2(x) := \sum_{k=0}^{\infty} \frac{B_{k+2}}{(k+2)k!} x^k = \frac{1}{x^2} - \frac{1}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2},$$

$$f_3(x) := \sum_{k=0}^{\infty} \frac{L_k}{k!} x^k = e^{\alpha x} + e^{\beta x},$$

$$f_4(x) := \sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x,$$

$$f_5(x) := \sum_{k=0}^{\infty} \frac{L_{k+2}}{(k+2)!} x^k = \frac{e^{\alpha x} - \alpha x - 1}{x^2} + \frac{e^{\beta x} - \beta x - 1}{x^2},$$

$$f_6(x) := \sum_{k=0}^{\infty} \frac{1}{(k+2)!} x^k = \frac{e^x - x - 1}{x^2}.$$

Therefore, we can calculate

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \left(2^k L_k - 2\right) 5^{(n-k)/2} \frac{B_{n-k+2}}{n-k+2} \right] \frac{x^n}{n!} \\
&= f_2\left(\sqrt{5}x\right) \left[f_3(2x) - 2f_4(x)\right] \\
&= \left(\frac{1}{5x^2} - \frac{1}{\left(e^{\frac{\sqrt{5}x}{2}} - e^{-\frac{\sqrt{5}x}{2}}\right)^2} \right) \cdot \left(e^{2\alpha x} + e^{2\beta x} - 2e^x\right) \\
&= \left(\frac{1}{5x^2} - \frac{e^x}{\left(e^{\alpha x} - e^{\beta x}\right)^2} \right) \cdot \left(e^{2\alpha x} + e^{2\beta x} - 2e^x\right) \\
&= \frac{e^{2\alpha x} + e^{2\beta x} - 2e^x}{5x^2} - e^x \\
&= \frac{e^{2\alpha x} + e^{2\beta x} - 2e^x + 2(1 - \alpha - \beta)x}{5x^2} - e^x \\
&= \frac{e^{2\alpha x} - 2\alpha x - 1}{5x^2} + \frac{e^{2\beta x} - 2\beta x - 1}{5x^2} + \frac{2x + 2 - 2e^x}{5x^2} - e^x \\
&= \frac{4}{5} \left(\frac{e^{2\alpha x} - 2\alpha x - 1}{4x^2} + \frac{e^{2\beta x} - 2\beta x - 1}{4x^2} \right) - \frac{2}{5} \left(\frac{e^x - x - 1}{x^2} \right) - e^x \\
&= \frac{4}{5} f_5(2x) - \frac{2}{5} f_6(x) - f_4(x) \\
&= \sum_{n=0}^{\infty} \left[\frac{2^{n+2} L_{n+2} - 2}{5(n+1)(n+2)} - 1 \right] \frac{x^n}{n!}.
\end{aligned}$$

This implies the result by comparing the coefficients $\left[\frac{x^n}{n!}\right]$ on both sides.
In the above calculation, we have used the identities $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, and $\alpha + \beta = 1$.

Furthermore, we have also used that

$$\begin{aligned}\left(\frac{1}{5x^2}-\frac{e^x}{\left(e^{\alpha x}-e^{\beta x}\right)^2}\right)\cdot\left(e^{2\alpha x}+e^{2\beta x}-2e^x\right) &= \frac{\left(e^{\alpha x}-e^{\beta x}\right)^2-5x^2e^x}{5x^2\left(e^{\alpha x}-e^{\beta x}\right)^2}\cdot\left(e^{2\alpha x}+e^{2\beta x}-2e^{(\alpha+\beta)x}\right)\\ &= \frac{e^{2\alpha x}+e^{2\beta x}-2e^{(\alpha+\beta)x}-5x^2e^x}{5x^2\left(e^{\alpha x}-e^{\beta x}\right)^2}\cdot\left(e^{\alpha x}-e^{\beta x}\right)^2\\ &= \frac{e^{2\alpha x}+e^{2\beta x}-2e^{x}-5x^2e^x}{5x^2\left(e^{\alpha x}-e^{\beta x}\right)^2}\cdot\left(e^{\alpha x}-e^{\beta x}\right)^2\\ &= \frac{e^{2\alpha x}+e^{2\beta x}-2e^{x}-5x^2e^x}{5x^2}\\ &= \frac{e^{2\alpha x}+e^{2\beta x}-2e^{x}}{5x^2}-e^x.\end{aligned}$$

Also solved by Dmitry Fleischman, Albert Stadler, and the proposer.

The sum of products of two consecutive generalized Tribonacci numbers

H-861 Proposed by David Terr, Oceanside, CA
(Vol. 58, No. 3, August 2020)

For arbitrary constants a, b, c define the sequence $(G_n)_{n\geq 0}$ by $G_0=a$, $G_1=b$, $G_2=c$ and the recurrence $G_n=G_{n-1}+G_{n-2}+G_{n-3}$ for $n\geq 3$. Find a closed form expression for

$$\sum_{j=0}^{n} G_{2j}G_{2j+1} \quad \text{valid for all} \quad n\geq 0.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

We have

$$2G_{2j}=G_{2j+1}+G_{2j}-G_{2j-1}-G_{2j-2} \tag{1}$$

and

$$2G_{2j+1}=G_{2j+1}+G_{2j}+G_{2j-1}+G_{2j-2}. \tag{2}$$

We have

$$\begin{aligned}\sum_{j=0}^{n} G_{2j}G_{2j+1} &= G_0G_1+\sum_{j=1}^{n} G_{2j}G_{2j+1}\\ &= ab+\frac{1}{4}\sum_{j=1}^{n}\left((G_{2j+1}+G_{2j})^2-(G_{2j-1}+G_{2j-2})^2\right) \quad \text{(by (1) and (2))}\\ &= ab+\frac{1}{4}\left((G_{2n+1}+G_{2n})^2-(G_1+G_0)^2\right)\\ &= \frac{(G_{2n+1}+G_{2n})^2-(b+a)^2+4ab}{4}=\frac{(G_{2n+1}+G_{2n})^2-(a-b)^2}{4}.\end{aligned}$$

Editorial comment: An equivalent form of the above identity was noticed before the problem was published by Kenneth B. Davenport as a generalization of Advanced Problem **H-833**. Davenport's letter is dated March 4, 2020, whereas Terr's proposal arrived on March 11, 2019.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Robert Frontczak, Raphael Schumacher, Albert Stadler, Andrés Ventas, and the proposer.

Sums of generalized Fibonacci and Lucas numbers

H-862 Proposed by Ángel Plaza, Gran Canaria, Spain (Vol. 58, No. 3, August 2020)

Let $(F_{k,n})_{n\in\mathbb{Z}}$ and $(L_{k,n})_{n\in\mathbb{Z}}$ denote the k-Fibonacci and k-Lucas numbers given by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $n \geq 1$ with $F_{k,0} = 0$, $F_{k,1} = 1$, $L_{k,0} = 2$, $L_{k,1} = k$. Prove that for integers $m \geq 1$ and $j \geq 0$, we have

(i) $\displaystyle\sum_{n=1}^{m} F_{k,n\pm j}L_{k,n\mp j} = \frac{F_{k,2m+1}-1}{k} + \begin{cases} 0, & \text{if } m \equiv 0 \pmod 2; \\ (-1)^j F_{k,\pm 2j}, & \text{if } m \equiv 1 \pmod 2. \end{cases}$

(ii) $\displaystyle\sum_{n=1}^{m} F_{k,n+j}F_{k,n-j}L_{k,n+j}L_{k,n-j} = \frac{F_{k,4m+2}/k - 1 - mL_{k,4j}}{k^2+4}.$

Solution by the proposer

We will use the Binet's formulas for these numbers $F_{k,n} = \dfrac{\alpha_k^n - \beta_k^n}{\alpha_k - \beta_k}$ and $L_{k,n} = \alpha_k^n + \beta_k^n$, where $\alpha_k = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\beta_k = \dfrac{k-\sqrt{k^2+4}}{2}$. Note that $\alpha_k \cdot \beta_k = -1$, $\alpha_k + \beta_k = k$, and $\alpha_k - \beta_k = \sqrt{k^2+4}$.

Therefore, for (i),

$$\begin{aligned}
\sum_{n=1}^{m} F_{k,n\pm j}L_{k,n\mp j} &= \frac{1}{\alpha_k-\beta_k}\sum_{n=1}^{m}\left(\alpha_k^{n\pm j}-\beta_k^{n\pm j}\right)\left(\alpha_k^{n\mp j}+\beta_k^{n\mp j}\right) \\
&= \frac{1}{\alpha_k-\beta_k}\sum_{n=1}^{m}\left(\alpha_k^{2n}-\beta_k^{2n}\right) + \frac{1}{\alpha_k-\beta_k}\sum_{n=1}^{m}(-1)^k\left(\left(\frac{\alpha_k}{\beta_k}\right)^{\pm j} - \left(\frac{\beta_k}{\alpha_k}\right)^{\pm j}\right) \\
&= \frac{1}{\alpha_k-\beta_k}\left(\frac{\alpha_k^2-\alpha_k^{2m+2}}{1-\alpha_k^2} - \frac{\beta_k^2-\beta_k^{2m+2}}{1-\beta_k^2}\right) + \sum_{n=1}^{m}(-1)^n(-1)^{\pm j}F_{k,\pm 2j} \\
&= \frac{1}{\alpha_k+\beta_k}\left(\frac{\alpha_k^{2m+1}-\beta_k^{2m+1}-(\alpha_k-\beta_k)}{\alpha_k-\beta_k}\right) + (-1)^{\pm j}F_{k,\pm 2j}\sum_{n=1}^{m}(-1)^n \\
&= \frac{F_{k,2m+1}-1}{k} + \begin{cases} 0, & \text{if } m \text{ is even;} \\ (-1)^{\pm j+1}F_{k,\pm 2j}, & \text{if } m \text{ is odd.} \end{cases}
\end{aligned}$$

Now, for (ii), we use that for any integer m, $F_{k,m}L_{k,m} = F_{k,2m}$, so

$$\begin{aligned}
\sum_{n=1}^{m} F_{k,n+j}F_{k,n-j}L_{k,n+j}L_{k,n-j} &= \sum_{n=1}^{m} F_{k,2n+2j}F_{k,2n-2j} \\
&= \frac{1}{(\alpha_k-\beta_k)^2}\sum_{n=1}^{m}\left(\alpha_k^{2n+2j}-\beta_k^{2n+2j}\right)\left(\alpha_k^{2n-2j}-\beta_k^{2n-2j}\right) \\
&= \frac{1}{(\alpha_k-\beta_k)^2}\sum_{n=1}^{m}\left[\left(\alpha_k^{4n}+\beta_k^{4n}\right)-\left(\frac{\alpha_k}{\beta_k}\right)^{2j}-\left(\frac{\beta_k}{\alpha_k}\right)^{2j}\right] \\
&= \frac{1}{(\alpha_k-\beta_k)^2}\sum_{n=1}^{m} L_{k,4n} - \frac{1}{(\alpha_k-\beta_k)^2}\sum_{n=1}^{m}\left(\alpha_k^{4j}+\beta_k^{4j}\right) \\
&= \frac{\frac{F_{k,4m+2}}{k}-1-mL_{k,4j}}{k^2+4},
\end{aligned}$$

where we have used that $\sum_{n=1}^{m} L_{k,4n} = \frac{F_{k,4m+2}}{k} - 1$.

Also solved by Brian Bradie, Dmitry Fleischman, Albert Stadler, David Terr, and Andrés Ventas.

Sums of series involving values of the Riemann zeta function

H-863 Proposed by Kenneth B. Davenport, Dallas, PA (Vol. 58, No. 4, November 2020)

Show that

$$\sum_{n\geq 1}\frac{\zeta(2n+1)-1}{2n+1} = 1-\gamma-\frac{\ln 2}{2} \quad \text{and} \quad \sum_{n\geq 1}\frac{\zeta(2n)-1}{n(n+1)} = \ln(2\pi)-\frac{3}{2},$$

where $\zeta(n)$ is the Riemann zeta function.

Solution by Andrés Ventas, Santiago de Compostela, Spain

The first formula appears in the work of Srivastava (formula 2.31 in [1]), and he mentions that it is a known result from Legendre.

For the second formula, we are going to use another formula from Srivastava (formula 2.32 in [2])

$$\sum_{n\geq 1}\frac{\zeta(2n)-1}{n} = \ln(2)$$

and a second formula from Choi, Srivastava, and Quine (formula 2.17 in [2])

$$\sum_{n\geq 1}\frac{\zeta(2n)-1}{n+1} = \frac{3}{2}-\ln(\pi).$$

Thus, we have

$$\sum_{n\geq 1}\frac{\zeta(2n)-1}{n(n+1)} = \sum_{n\geq 1}\frac{\zeta(2n)-1}{n} - \sum_{n\geq 1}\frac{\zeta(2n)-1}{n+1} = \ln(2)-\left(\frac{3}{2}-\ln(\pi)\right) = \ln(2\pi)-\frac{3}{2}.$$

References

[1] H. M. Srivastava, *A unified presentation of certain classes of series of the Riemann Zeta function*, Riv. Mat. Univ. Parma, **14.4** (1988), 1–23.

[2] J. Choi, H. M. Srivastava, and J. R. Quine, *Some series involving the zeta function*, Bull. Austral. Math. Soc., **51** (1995), 383–393.

Also solved by Michel Bataille, Brian Bradie, Khristo N. Boyadzhiev, Alejandro Cardona Castrillón, Dmitry Fleischman, Robert Frontczak, Won Kyun Jeong, Raphael Schumacher, Albert Stadler, Séan M. Stewart, David Terr, and the proposer.

A series involving inverse tangents of reciprocals of Pell numbers

H-864 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 58, No. 4, November 2020)

The Pell numbers $\{P_n\}_{n\geq 0}$ satisfy $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{\sqrt{2}P_n}\tan^{-1}\frac{1}{\sqrt{2}P_{n+1}} = \frac{\pi}{4}\tan^{-1}\frac{1}{2\sqrt{2}}.$$

Solution by Andrés Ventas, Santiago de Compostela, Spain

This can be solved in the same way as the solution from Jason L. Smith to the Advanced Problem **H-821** (**Vol. 58, No. 2, May 2020**)

$$\text{formula for } \textbf{H-821}\text{:}\ \tan^{-1}\left(\frac{1}{F_n}\right) = \tan^{-1}\left(\frac{1}{F_{n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{n+2}}\right).$$

$$\text{formula for } \textbf{H-864}\text{:}\ \tan^{-1}\left(\frac{1}{\sqrt{2}P_n}\right) = 2\tan^{-1}\left(\frac{1}{\sqrt{2}P_{n+1}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}P_{n+2}}\right).$$

Using the same notation as the one used by Jason L. Smith,

$$\begin{aligned}
S &= t_1t_2 + \sum_{m\geq 1} t_{2m}t_{2m+1} + t_{2m+1}t_{2m+2} = t_1t_2 + \sum_{m\geq 1} t_{2m+1}(t_{2m} + t_{2m+2}) \\
&= t_1t_2 + \frac{1}{2}\sum_{m\geq 1}(t_{2m} - t_{2m+2})(t_{2m} + t_{2m+2}) = t_1t_2 + \frac{1}{2}\sum_{m\geq 1}(t_{2m}^2 - t_{2m+2}^2) \\
\text{(telescoping)} &= t_1t_2 + \frac{1}{2}t_2^2 = \left(t_1 + \frac{1}{2}t_2\right)t_2 \\
&= \left(\tan^{-1}\frac{1}{\sqrt{2}P_1} + \frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}P_2}\right)\tan^{-1}\frac{1}{\sqrt{2}P_2} \\
&= \left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)\right)\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = \frac{\pi}{4}\tan^{-1}\frac{1}{2\sqrt{2}}.
\end{aligned}$$

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Ángel Plaza, Albert Stadler, and the proposer.

A limit with nth roots of products of Fibonacci and Lucas numbers

H-865 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania (Vol. 58, No. 4, November 2020)

Let $\{x_n\}_{n\geq 0}$ be the sequence given by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n \quad \text{for} \quad n \geq 0.$$

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}} - \sqrt[n]{F_nL_nx_n}\right).$$

Solution by Ángel Plaza, Gran Canaria, Spain

After solving the recurrrence relation, for example by generating functions, we have $x_n = \dfrac{n(n+3)!}{4(n+1)(n+2)} = \dfrac{n(n+3)n!}{4}$. Now by the Cezaro-Stolz lemma the proposed limit, say L, is

$$\begin{aligned} L &= \lim_{n\to\infty}\frac{\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}}}{n+1} \\ &= \lim_{n\to\infty}\frac{\sqrt[n+1]{\alpha^{2(n+1)}(n+1)!}}{n+1} \\ &= \lim_{n\to\infty}\frac{\alpha^2(n+1)e^{-1}}{n+1} \\ &= \frac{\alpha^2}{e}. \end{aligned}$$

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Albert Stadler, Andrés Ventas, and the proposer.